Row Reduction

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Linear Equations

$$
\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & b_1 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_m
\end{array}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix}
= \\ = \\ \\ =
\end{matrix}
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

Linear Equations

$$\begin{matrix} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & b_1 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & b_m \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{1j} \cdot x_j = b_1$$

$$\begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_2 \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{2j} \cdot x_j = b_2$$

$$\begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = b_m$$

$$\sum_{j=1}^{n} a_{mj} \cdot x_j = b_m$$

Example

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

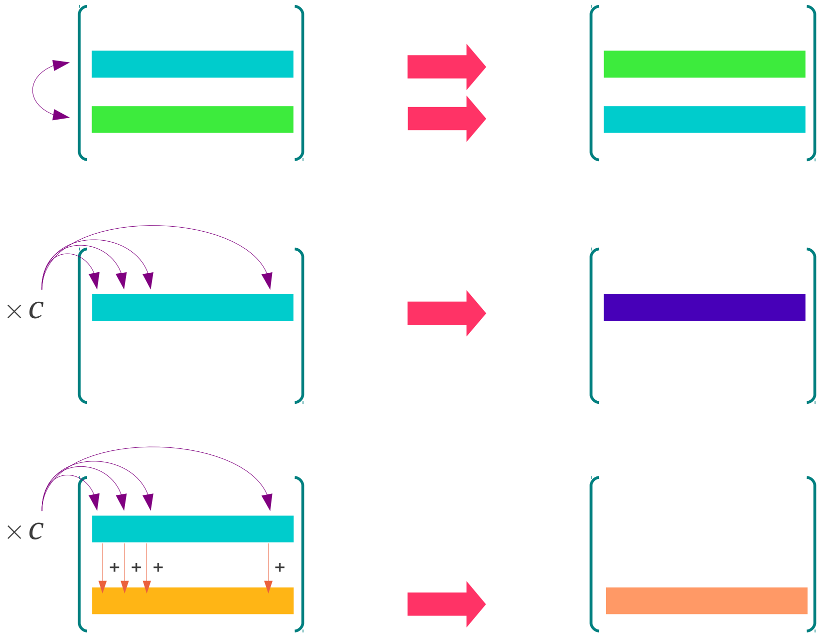
$$
\begin{array}{cccccccc}
2\,x_1 & + & 1\,x_2 & - & 1\,x_3 & = & +8 \\
-3\,x_1 & - & 1\,x_2 & + & 2\,x_3 & = & -11 \\
-2\,x_1 & + & 1\,x_2 & + & 2\,x_3 & = & -3
\end{array}
$$

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{pmatrix}
$$

$$
\begin{pmatrix}
+2 & +1 & -1 \\
-3 & -1 & +2 \\
-2 & +1 & +2
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ = \end{matrix}
\begin{pmatrix}
+8 \\ -11 \\ -3
\end{pmatrix}
$$

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



Gauss-Jordan Elimination – Step 1

$$
\begin{aligned}
+2x_1 + x_2 - x_3 &= 8 && (L_1)\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\left[\begin{array}{ccc|c}
\enclose{circle}{+2} & +1 & -1 & +8\\
-3 & -1 & +2 & -11\\
-2 & +1 & +2 & -3
\end{array}\right]
$$

$$
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1) \qquad\qquad +2/2 \quad +1/2 \quad -1/2 \qquad +8/2
$$

$$
\begin{aligned}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= 4 && (\tfrac{1}{2} \times L_1)\\
-3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\
-2x_1 + x_2 + 2x_3 &= -3 && (L_3)
\end{aligned}
\qquad
\left[\begin{array}{ccc|c}
\enclose{circle}{+1} & +1/2 & -1/2 & +4\\
-3 & -1 & +2 & -11\\
-2 & +1 & +2 & -3
\end{array}\right]
$$

Gauss-Jordan Elimination – Step 2

$$\begin{aligned} +1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 && (L_1)\\ -3x_1 - x_2 + 2x_3 &= -11 && (L_2)\\ -2x_1 + x_2 + 2x_3 &= -3 && (L_3) \end{aligned}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$\begin{aligned} +3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 &= +12 && (\boxed{3 \times L_1})\\ -3x_1 - x_2 + 2x_3 &= -11 && (L_2) \end{aligned}$$

$$\begin{array}{cccc} +3 & +3/2 & -3/2 & +12 \\ -3 & -1 & +2 & -11 \end{array}$$

$$\begin{aligned} +2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 &= +8 && (\boxed{2 \times L_1})\\ -2x_1 + x_2 + 2x_3 &= -3 && (L_3) \end{aligned}$$

$$\begin{array}{cccc} +2 & +2/2 & -2/2 & +8 \\ -2 & +1 & +2 & -3 \end{array}$$

$$\begin{aligned} +1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= +4 && (L_1)\\ 0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 &= +1 && (\boxed{3 \times L_1} + L_2)\\ 0x_1 + 2x_2 + 1x_3 &= +5 && (\boxed{2 \times L_1} + L_3) \end{aligned}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)$$

Gauss-Jordan Elimination – Step 3

$$
\begin{array}{lr}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 & (L_1) \\
0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 & (L_2) \\
0x_1 + 2x_2 + 1x_3 = +5 & (L_3)
\end{array}
\qquad
\left(\begin{array}{ccc|c}
+1 & +1/2 & -1/2 & +4 \\
0 & \enclose{circle}{+1/2} & +1/2 & +1 \\
0 & +2 & +1 & +5
\end{array}\right)
$$

$$
0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2) \qquad\qquad
\begin{array}{cccc}
0 & +1 & +1 & +2
\end{array}
$$

$$
\begin{array}{lr}
+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 & (L_1) \\
0x_1 + 1x_2 + 1x_3 = +2 & (2 \times L_2) \\
0x_1 + 2x_2 + 1x_3 = +5 & (L_3)
\end{array}
\qquad
\left(\begin{array}{ccc|c}
+1 & +1/2 & -1/2 & +4 \\
0 & \enclose{circle}{+1} & +1 & +2 \\
0 & +2 & +1 & +5
\end{array}\right)
$$

Gauss-Jordan Elimination – Step 4

$$\begin{array}{ll} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \\ 0x_1 + 2x_2 + 1x_3 = +5 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$\begin{array}{ll} 0x_1 - 2x_2 - 2x_3 = -4 & \boxed{(-2 \times L_2)} \\ 0x_1 + 2x_2 + 1x_3 = +5 & (L_3) \end{array}$$

$$\begin{array}{cccc} 0 & -2 & -2 & -4 \\ 0 & +2 & +1 & +5 \end{array}$$

$$\begin{array}{ll} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \\ 0x_1 + 0x_2 - 1x_3 = +1 & (\boxed{-2 \times L_2} + L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 5

$$\begin{array}{ll} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \\ 0x_1 + 0x_2 - 1x_3 = +1 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right)$$

$$0x_1 - 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\begin{array}{cccc} 0 & 0 & +1 & -1 \end{array}$$

$$\begin{array}{ll} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \\ 0x_1 + 0x_2 + 1x_3 = -1 & (-1 \times L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Forward Phase

$$
\left(\begin{array}{ccc|c} \textcircled{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)
\Rightarrow
\left(\begin{array}{ccc|c} \textcircled{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right)
\Rightarrow
\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)
\Rightarrow
$$

$$
\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & \textcircled{+1} & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right)
\Rightarrow
\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)
\Rightarrow
\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \textcircled{+1} & -1 \end{array}\right)
\Rightarrow
$$

Forward Phase - Gaussian Elimination

Gauss-Jordan Elimination – Step 6

$$\begin{array}{lr} +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \\ 0x_1 + 0x_2 + 1x_3 = -1 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$\begin{array}{lr} 0x_1 + 0x_2 + \frac{1}{2}x_3 = -\frac{1}{2} & \boxed{(+\frac{1}{2} \times L_3)} \\ +1x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = +4 & (L_1) \end{array}$$

$$\begin{array}{cccc} 0 & 0 & +1/2 & -1/2 \\ +1 & +1/2 & -1/2 & +4 \end{array}$$

$$\begin{array}{lr} 0x_1 + 0x_2 - 1x_3 = +1 & \boxed{(-1 \times L_3)} \\ 0x_1 + 1x_2 + 1x_3 = +2 & (L_2) \end{array}$$

$$\begin{array}{cccc} 0 & 0 & -1 & +1 \\ 0 & +1 & +1 & +2 \end{array}$$

$$\begin{array}{lr} +1x_1 + \frac{1}{2}x_2 + 0x_3 = +\frac{7}{2} & (+\frac{1}{2} \times L_3 + L_1) \\ 0x_1 + 1x_2 + 0x_3 = +3 & (-1 \times L_3 + L_2) \\ 0x_1 + 0x_2 + 1x_3 = -1 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Gauss-Jordan Elimination – Step 7

$$\begin{array}{lr} +1x_1 + \tfrac{1}{2}x_2 + 0x_3 \;=\; +\tfrac{7}{2} & (L_1) \\ 0x_1 + 1x_2 + 0x_3 \;=\; +3 & (L_2) \\ 0x_1 + 0x_2 + 1x_3 \;=\; -1 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$\begin{array}{lr} 0x_1 - \tfrac{1}{2}x_2 + 0x_3 \;=\; -\tfrac{3}{2} & \boxed{(-\tfrac{1}{2} \times L_2)} \\ +1x_1 + 0x_2 - 0x_3 \;=\; +2 & (L_1) \end{array}$$

$$\begin{array}{cccc} 0 & -1/2 & 0 & -3/2 \\ +1 & +1/2 & 0 & +7/2 \end{array}$$

$$\begin{array}{lr} +1x_1 + 0x_2 - 0x_3 \;=\; +2 & (-\tfrac{1}{2} \times L_2 + L_1) \\ 0x_1 + 1x_2 + 0x_3 \;=\; +3 & (L_2) \\ 0x_1 + 0x_2 + 1x_3 \;=\; -1 & (L_3) \end{array}$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

Gauss-Jordan Elimination

Forward Phase – Gaussian Elimination

$$\left(\begin{array}{ccc|c} \enclose{circle}{+2} & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} \enclose{circle}{+1} & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & \enclose{circle}{+1} & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & \enclose{circle}{+1} & -1 \end{array}\right) \Rightarrow$$

Backward Phase – Guass-Jordan Elimination

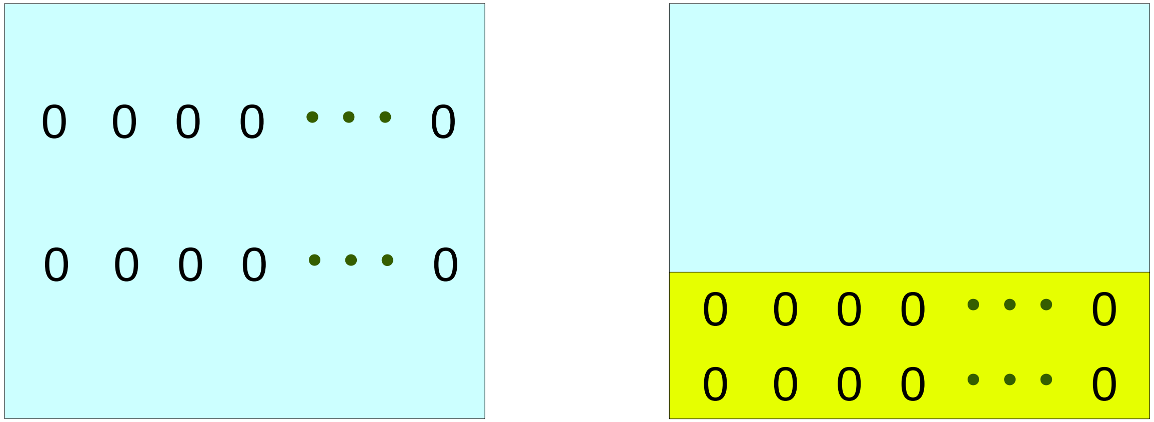
$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

REF: Row Echelon Forms (1)

zero rows	➡	Should be grouped at the bottom
non-zero row	➡	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows	➡	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row

REF: Row Echelon Forms (2)

zero rows → Should be grouped at the bottom



$0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0$

$0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0$

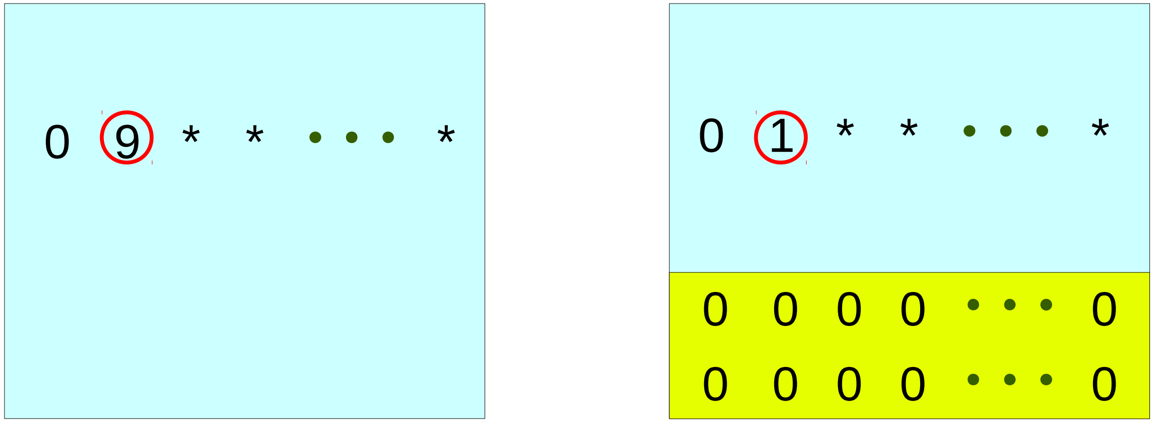
$0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0$

$0 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0$

REF: Row Echelon Forms (3)

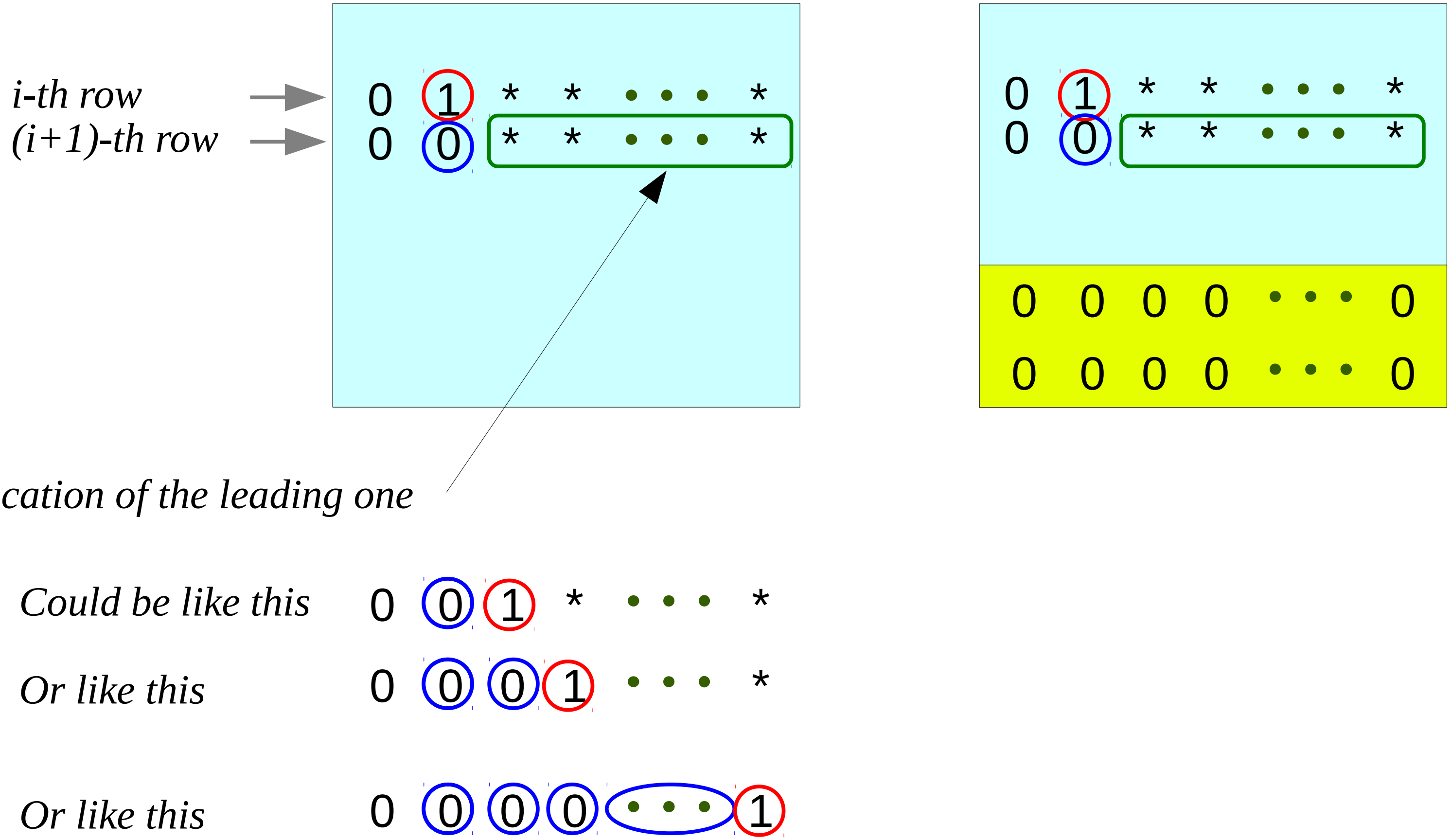
non-zero row → A leading one

The 1st non-zero element should be one



REF: Row Echelon Forms (4)

Any successive non-zero rows → The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row



The possible location of the leading one

Could be like this 0 0 1 * • • • *

Or like this 0 0 0 1 • • • *

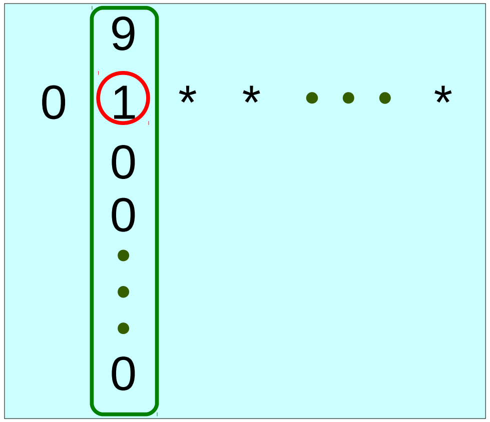
Or like this 0 0 0 0 • • • 1

RREF: Reduced Row Echelon Forms (1)

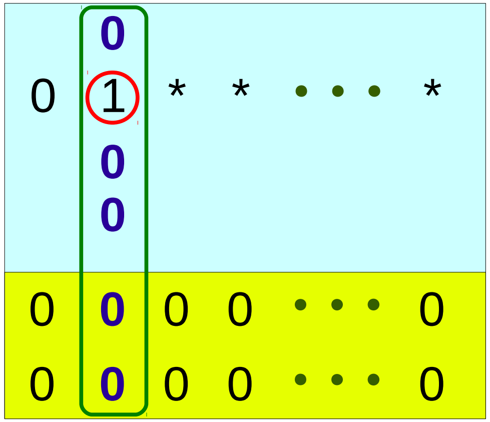
zero rows	➡	Should be grouped at the bottom
non-zero row	➡	A leading **1** The 1st non-zero element should be one
Any successive non-zero rows	➡	The leading **1** of the lower row should be farther to the **right** than the leading **1** of the higher row
Any column that contains a leading **1**	➡	All other elements except the leading one are **all zeros**

RREF: Reduced Row Echelon Forms (2)

Any column that contains a leading one → All other elements except the leading one are **all zeros**



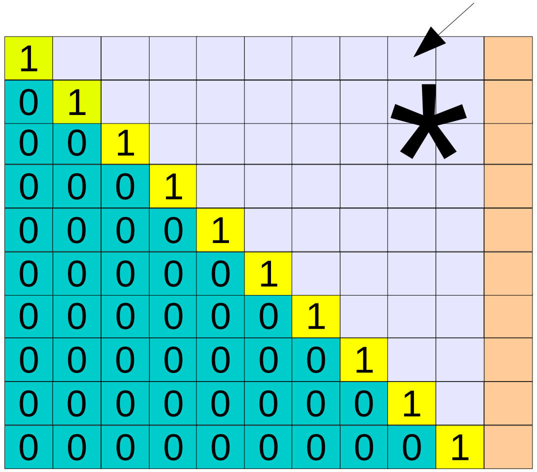
$$\begin{array}{cccccc} & 9 & & & & \\ 0 & 1 & * & * & \cdots & * \\ & 0 & & & & \\ & 0 & & & & \\ & \vdots & & & & \\ & 0 & & & & \end{array}$$



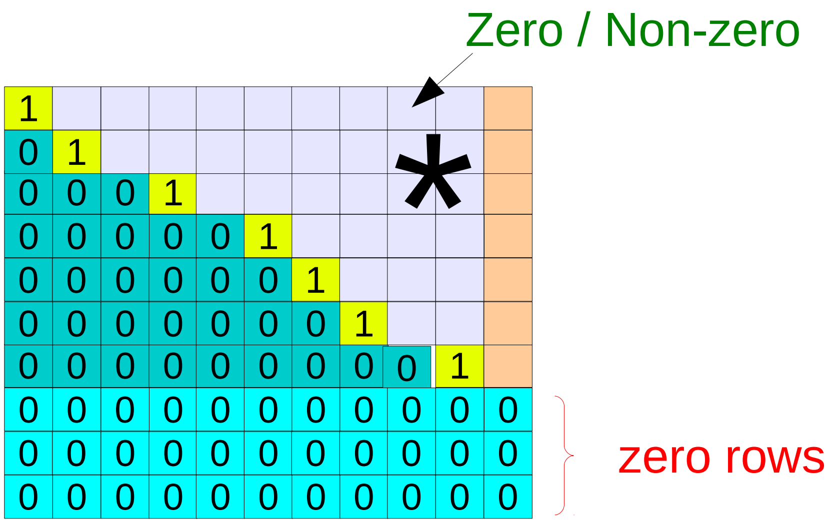
$$\begin{array}{cccccc} & \mathbf{0} & & & & \\ 0 & 1 & * & * & \cdots & * \\ & \mathbf{0} & & & & \\ & \mathbf{0} & & & & \\ 0 & \mathbf{0} & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{0} & 0 & 0 & \cdots & 0 \end{array}$$

Examples

Row Echelon Form

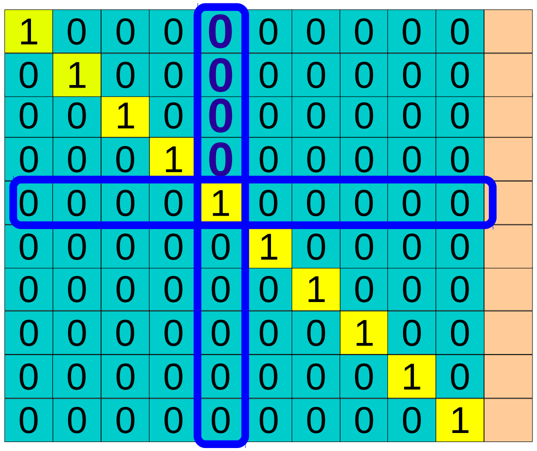


1										
0	1									
0	0	1								
0	0	0	1							
0	0	0	0	1						
0	0	0	0	0	1					
0	0	0	0	0	0	1				
0	0	0	0	0	0	0	1			
0	0	0	0	0	0	0	0	1		
0	0	0	0	0	0	0	0	0	1	

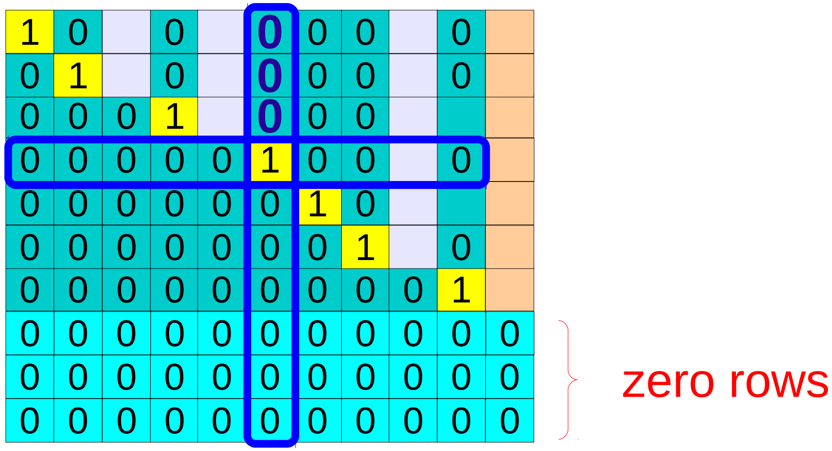


1										
0	1									
0	0	0	1							
0	0	0	0	0	1					
0	0	0	0	0	0	1				
0	0	0	0	0	0	0	1			
0	0	0	0	0	0	0	0	0	1	
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Reduced Row Echelon Form



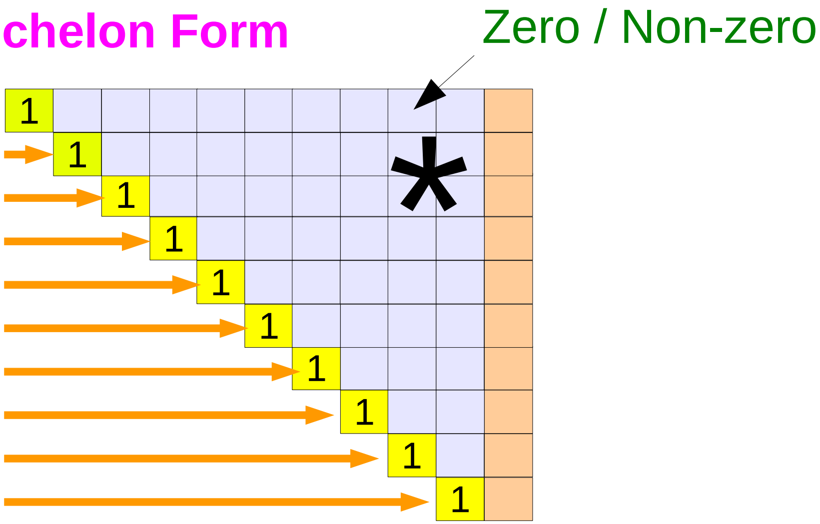
1	0	0	0	0	0	0	0	0	0	
0	1	0	0	0	0	0	0	0	0	
0	0	1	0	0	0	0	0	0	0	
0	0	0	1	0	0	0	0	0	0	
0	0	0	0	1	0	0	0	0	0	
0	0	0	0	0	1	0	0	0	0	
0	0	0	0	0	0	1	0	0	0	
0	0	0	0	0	0	0	1	0	0	
0	0	0	0	0	0	0	0	1	0	
0	0	0	0	0	0	0	0	0	1	

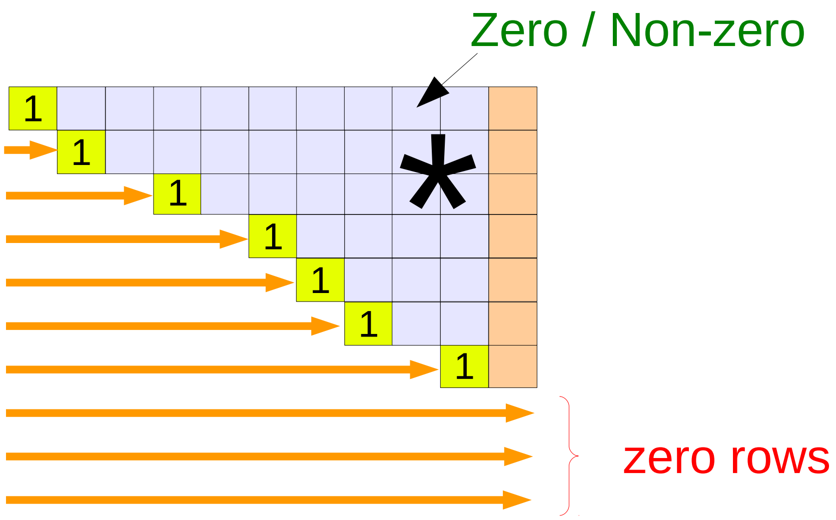


1	0		0		0	0	0		0	
0	1		0		0	0	0		0	
0	0	0	1		0	0	0			
0	0	0	0	0	1	0	0		0	
0	0	0	0	0	0	1	0			
0	0	0	0	0	0	0	1		0	
0	0	0	0	0	0	0	0	0	1	
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

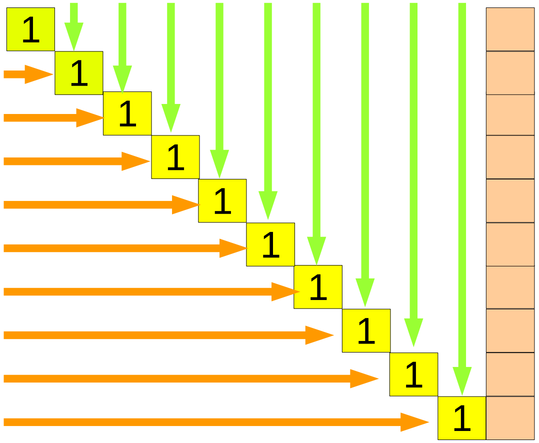
Examples

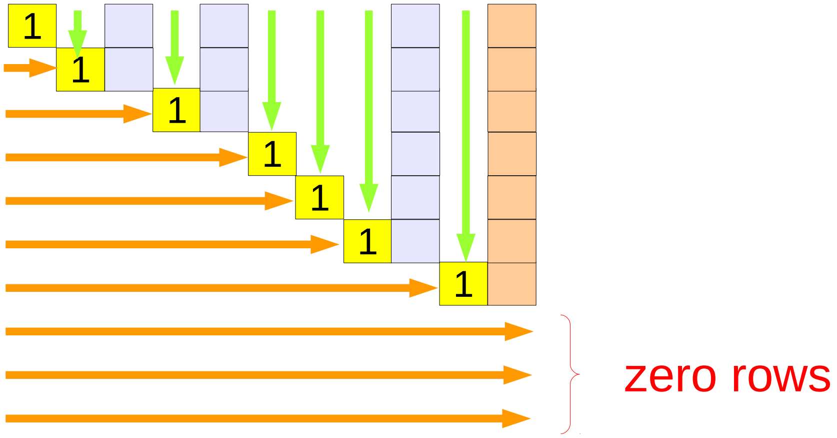
Row Echelon Form





Reduced Row Echelon Form



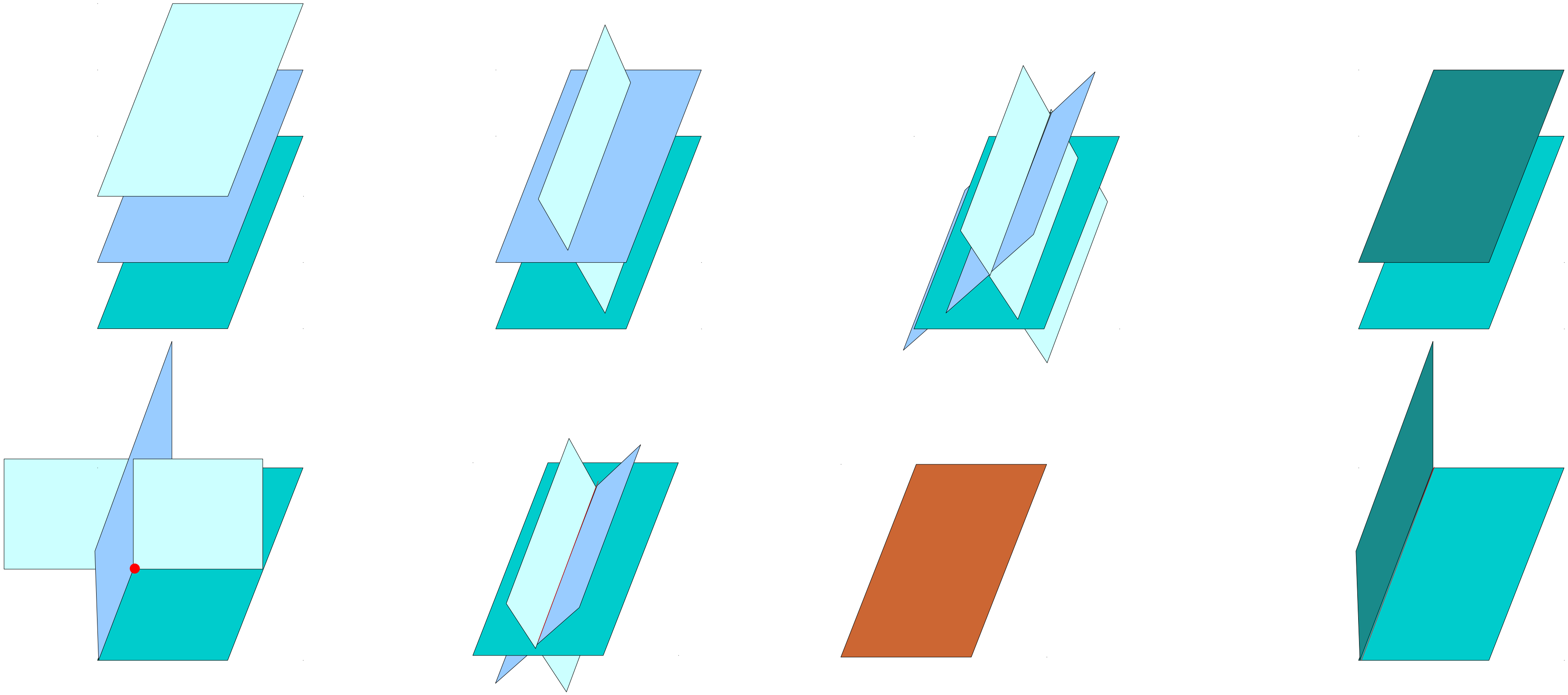


Linear Systems of 3 Unknowns

(Eq 1) ➡ $a_{11}\ x_1 + a_{12}\ x_2 + a_{13}\ x_3 = b_1$

(Eq 2) ➡ $a_{21}\ x_1 + a_{22}\ x_2 + a_{23}\ x_3 = b_2$

(Eq 3) ➡ $a_{31}\ x_1 + a_{32}\ x_2 + a_{33}\ x_3 = b_3$



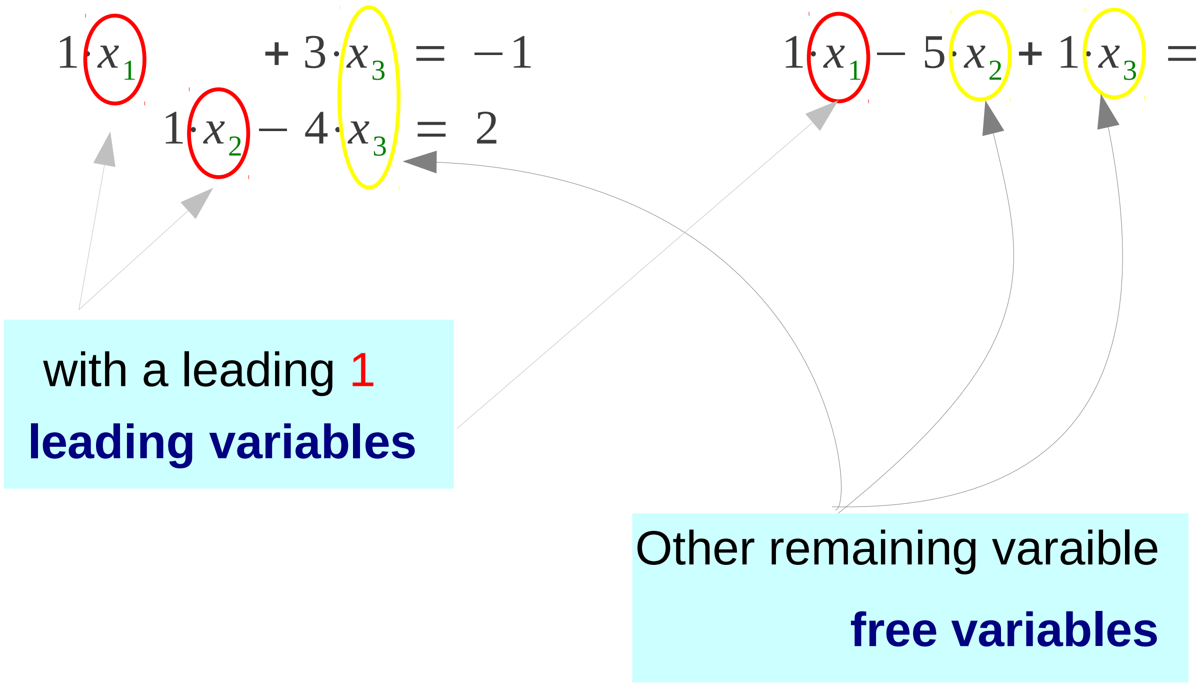
Leading and Free Variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array}\right)$$

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$



$$\begin{aligned} 1\cdot x_1 \qquad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned}$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

with a leading 1

leading variables

Other remaining varaible

free variables

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$0\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 = 1$$

$$\xcancel{0 = 1}$$

Free Variables as Parameters (1)

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned}$$

$$\begin{aligned} 1\cdot x_1 \qquad\quad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned}$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

Solve for a leading variable

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3\cdot x_3 \\ x_2 = 2 + 4\cdot x_3 \end{cases}$$

$$x_1 = 4 + 5\cdot x_2 - 1\cdot x_3$$

Treat a free variable as a parameter

$$x_3 = t$$

$$x_2 = s \qquad x_3 = t$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{cases} x_1 = 4 + 5\cdot s - 1\cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

Parametric Solutions (2)

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned}$$

$$\left\{ \begin{aligned} x_1 &= 5 \\ x_2 &= 7 \\ x_3 &= 9 \end{aligned} \right.$$

many other forms of parametric solutions

$$\begin{aligned} 1\cdot x_1 \qquad\quad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned}$$

$$\left\{ \begin{aligned} x_1 &= t \\ x_2 &= \tfrac{1}{3}(-4t+2) \\ x_3 &= \tfrac{1}{3}(-t-1) \end{aligned} \right.$$

$$\left\{ \begin{aligned} x_1 &= \tfrac{1}{4}(-3t+2) \\ x_2 &= t \\ x_3 &= \tfrac{1}{4}(t-2) \end{aligned} \right.$$

$$\left\{ \begin{aligned} x_1 &= -1 - 3t \\ x_2 &= 2 + 4t \\ x_3 &= t \end{aligned} \right.$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

$$\left\{ \begin{aligned} x_1 &= s \\ x_2 &= t \\ x_3 &= -s + 5t + 4 \end{aligned} \right.$$

$$\left\{ \begin{aligned} x_1 &= s \\ x_2 &= \tfrac{1}{5}(s + t - 4) \\ x_3 &= t \end{aligned} \right.$$

$$\left\{ \begin{aligned} x_1 &= 4 + 5\cdot s - 1\cdot t \\ x_2 &= s \\ x_3 &= t \end{aligned} \right.$$

Many Solutions (3)

$$\begin{aligned} 1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3 &= 5 \\ 0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3 &= 7 \\ 0\cdot x_1 + 0\cdot x_2 + 1\cdot x_3 &= 9 \end{aligned}$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

$$\begin{aligned} 1\cdot x_1 \qquad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned}$$

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$$\begin{aligned} &(-1,\ +2,\ 0) \\ &(-4,\ +6,\ 1) \\ &(-7,\ +10,\ 2) \\ &(\underbrace{-10,\ +14}_{},\ 2) \end{aligned}$$

$$x_2 = -\frac{4}{3}x_1 + \frac{2}{3}$$

$$1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

$$\begin{cases} x_1 = 4 + 5\cdot s - 1\cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

Solutions in R³ (4)

$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

free variable ($x_3 = t$)

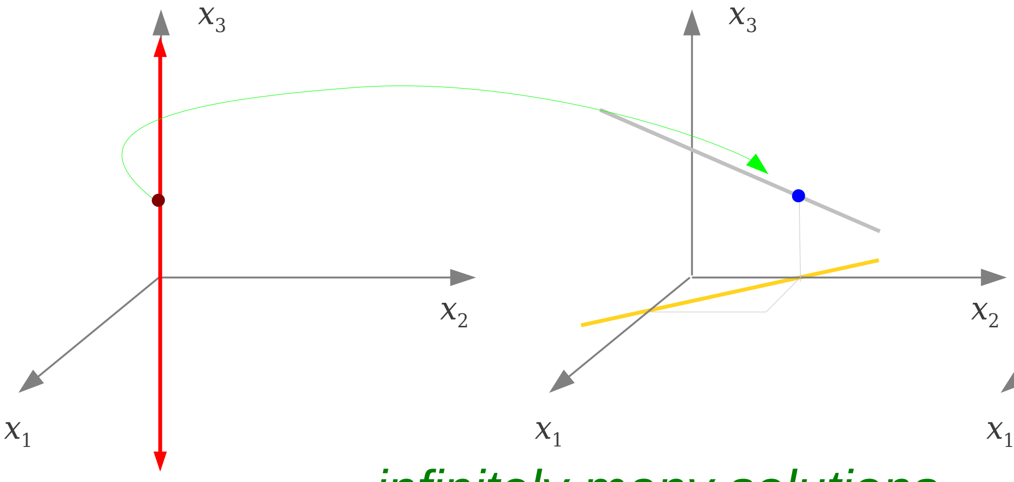
$$\begin{cases} x_1 = 4 + 5\cdot s - 1\cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

free variable ($x_2 = s$)

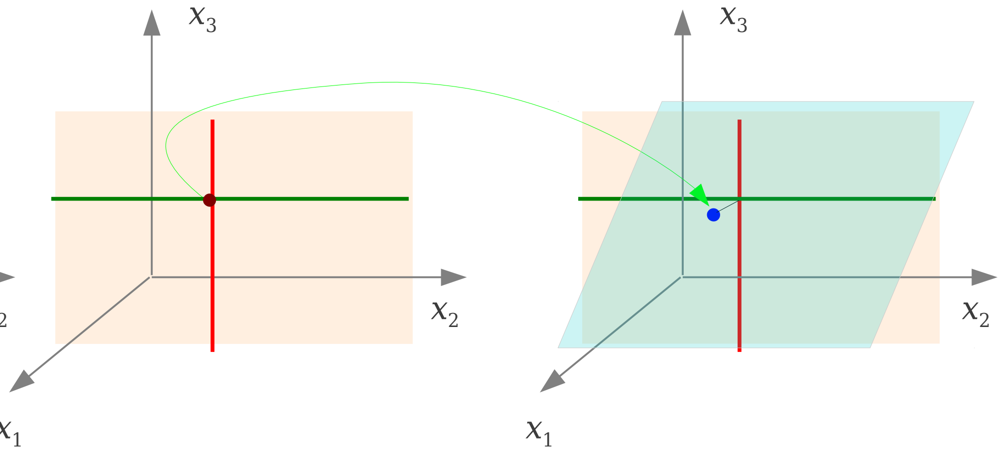
free variable ($x_3 = t$)

$$4x_1 + 3x_2 = 2$$

$$x_1 - 5x_2 + x_3 = 4$$



infinitely many solutions



infinitely many solutions

Solutions in R^3 and No of Free Variables (5)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 = 5 \\ x_2 = 7 \\ x_3 = 9 \end{cases}$$

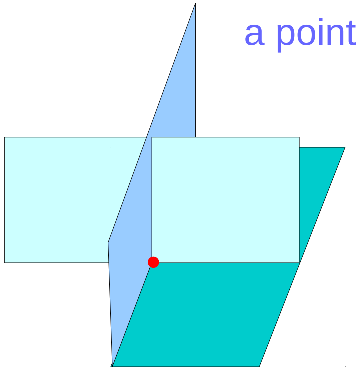
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

$x_3 = t$ ⇐ free variable

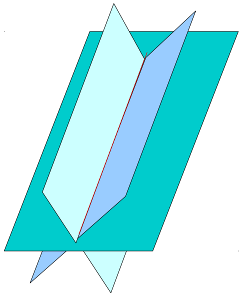
$$\begin{cases} x_1 = 4 + 5 \cdot s - 1 \cdot t \\ x_2 = s \\ x_3 = t \end{cases}$$

$x_2 = s$ ⇐ free variable

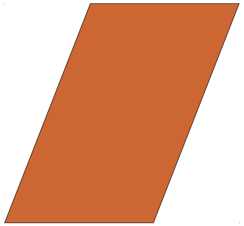
$x_3 = t$ ⇐ free variable



a point in R^3



a line in R^3



a plane in R^3

Consistent Linear System

A linear system with at least one solution

➡ A Consistent Linear System

A linear system with no solutions

➡ A Inconsistent Linear System

General Solution

A linear system with infinitely many solutions

Solve for a leading variable

Treat a free variable as a parameter

➡ A set of parametric equations

All solutions can be obtained
by assigning numerical values to those parameters

➡ Called a general solution

Homogeneous System

$$
\begin{array}{ccccccccc}
a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\
a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0
\end{array}
$$

All constant terms are zero

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{pmatrix}
\begin{matrix} = \\ = \\ \\ = \end{matrix}
\begin{pmatrix}
0 \\ 0 \\ \vdots \\ 0
\end{pmatrix}
$$

All constant terms are zero

Solutions of a Homogeneous System

All homogeneous systems
pass through the origin

$$\begin{matrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{matrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The homogeneous system has

* only the trivial solution

* many solutions
in addition to the trivial solution

Trivial Solution

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

satisfies all homogeneous equations

$$\begin{array}{ccccccccc} a_{11} x_1 & + & a_{12} x_2 & + & \cdots & + & a_{1n} x_n & = & 0 \\ a_{21} x_1 & + & a_{22} x_2 & + & \cdots & + & a_{2n} x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1} x_1 & + & a_{m2} x_2 & + & \cdots & + & a_{mn} x_n & = & 0 \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

All homogeneous systems pass through the origin

Impossible Solution

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ **rank =2**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$ **rank =3**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$ **rank =2**

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$ **rank =3**

$$0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 1$$

$$\xcancel{0 = 1}$$

$$rank(\mathbf{A}) < rank(\mathbf{A}|\mathbf{b})$$

Linear System Ax = B

$$\boldsymbol{A} \ \boldsymbol{x} = \boldsymbol{0}$$

Always consistent

$rank(\boldsymbol{A}) = n$

unique solution $\boldsymbol{x} = \boldsymbol{0}$

$rank(\boldsymbol{A}) < n$

Infinitely many solution

$n - r$ parameters

$$A = [a_{ij}]_{m \times n}$$

m equations

n unknowns

$$\boldsymbol{A} \ \boldsymbol{x} = \boldsymbol{b}$$

$rank(\boldsymbol{A}) = rank(\boldsymbol{A}|\boldsymbol{b})$

: Consistent

$rank(\boldsymbol{A}) = n$

unique solution $\boldsymbol{x} \neq \boldsymbol{0}$

$rank(\boldsymbol{A}) < n$

Infinitely many solution

$n - r$ parameters

$rank(\boldsymbol{A}) < rank(\boldsymbol{A}|\boldsymbol{b})$

: Inconsistent

Augmented Matrix

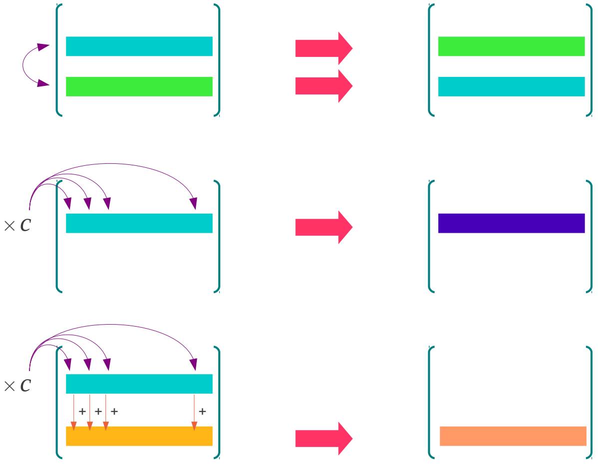
$$\begin{array}{ccccccccc} a_{11} & x_1 & + & a_{12} & x_2 & + & \cdots & + & a_{1n} & x_n & = & 0 \\ a_{21} & x_1 & + & a_{22} & x_2 & + & \cdots & + & a_{2n} & x_n & = & 0 \\ \vdots & & & \vdots & & & & & \vdots & & & \vdots \\ a_{m1} & x_1 & + & a_{m2} & x_2 & + & \cdots & + & a_{mn} & x_n & = & 0 \end{array}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}$$

Augmented matrix of a homogeneous system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

Reduced Row Echelon Form



Elementary row operations do not alter the zero column of a matrix

homogeneous system

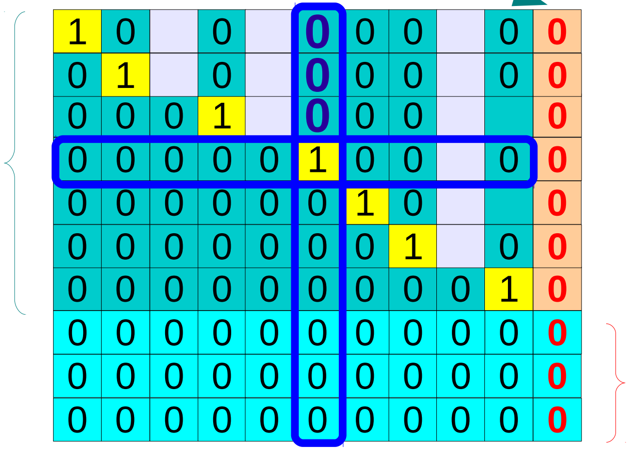
The augmented zero column is preserved in the reduced row echelon form

Reduced Echelon Form

1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0
		0	0	0	1	0	0	0	0	0
		0	0	0	0	1	0	0	0	0
		0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	1	0

m
leading
variables

r
leading
variables

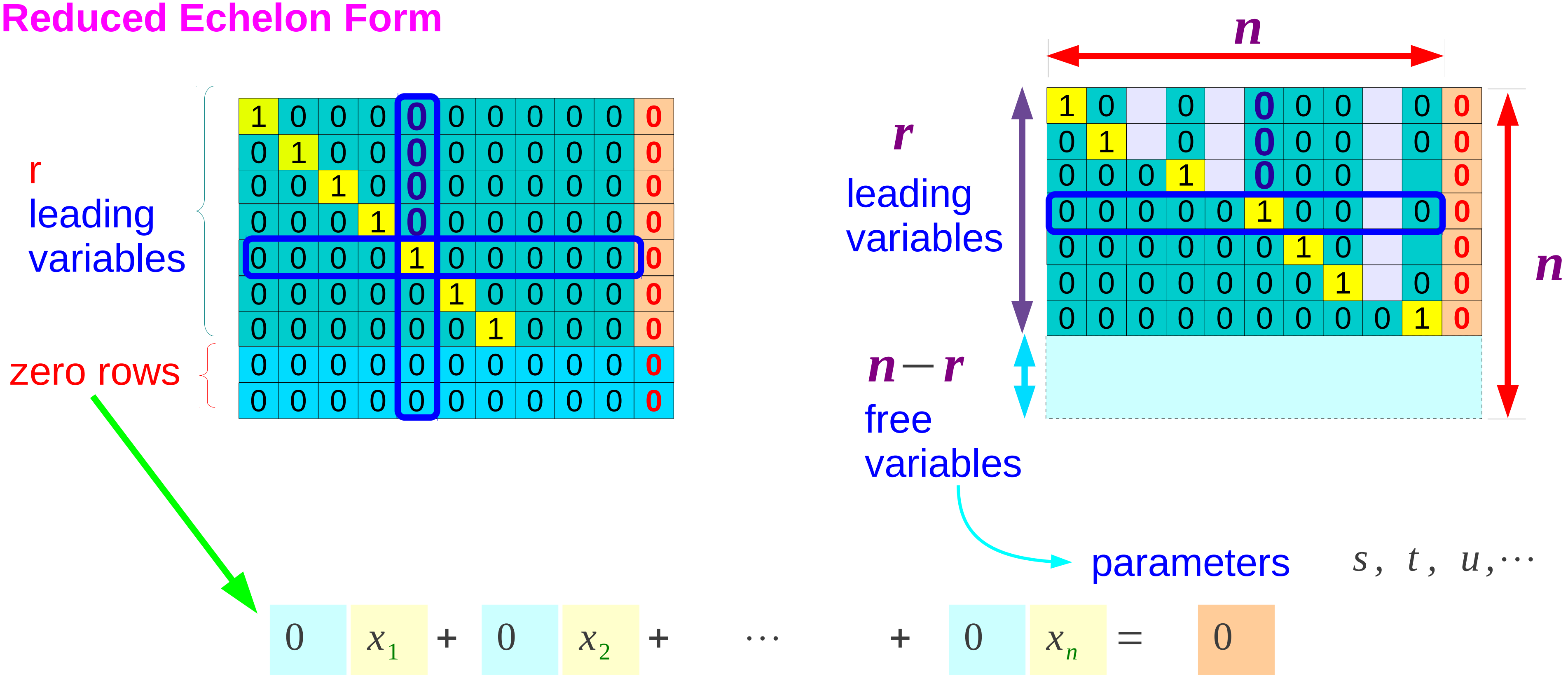


1	0		0		0	0	0		0	0
0	1		0		0	0	0		0	0
0	0	0	1		0	0	0			0
0	0	0	0	0	1	0	0		0	0
0	0	0	0	0	0	1	0			0
0	0	0	0	0	0	0	1		0	0
0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

zero rows

Free Variable Theorem

Reduced Echelon Form



$$\left[\begin{array}{cccccccccc|c}
1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{array}\right]$$

$$\left[\begin{array}{cccccccccc|c}
1&0&&0&&0&0&0&&0&0\\
0&1&&0&&0&0&0&&0&0\\
0&0&0&1&&0&0&0&&&0\\
0&0&0&0&0&1&0&0&&0&0\\
0&0&0&0&0&0&1&0&&&0\\
0&0&0&0&0&0&0&1&&0&0\\
0&0&0&0&0&0&0&0&0&1&0
\end{array}\right]$$

$$0\ x_1 + 0\ x_2 + \cdots + 0\ x_n = 0$$

A homogeneous linear system with ***n*** **unknowns**

If the reduced row echelon form of its augmented matrix has

r non-zero rows ➡ ***n – r*** free variables ➡ *infinitely many solutions*

Free Variable Theorem Example

Reduced Echelon Form

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{aligned} 1\cdot x_1 \qquad + 3\cdot x_3 &= -1 \\ 1\cdot x_2 - 4\cdot x_3 &= 2 \end{aligned} \qquad 1\cdot x_1 - 5\cdot x_2 + 1\cdot x_3 = 4$$

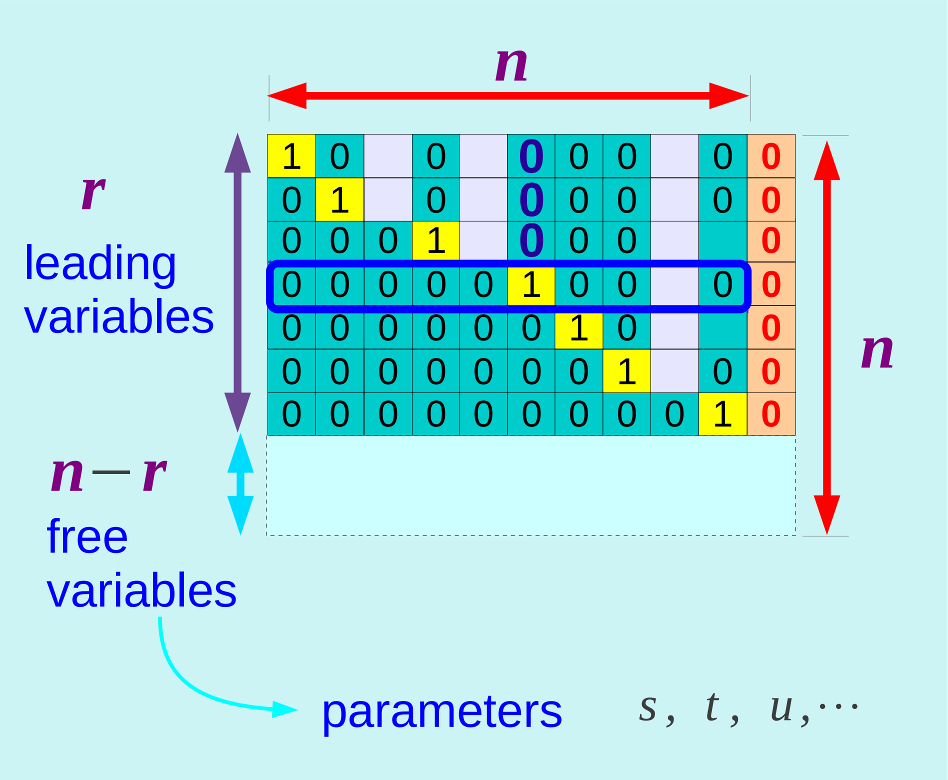
$$\begin{cases} x_1 = -1 - 3t \\ x_2 = 2 + 4t \\ x_3 = t \end{cases}$$

⇐ free variable

$$\begin{cases} x_1 = 4 + 5\cdot x_2 - 1\cdot x_3 \\ x_2 = s \\ x_3 = t \end{cases}$$

⇐ free variable

⇐ free variable



1	0		0		0	0	0		0	0
0	1		0		0	0	0		0	0
0	0	0	1		0	0	0			0
0	0	0	0	0	1	0	0		0	0
0	0	0	0	0	0	1	0			0
0	0	0	0	0	0	0	1		0	0
0	0	0	0	0	0	0	0	0	1	0

A homogeneous linear system with *n* unknowns

If the reduced row echelon form of its augmented matrix has

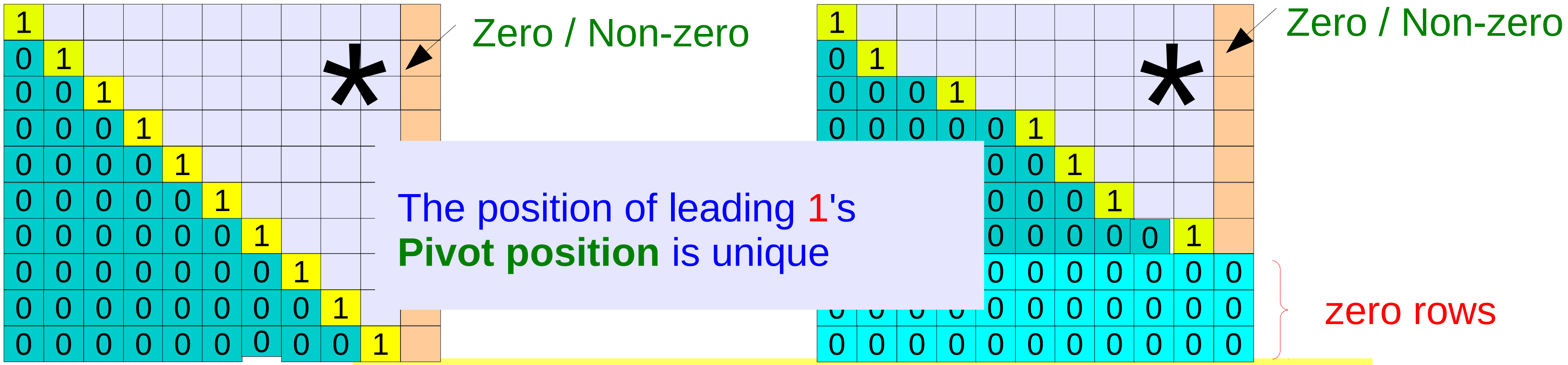
r non-zero rows ⇨ *n* – *r* free variables ⇨ *infinitely many solutions*

Pivot Positions

Row Echelon Form

→ **Not unique**

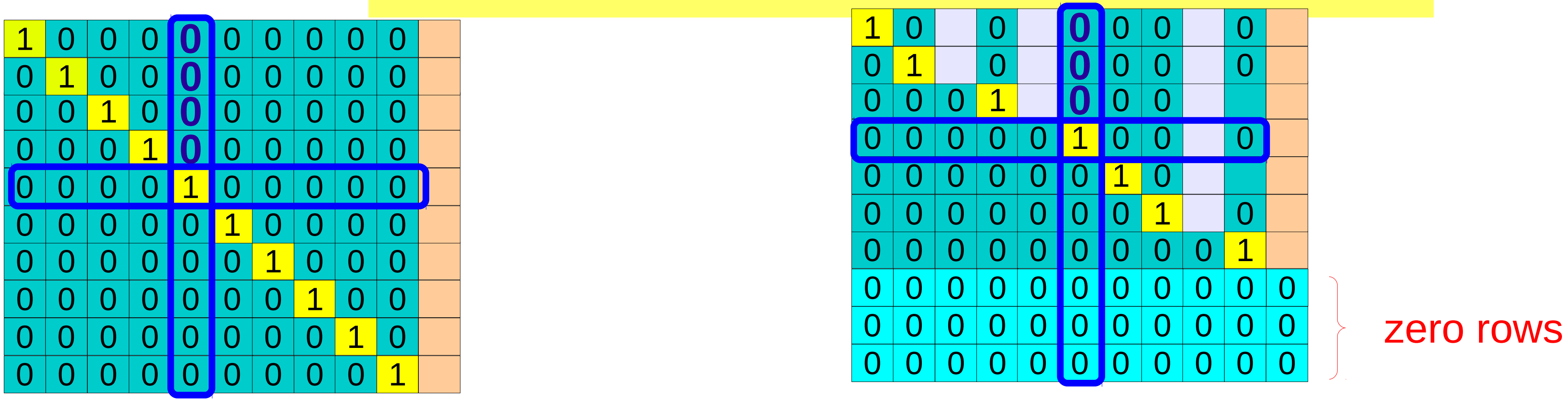
Depend on the sequence of elementary row operations



The position of leading 1's
Pivot position is unique

Reduced Row Echelon Form

→ **Unique**



Pulse

References

[1] http://en.wikipedia.org/
[2] Anton & Busby, “Contemporary Linear Algebra”
[3] Anton & Rorres, “Elementary Linear Algebra”