Laplace Transform & LTI System (5A)

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Laplace Transform and ODE's

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \longleftrightarrow Y(s)$$

$$y'(x) \longleftrightarrow sY(s) - y(0)$$

$$y''(x) \longleftrightarrow s^2 Y(s) - s\,y(0) - y'(0)$$

$$e^{-3x} \longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + \frac{1}{s+3}$$

Partitioning

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[s\,Y(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)\,Y(s) = (k_1 s + k_2 + 3k_1) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2

depends only on input e^{-3x}

Decomposed Y(s)

output ⇧ ⇩ input

$$(s^2 + 3s + 2) Y_{zi}(s) = (k_1 s + k_2 + 3k_1)$$

depends only on initial conditions k_1, k_2

No Input

output ⇧ ⇩ input

$$(s^2 + 3s + 2) Y_{zs}(s) = \frac{1}{s+3}$$

depends only on input e^{-3x}

No State

output ⇧ ⇩ input

$$(s^2 + 3s + 2) Y(s) = (k_1 s + k_2 + 3k_1) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2

depends only on input e^{-3x}

ZIR & ZSR

$y_{zi}(x) \Longleftrightarrow Y_{zi}(s) = \dfrac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$ **Zero Input Response**

$y_{zs}(x) \Longleftrightarrow Y_{zs}(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$ **Zero State Response**

$$y(x) \Longleftrightarrow Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)(s+3)}$$

$$y(x) \Longleftrightarrow Y(s) = Y_{zi}(s) + Y_{zs}(s)$$

Laplace Transform and IVP's

$$y'' + 3y' + 2y = 0 \qquad y(0) = k_1, \quad y'(0) = k_2$$

ZIR IVP

$$y_{zi}(x) \Longleftrightarrow Y_{zi}(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = 0, \quad y'(0) = 0$$

ZSR IVP

$$y_{zs}(x) \Longleftrightarrow Y_{zs}(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

Unilateral and Bilateral Laplace Transforms

Unilateral Laplace Transform

Including an Impulse at the origin

$$F_-(s) = \int_{0^-}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow s F_-(s) - f(0^-)$$

Excluding an Impulse at the origin

$$F_+(s) = \int_{0^+}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow s F_+(s) - f(0^+)$$

Bilateral Laplace Transform

$$F_2(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow s F_2(s) - f(0)$$

To include impulse inputs

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \longleftrightarrow Y(s)$$

$$y'(x) \longleftrightarrow sY(s) - y(0^-)$$

$$y''(x) \longleftrightarrow s^2 Y(s) - s\,y(0^-) - y'(0^-)$$

$$e^{-3x} \longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

ODEs with an input g(x)

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

usually known i.c. $\qquad y(0^-) = k_1, \quad y'(0^-) = k_2$

i.c. to be calculated $\qquad y(0^+) = m_1, \quad y'(0^+) = m_2$

solution to be found $\qquad y(t) \quad (t > 0)$

ODEs with an input g(x)

Non-homogeneous Eq

$$y'' + 3y' + 2y = e^{+x}$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \ -2$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = A e^{+x}$$

$$Ae^{+x} + 3Ae^{+x} + 2Ae^{+x} = e^{+x}$$

$$6Ae^{+x} = e^{+x} \qquad A = 1/6$$

$$y_p = \frac{1}{6} e^{+x}$$

General Solution

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^{+x}$$

IVP with nonzero IC

$$y(0) = 1, \ \ y'(0) = 2$$

$$1 = c_1 e^0 + c_2 e^0 + \frac{1}{6} e^0$$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x} + \frac{1}{6} e^{+x}$$

$$2 = -c_1 e^0 - 2c_2 e^0 + \frac{1}{6} e^0$$

$$c_1 + c_2 + \frac{1}{6} = 1$$

$$-c_1 - 2c_2 + \frac{1}{6} = 2$$

$$c_2 = -\frac{8}{3}$$

$$c_1 = \frac{5}{6} + \frac{8}{3} = \frac{21}{6} = \frac{7}{2}$$

$$y = \frac{7}{2} e^{-x} - \frac{8}{3} e^{-2x} + \frac{1}{6} e^{+x}$$

Zero State Response

$$y(0) = 0, \ \ y'(0) = 0$$

$$c_1 + c_2 + \frac{1}{6} = 0$$

$$-c_1 - 2c_2 + \frac{1}{6} = 0$$

$$c_2 = +\frac{1}{3}$$

$$c_1 = -\frac{1}{3} - \frac{1}{6} = -\frac{1}{2}$$

$$y = -\frac{1}{2} e^{-x} + \frac{1}{3} e^{-2x} + \frac{1}{6} e^{+x}$$

ODEs without an input g(x)

Homogeneous Eq

$$y'' + 3y' + 2y = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \ -2$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Homogeneous Solution

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Zero Input Response

$$y(0) = 1, \ y'(0) = 2$$

$$1 = c_1 e^0 + c_2 e^0$$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$2 = -c_1 e^0 - 2c_2 e^0$$

$$c_1 + c_2 = 1$$

$$-c_1 - 2c_2 = 2$$

$$c_2 = -3$$

$$c_1 = 1 - (-3) = 4$$

$$y = 4e^{-x} - 3e^{-2x}$$

$$y(0) = 0, \ y'(0) = 0$$

$$c_1 + c_2 = 0$$

$$-c_1 - 2c_2 = 0$$

$$c_2 = 0$$

$$c_1 = 0$$

$$y = 0$$

ODEs with an input g(x)

$$y'' + 3y' + 2y = e^{+x} \qquad y(0) = k_1,\ \ y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + \frac{1}{s-1}$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

IVP with nonzero IC

$$y(0) = 1,\ \ y'(0) = 2$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

$$= -\frac{8}{3}\frac{1}{(s+2)} + \frac{7}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\Longleftrightarrow \quad y = \frac{7}{2}e^{-x} - \frac{8}{3}e^{-2x} + \frac{1}{6}e^{+x}$$

Zero State Response

$$y(0) = 0,\ \ y'(0) = 0$$

$$Y(s) = \frac{1}{(s-1)(s+1)(s+2)}$$

$$= +\frac{1}{3}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\Longleftrightarrow \quad y = -\frac{1}{2}e^{-x} + \frac{1}{3}e^{-2x} + \frac{1}{6}e^{+x}$$

Homogeneous Solution

$$y'' + 3y' + y = 0 \qquad y(0) = k_1,\ \ y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = 0$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} \qquad 0$$

homogeneous solution

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$\Longleftrightarrow \quad Y_h(s) = \frac{c_1}{(s+1)} + \frac{c_2}{(s+2)} = \frac{c_1(s+2) + c_2(s+1)}{(s+1)(s+2)} = \frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)}$$

for every initial value of y_h

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} = \frac{y(0)s + y'(0) + 3y(0)}{(s+1)(s+2)} = \frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)}$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_h' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$y_h(0) = c_1 + c_2$$

$$y_h'(0) = -c_1 - 2c_2$$

$$y(0) \leftarrow y_h(0)$$

$$y'(0) \leftarrow y_h'(0)$$

$$(s^2+3s+2)Y_h(s) - y_h(0)s - y_h'(0) - 3y_h(0) = 0 \Rightarrow (s+1)(s+2)\frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)} - (c_1+c_2)s + (2c_1+c_2) = 0$$

Homogeneous Solution

$$y'' + 3y' + y = 0 \qquad y(0) = k_1,\ \ y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = 0$$

$$(s^2 + 3s + 2)\,Y(s) = k_1 s + k_2 + 3k_1 = 0$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

Zero Input Response

$$y(0) = 1,\ \ y'(0) = 2$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)}$$

$$= +4\frac{1}{(s+1)} - 3\frac{1}{(s+2)}$$

$$\Longleftrightarrow \quad y = 4e^{-x} - 3e^{-2x}$$

Zero Input & Zero State Response

$$y(0) = 0,\ \ y'(0) = 0$$

$$Y(s) = 0$$

$$\Longleftrightarrow \quad y = 0$$

ODE : y''+3y'+2y with various inputs

$$y'' + 3y' + 2y = 0 \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} \Longleftrightarrow Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)}$$

$$y'' + 3y' + 2y = e^{+x} \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^{+x} \Longleftrightarrow Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{1}{6} \frac{1}{(s-1)}$$

$$y'' + 3y' + 2y = e^{-x} \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + (x-1) e^{-x} \Longleftrightarrow Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{t}{(s+1)^2} - \frac{1}{(s+1)}$$

$$y'' + 3y' + 2y = e^{+ix} \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{1+3i} e^{+ix} \Longleftrightarrow Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{1}{(1+3i)} \frac{1}{(s-i)}$$

$$y'' + 3y' + 2y = e^{-ix} \Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{1-3i} e^{-ix} \Longleftrightarrow Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{1}{(1-3i)} \frac{1}{(s+i)}$$

ODE : y''+y with various inputs

$$y'' + y = 0 \quad \Rightarrow \quad y = c_1 e^{+ix} + c_2 e^{-ix} \quad \longleftrightarrow \quad Y(s) = c_1 \frac{1}{(s-i)} + c_2 \frac{1}{(s+i)}$$

$$y'' + y = e^{+x} \quad \Rightarrow \quad y = c_1 e^{+ix} + c_2 e^{-ix} + \frac{1}{2} e^{+x} \quad \longleftrightarrow \quad Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{1}{2} \frac{1}{(s-1)}$$

$$y'' + y = e^{-x} \quad \Rightarrow \quad y = c_1 e^{+ix} + c_2 e^{-ix} + \frac{1}{2} e^{-x} \quad \longleftrightarrow \quad Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + \frac{1}{2} \frac{1}{(s+1)}$$

$$y'' + y = e^{+ix} \quad \Rightarrow \quad y = c_1 e^{+ix} + c_2 e^{-ix} - 2i e^{+ix} \quad \longleftrightarrow \quad Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} - 2i \frac{1}{(s-i)}$$

$$y'' + y = e^{-ix} \quad \Rightarrow \quad y = c_1 e^{+ix} + c_2 e^{-ix} + 2i e^{+ix} \quad \longleftrightarrow \quad Y(s) = c_1 \frac{1}{(s+1)} + c_2 \frac{1}{(s+2)} + 2i \frac{1}{(s+i)}$$

References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] E. Kreyszig, “Advanced Engineering Mathematics”
[5] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[6] T. J. Cavicchi, “Digital Signal Processing”
[7] F. Waleffe, Math 321 Notes, UW 2012/12/11
[8] J. Nearing, University of Miami
[9] http://scipp.ucsc.edu/~haber/ph116A/ComplexFunBranchTheory.pdf