Coordinate Systems

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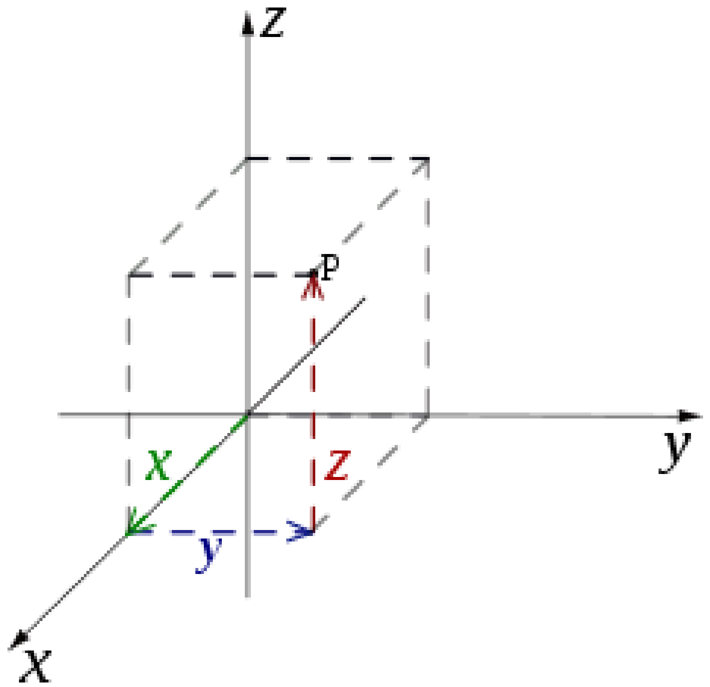
Cartesian Coordinates

Cylindrical Coordinate

Spherical Coordinate

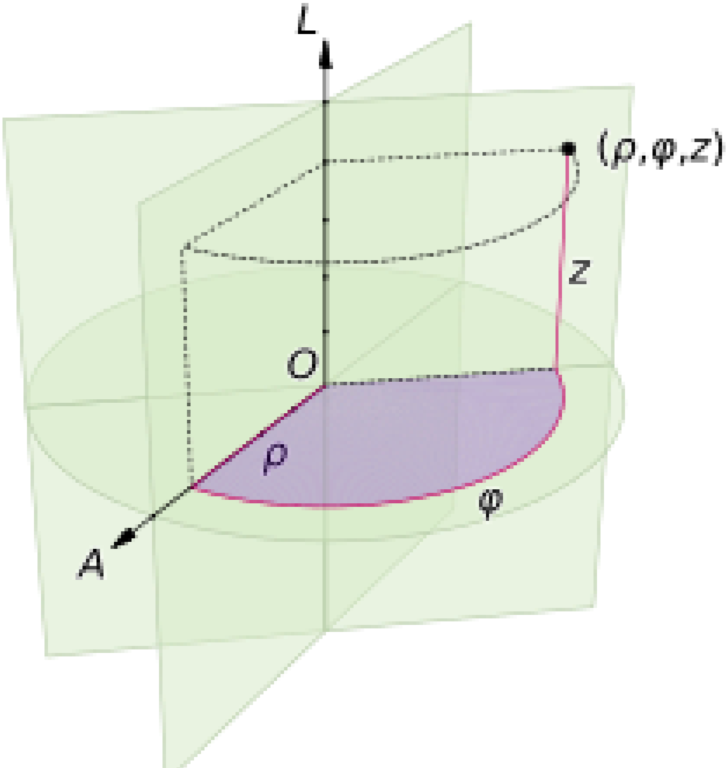
Cartesian Coordinate System

https://en.wikipedia.org/wiki/Coordinate_system#/media/File:Rectangular_coordinates.svg

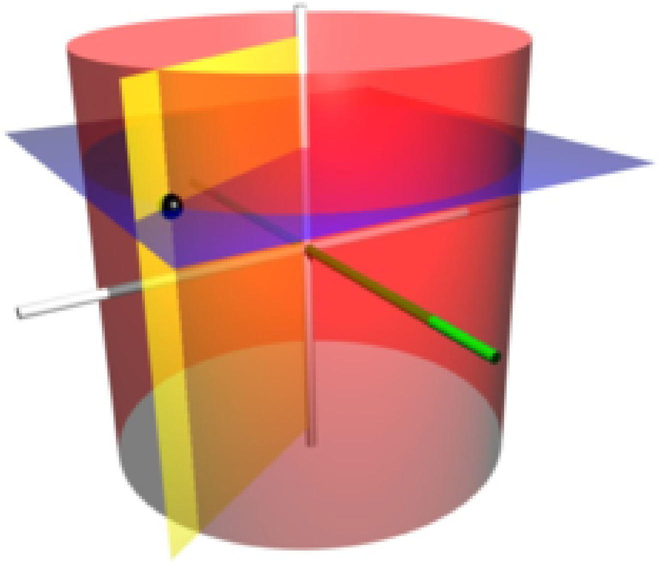


Spherical Coordinate System

https://en.wikipedia.org/wiki/Cylindrical_coordinate_system#/media/File:Coord_system_CY_1.svg
https://en.wikipedia.org/wiki/Cylindrical_coordinate_system#/media/File:Cylindrical_coordinate_surfaces.png

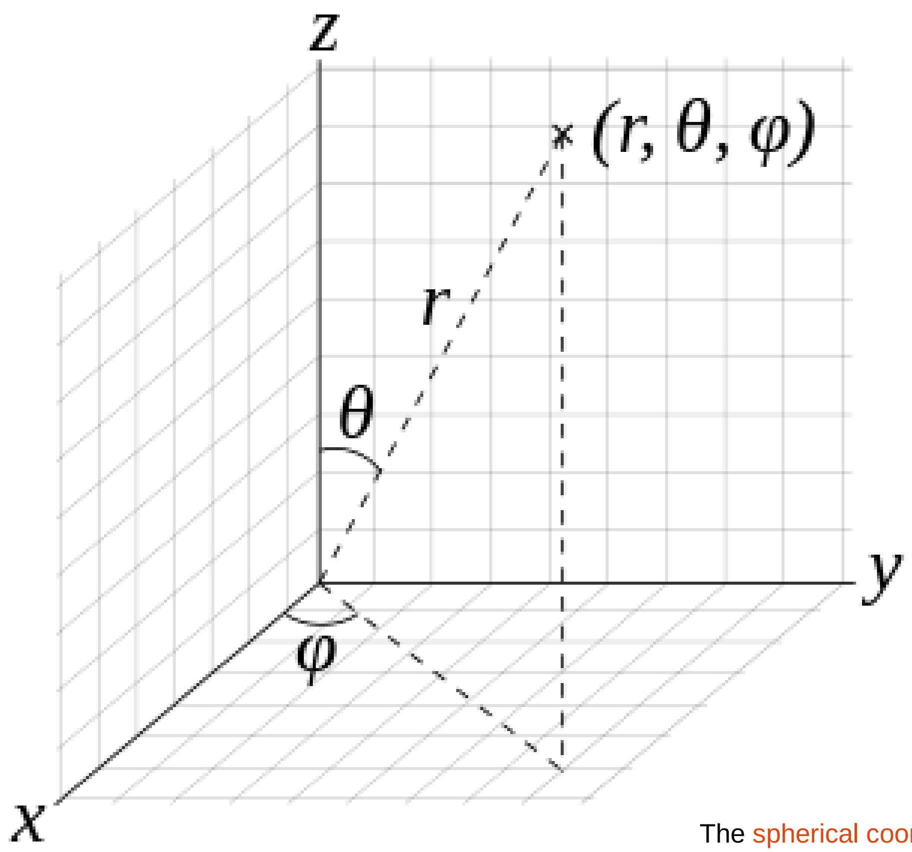


A cylindrical coordinate system with origin O, polar axis A, and longitudinal axis L. The dot is the point with radial distance ρ = 4, angular coordinate φ = 130°, and height z = 4.



The coordinate surfaces of the cylindrical coordinates (ρ, φ, z). The red cylinder shows the points with ρ=2, the blue plane shows the points with z=1, and the yellow half-plane shows the points with φ=−60°. The z-axis is vertical and the x-axis is highlighted in green. The three surfaces intersect at the point P with those coordinates (shown as a black sphere); the Cartesian coordinates of P are roughly (1.0, −1.732, 1.0).

Spherical Coordinate System



The spherical coordinate system is commonly used in physics. It assigns three numbers (known as coordinates) to every point in Euclidean space: radial distance r, polar angle θ (theta), and azimuthal angle φ (phi). The symbol ρ (rho) is often used instead of r.

The Euler constan

Cylindrical Coordinate System

https://en.wikipedia.org/wiki/Cylindrical_coordinate_system

The line element is

$$\mathrm{d}\mathbf{r} = \mathrm{d}\rho\,\boldsymbol{\hat\rho} + \rho\,\mathrm{d}\varphi\,\boldsymbol{\hat\varphi} + \mathrm{d}z\,\mathbf{\hat z}.$$

The volume element is

$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z.$$

The surface element in a surface of constant radius ρ (a vertical cylinder) is

$$\mathrm{d}S_\rho = \rho\,\mathrm{d}\varphi\,\mathrm{d}z.$$

The surface element in a surface of constant azimuth φ (a vertical half-plane) is

$$\mathrm{d}S_\varphi = \mathrm{d}\rho\,\mathrm{d}z.$$

The surface element in a surface of constant height z (a horizontal plane) is

$$\mathrm{d}S_z = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

The del operator in this system leads to the following expressions for gradient, divergence, curl and Laplacian:

$$\nabla f = \frac{\partial f}{\partial \rho}\boldsymbol{\hat\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \varphi}\boldsymbol{\hat\varphi} + \frac{\partial f}{\partial z}\mathbf{\hat z},$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \boldsymbol{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\boldsymbol{\hat\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\boldsymbol{\hat\varphi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right)\mathbf{\hat z}$$

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}.$$

Cylindrical Coordinate System

The following equations assume that θ is inclination from the z (polar) axis (ambiguous since *x*, *y*, and *z* are mutually normal):

The line element for an infinitesimal displacement from (r, θ, φ) to $(r + \mathrm{d}r,\ \theta + \mathrm{d}\theta,\ \varphi + \mathrm{d}\varphi)$ is

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\,\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}}.$$

where

$$\begin{aligned}\hat{\boldsymbol{r}} &= \sin\theta\cos\varphi\,\hat{\boldsymbol{x}} + \sin\theta\sin\varphi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\varphi\,\hat{\boldsymbol{x}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}}\\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\boldsymbol{x}} + \cos\varphi\,\hat{\boldsymbol{y}}\end{aligned}$$

are the local orthogonal unit vectors in the directions of increasing r, θ, and φ, respectively, and $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are the unit vectors in Cartesian coordinates.

The surface element spanning from θ to $\theta + \mathrm{d}\theta$ and φ to $\varphi + \mathrm{d}\varphi$ on a spherical surface at (constant) radius r is

$$\mathrm{d}S_r = r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Thus the differential solid angle is

$$\mathrm{d}\Omega = \frac{\mathrm{d}S_r}{r^2} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

The surface element in a surface of polar angle θ constant (a cone with vertex the origin) is

$$\mathrm{d}S_\theta = r\sin\theta\,\mathrm{d}\varphi\,\mathrm{d}r.$$

The surface element in a surface of azimuth φ constant (a vertical half-plane) is

$$\mathrm{d}S_\varphi = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

The volume element spanning from r to $r + \mathrm{d}r$, θ to $\theta + \mathrm{d}\theta$, and φ to $\varphi + \mathrm{d}\varphi$ is

$$\mathrm{d}V = r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

https://en.wikipedia.org/wiki/Spherical_coordinate_system

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
[3] E. Kreyszig, "Advanced Engineering Mathematics"
[4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"