Characteristics of Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Bivariate Gaussian Density

two random variables

Definition

The two random variables X and Y are said to be jointly Gaussian, if their joint density function is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$exp\left\{\frac{-1}{2(1-\rho^2)} \cdot \left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$\overline{X} = E[X], \qquad \sigma_X^2 = E[(X-\overline{X})^2],$$

$$Y = E[Y], \qquad \sigma_Y^2 = E[(Y-\overline{Y})^2],$$

$$\rho = E[(X-\overline{X})(Y-\overline{Y})]/\sigma_X\sigma_Y$$

Bivariate Gaussian Density - Maximum value

two random variables

$$f_{X,Y}(x,y) \le f_{X,Y}(\overline{X},\overline{Y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

Bivariate Gaussian Density - Uncorrelated

two random variables

$f_{X,Y}(x,y) = f_X(x)f_Y(x)$ is sufficient to guarantee that X and Y are statistically independent. Any uncorrelated Guassian random variables are also statistically independent a coordinate rotation (linear transformation of X and Y) through the angle

$$\theta = \frac{1}{2} tan^{-1} \left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2\sigma_Y^2} \right]$$

is sufficient to convert correlated random variables X and Y having σ_X^2 and σ_Y^2 , respectively, correlation coefficient ρ, and the joint densityof $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot exp[\cdots]$ into two statistically independent Gaussian random variables

Multi-variate Gaussian Density

N random variables

N random variables $X_1, X_2, ..., X_N$ are called jointly Gaussian if their joint density function can be written as

$$f_{X_1,\cdots,X_N}(x_1,\cdots,x_N) = \frac{\left|[C_X|^{-1}\right|^{1/2}}{(2\pi)^{N/2}} exp\left\{-\frac{[x-\overline{X}]^t[C_X][x-\overline{X}]}{2}\right\}$$

where

$$[x-\overline{X}] = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ \\ x_N - \overline{X}_N \end{bmatrix}, \qquad [C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

Multi-variate Gaussian Density - notations

N random variables

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where $[\bullet]^t$denotes a matrix transposition,
$[\bullet]^{-1}$denotes a matrix inversion
$|\bullet|$ denotes a matrix determinant

Covariance Matrix
N random variables

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where $[C_x]$ is called the covariance matrix of N random variables

$$C_{ij} = E[(X_i - \overline{X}_i)(X_j - \overline{X}_j)] = \begin{cases} \sigma_{X_i}^2 & i = j \\ C_{X_i X_j} & i \neq j \end{cases}$$

Covariance Matrix ($N = 2$)

N random variables

$$f_{X_1 X_2}(x_1, x_2) = \frac{\left|[C_X|^{-1}\right|^{1/2}}{(2\pi)^{2/2}} exp\left\{-\frac{[x-\overline{X}]^t [C_X][x-\overline{X}]}{2}\right\}$$

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{X_1}^2 & \rho\sigma_{X_1}\sigma_{X_2} \\ \rho\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$[C_X]^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} \sigma_{X_1}^2 & -\rho/\sigma_{X_1}\sigma_{X_2} \\ -\rho/\sigma_{X_1}\sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$\left|[C_X]^{-1}\right| = 1/\sigma_{X_1}^2\sigma_{X_2}^2(1-\rho^2)$$

Properties of Gaussian Random Variables
N random variables

- completely defined thorugh only their first and second order mements - their means, variances, and covariances
- if uncorrelated, then statistically independent
- a linear transformation of Gaussian random variables will produce Gaussian random variables
- any k-dimensional (k-variate) marginal density function obtained from the N-dimensional density function by integrating $N-k$ random variables will be Gaussian
- $X_1, ..., X_k$by integrating out $X_{k+1}, ..., X_N$
- the covariance of $X_1, ..., X_k$ is the $k \times k$ submatrix of the $N \times N$ covariance matrix of $X_1, ..., X_N$