Characteristics of Multiple Random Variables

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles, Jr. and B. Shi

Outline

1 Joint Guassian Random Variables



Bivariate Gaussian Density

two random variables

Definition

The two random variables X and Y are said to be jointly Gaussian, if their joint density function is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left\{\frac{-1}{2(1-\rho^2)} \cdot \left[\frac{(x-\overline{X})^2}{\sigma_X^2} - \frac{2\rho(x-\overline{X})(y-\overline{Y})}{\sigma_X\sigma_Y} + \frac{(y-\overline{Y})^2}{\sigma_Y^2}\right]\right\}$$

$$\overline{X} = E[X], \qquad \sigma_X^2 = E[(X - \overline{X})^2],$$

$$Y = E[Y], \qquad \sigma_Y^2 = E[(Y - \overline{Y})^2],$$

$$\rho = E[(X - \overline{X})(Y - \overline{Y})]/\sigma_X\sigma_Y$$

Bivariate Gaussian Density - Maximum value two random variables

$$f_{X,Y}(x, y) \le f_{X,Y}(\overline{X}, \overline{Y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

Bivariate Gaussian Density - Uncorrelated

 $f_{X,Y}(x,y) = f_X(x)f_Y(x)$ is sufficient to guarantee that X and Y are statistically independent. Any uncorrelated Guassian random variables are also statistically independent a coordinate rotation (linear transformation of X and Y) through the angle

$$\theta = \frac{1}{2} tan^{-1} \left[\frac{2\rho \sigma_X \sigma_Y}{\sigma_X^2 \sigma_Y^2} \right]$$

is sufficient to convert correlated random variables X and Y having σ_X^2 and σ_Y^2 , respectively, correlation coefficient ρ , and the joint densityof $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\cdot \exp\left[\cdots\right]$ into two statistically independent Gaussian random variables



Multi-variate Gaussian Density N random variables

N random variables $X_1, X_2, ..., X_N$ are called jointly Gaussian if their joint density function can be written as

$$f_{X_{1},\dots,X_{N}}(x_{1},\dots,x_{N}) = \frac{\left|\left[C_{X}\right|^{-1}\right|^{1/2}}{(2\pi)^{N/2}}exp\left\{-\frac{[x-\overline{X}]^{t}[C_{X}][x-\overline{X}]}{2}\right\}$$

$$[x-\overline{X}] = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ x_N - \overline{X}_N \end{bmatrix}, \quad [C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

Multi-variate Gaussian Density - notations N random variables

N random variables $X_1, X_2, ..., X_N$ are called jointly Gaussian if their joint density function can be written as

$$f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\left| [C_X|^{-1} \right|^{1/2}}{(2\pi)^{N/2}} exp \left\{ -\frac{[x-\overline{X}]^t [C_X][x-\overline{X}]}{2} \right\}$$

where $[\bullet]^t$ denotes a matrix transposition,

- [•]⁻¹denotes a matrix inversion
- | | denotes a matrix determinant



Covariance Matrix N random variables

N random variables $X_1, X_2, ..., X_N$ are called jointly Gaussian if their joint density function can be written as

$$f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\left| [C_X|^{-1} \right|^{1/2}}{(2\pi)^{N/2}} exp \left\{ -\frac{[x-\overline{X}]^t [C_X][x-\overline{X}]}{2} \right\}$$

where $[C_x]$ is called the covariance matrix of N random variables

$$C_{ij} = E[(X_i - \overline{X}_i)(X_j - \overline{X}_j)] = \begin{cases} \sigma_{X_i}^2 & i = j \\ C_{X_i X_i} & i \neq j \end{cases}$$

Covariance Matrix (N = 2) N random variables

$$f_{X_{1}X_{2}}(x_{1},x_{2}) = \frac{\left| \left[C_{X} \right|^{-1} \right|^{1/2}}{(2\pi)^{2/2}} exp \left\{ -\frac{\left[x - \overline{X} \right]^{t} \left[C_{X} \right] \left[x - \overline{X} \right]}{2} \right\}$$

$$\left[C_{X} \right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{X_{1}}^{2} & \rho \sigma_{X_{1}} \sigma_{X_{2}} \\ \rho \sigma_{X_{1}} \sigma_{X_{2}} & \sigma_{X_{2}}^{2} \end{bmatrix}$$

$$\left[C_{X} \right]^{-1} = \frac{1}{1 - \rho^{2}} \begin{bmatrix} \sigma_{X_{1}}^{2} & -\rho / \sigma_{X_{1}} \sigma_{X_{2}} \\ -\rho / \sigma_{X_{1}} \sigma_{X_{2}} & \sigma_{X_{2}}^{2} \end{bmatrix}$$

$$\left| \left[C_{X} \right]^{-1} \right| = 1 / \sigma_{X_{1}}^{2} \sigma_{X_{2}}^{2} (1 - \rho^{2})$$

Properties of Gaussian Random Variables N random variables

- completely defined thorugh only their first and second order mements - their means, variances, and covariances
- if uncorrelated, then statistically independent
- a linear transformation of Gaussian random variables will produce Gaussian random variables
- any k-dimensional (k-variate) marginal density function obtained from the N-dimensional density function by integrating N – k random variables will be Gaussian
- $X_1,...,X_k$ by integrating out $X_{k+1},...,X_N$
- the covariance of $X_1,...,X_k$ is the $k \times k$ submatrix of the $N \times N$ covariance matrix of $X_1,...,X_N$

