Complex Integration (2C)

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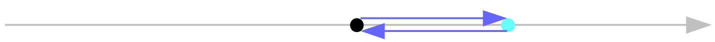
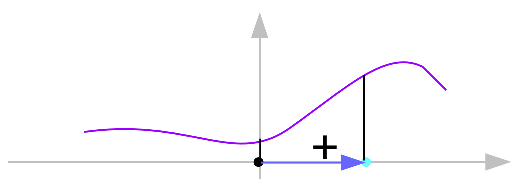
Fundamental Theorem

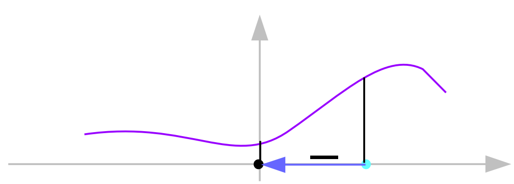
Real Domain



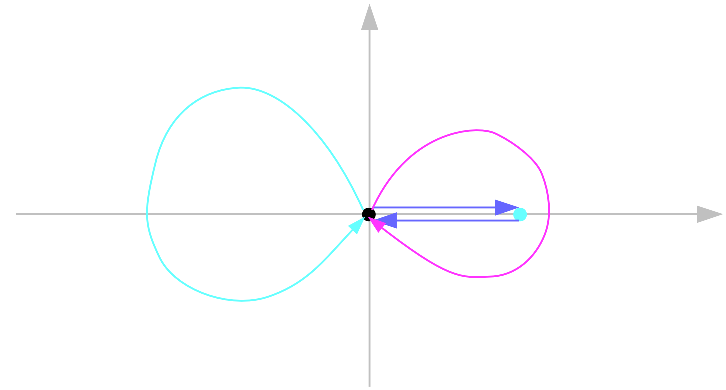
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_b^a f(x)\,dx = F(a) - F(b)$$





Complex Domain



$$\int_C f(z)\,dz = F(z_2) - F(z_1)$$

$$z_2 = z_1 \quad \Rightarrow \quad \int_C f(z)\,dz = 0$$

Contour Integration Evaluation f(z) = 1/z

(1) Indefinite Integration of Analytic Functions

$z_1 = z_0$ ➡ $\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) = 0$

But $f(z) = \frac{1}{z}$ not analytic at $z = 0$ ➡ cannot apply this method

(2) Integration by the Use of the Path

C : the unit circle ➡ $z(t) = \cos t + i\sin t = e^{it}$ $(0 \le t \le 2\pi)$

$z'(t) = -\sin t + i\cos t = i e^{it}$

$$\int_C f(z)\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$$

Polar Coordinates

$$z = re^{i\theta} \quad \Rightarrow \quad dz = e^{i\theta}(dr + ir\,d\theta)$$

$$z(a) = z_1 \qquad z(b) = z_2$$

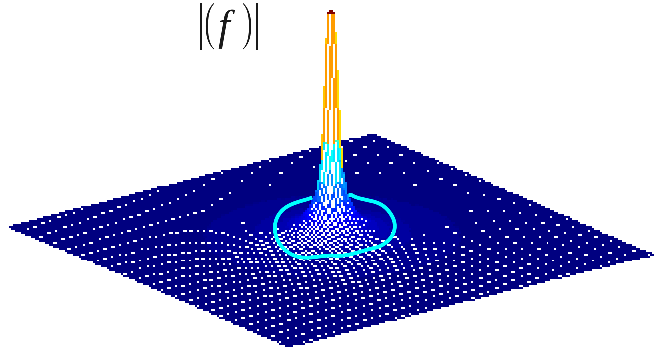
$$\int_{z_1}^{z_2} f(z)\,dz = \int_a^b f(re^{i\theta})\,e^{i\theta}\left(\frac{dr}{dt} + ir\frac{d\theta}{dt}\right)dt$$

$$f(re^{i\theta})e^{i\theta} \equiv f_\theta(r,\theta) \equiv u_\theta(r,\theta) + iv(r,\theta)$$

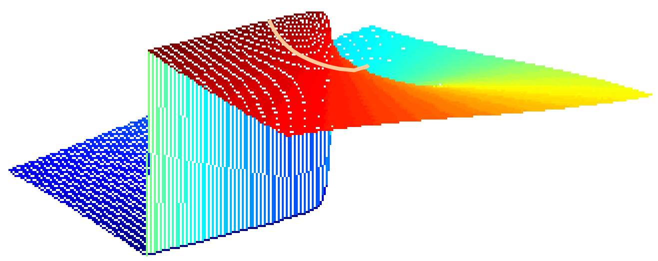
Complex Function f(z) = 1/z

$$f(z) = \frac{1}{z}$$

$|(f)|$

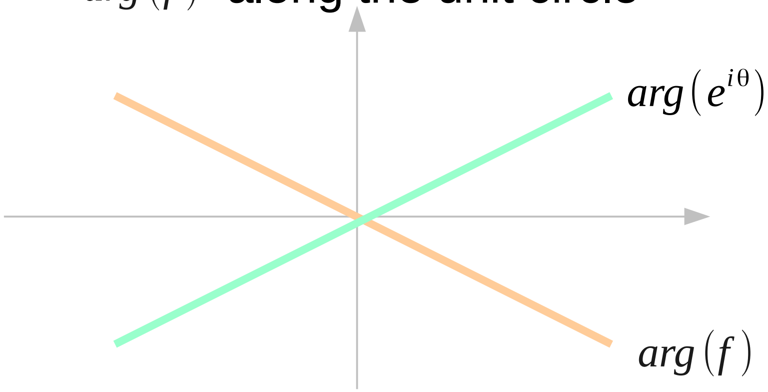


$arg(f)$

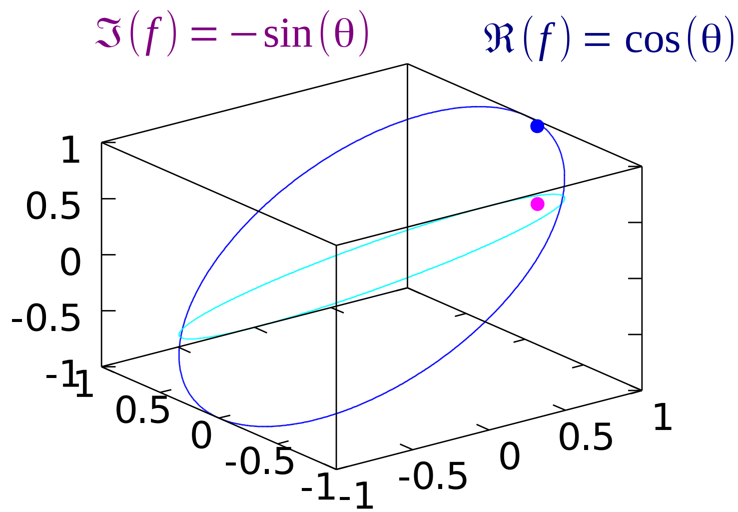


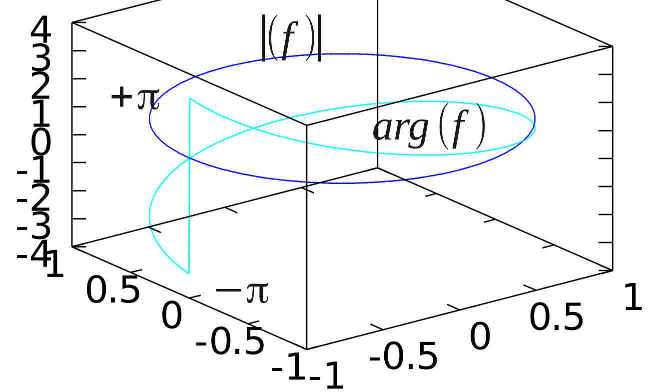
$|(f)|$ along the unit circle

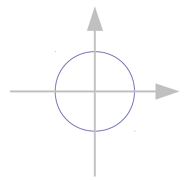
$arg(f)$ along the unit circle

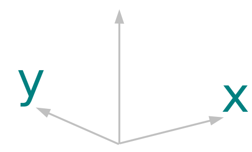


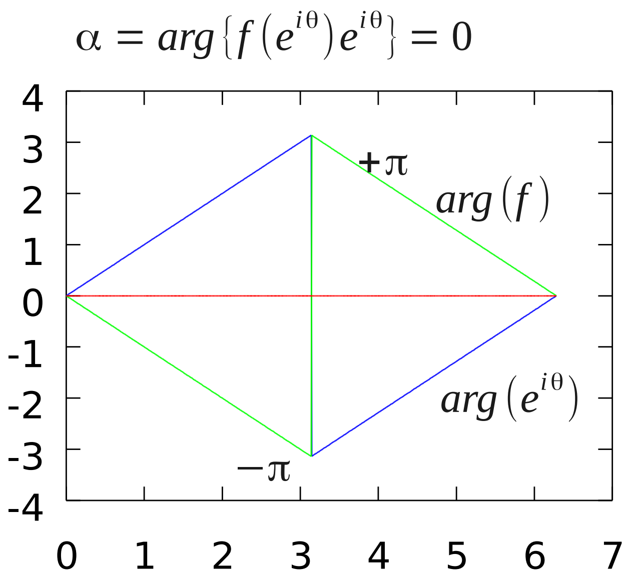
Contour Integration of f(z)=1/z

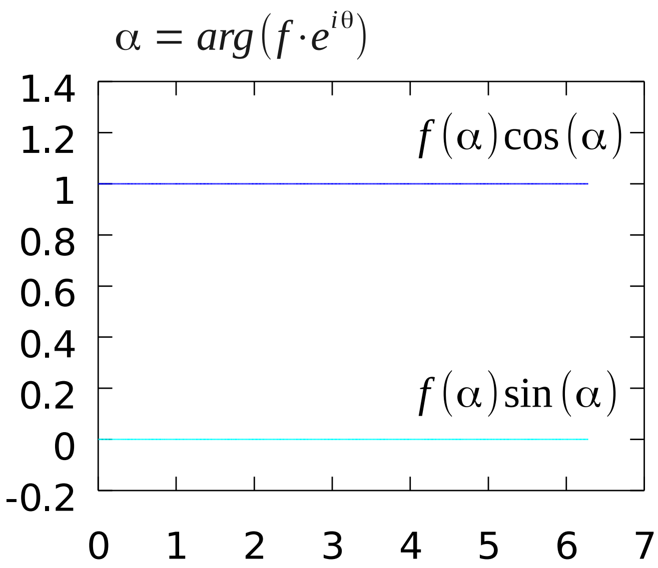












$$
\begin{aligned}
&\int_{-\pi}^{+\pi} f(r,\theta)\, i e^{i\theta}\, d\theta \\
&= \int_{-\pi}^{+\pi} e^{-i\theta}\, i e^{i\theta}\, d\theta \\
&= \int_{-\pi}^{+\pi} i\, d\theta \\
&= 2\pi i
\end{aligned}
$$

Contour Integration f(z) = z^2, z^1, z^0, z^{-1}, z^{-2}, z^{-3}

$$\int_C f(z)\,dz = \int_0^{2\pi} e^{mit}\, i e^{it}\,dt = \int_0^{2\pi} i e^{i(m+1)t}\,dt \qquad dz = i e^{it}\,dt$$

m=2	$\int_C z^2\,dz = \int_0^{2\pi} e^{i2t}\, i e^{it}\,dt = \int_0^{2\pi} i e^{i3t}\,dt = \left[\frac{1}{3} e^{i3t}\right]_0^{2\pi} = \frac{1}{3}(e^{6\pi} - e^0) = 0$	3
m=1	$\int_C z\,dz = \int_0^{2\pi} e^{it}\, i e^{it}\,dt = \int_0^{2\pi} i e^{i2t}\,dt = \left[\frac{1}{2} e^{i2t}\right]_0^{2\pi} = \frac{1}{2}(e^{4\pi} - e^0) = 0$	2
m=0	$\int_C 1\,dz = \int_0^{2\pi} i e^{it}\,dt = \int_0^{2\pi} i e^{it}\,dt = \left[e^{it}\right]_0^{2\pi} = (e^{2\pi} - e^0) = 0$	1
m=-1	$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} e^{-it}\, i e^{it}\,dt = \int_0^{2\pi} i\,dt = \left[i\right]_0^{2\pi} = i(2\pi - 0) = 2\pi i$	0
m=-2	$\int_C \frac{1}{z^2}\,dz = \int_0^{2\pi} e^{-i2t}\, i e^{it}\,dt = \int_0^{2\pi} i e^{-it}\,dt = \left[-e^{-it}\right]_0^{2\pi} = -(e^{-2\pi} - e^0) = 0$	-1
m=-3	$\int_C \frac{1}{z^3}\,dz = \int_0^{2\pi} e^{-i3t}\, i e^{it}\,dt = \int_0^{2\pi} i e^{-i2t}\,dt = \left[-\frac{1}{2} e^{-i2t}\right]_0^{2\pi} = -\frac{1}{2}(e^{-4\pi} - e^0) = 0$	-2

Contour Integration & Maclaurin Series

$$f(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$

$$\int_C f(z)\,dz = \int_C a_0 z^0\,dz + \int_C a_1 z^1\,dz + \int_C a_2 z^2\,dz + \cdots \qquad \int_C f(z)\,dz = 0$$

$$\int_C \frac{f(z)}{z}\,dz = \int_C \frac{a_0}{z}\,dz + \int_C a_1 z^0\,dz + \int_C a_2 z^1\,dz + \cdots \qquad \int_C \frac{f(z)}{z}\,dz = a_0 \cdot 2\pi i$$

$$\int_C \frac{f(z)}{z^2}\,dz = \int_C a_0 z^{-2}\,dz + \int_C \frac{a_1}{z}\,dz + \int_C a_2 z^0\,dz + \cdots \qquad \int_C \frac{f(z)}{z^2}\,dz = a_1 \cdot 2\pi i$$

$$\int_C \frac{f(z)}{z^3}\,dz = \int_C a_0 z^{-3}\,dz + \int_C a_1 z^{-2}\,dz + \int_C \frac{a_2}{z}\,dz + \cdots \qquad \int_C \frac{f(z)}{z^3}\,dz = a_2 \cdot 2\pi i$$

Continuous along C

Discrete a_i

Differentiation & MacLauren Series

$$f(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$

$$f(z) = a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots \qquad f(0) = a_0 \qquad \int_C \frac{f(z)}{z}\, dz = a_0 \cdot 2\pi i$$

$$f'(z) = a_1 z^0 + a_2 2 z^1 + a_3 3 z^2 + \cdots \qquad f'(0) = a_1 \qquad \int_C \frac{f(z)}{z^2}\, dz = a_1 \cdot 2\pi i$$

$$f''(z) = a_2 2 z^0 + a_3 3 \cdot 2 z^1 + a_4 4 \cdot 3 z^2 \cdots \qquad f''(0) = a_2 2 \qquad \int_C \frac{f(z)}{z^3}\, dz = a_2 \cdot 2\pi i$$

$$f^{(n)}(0) = a_n \cdot n! \qquad \int_C \frac{f(z)}{z^{n+1}}\, dz = a_n \cdot 2\pi i$$

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz \qquad a_n = \frac{1}{n!} f^{(n)}(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z^{n+1}}\, dz$$

Maclaurin Series

A **power series** in powers of z — non-negative powers

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

The **Maclaurin series** of a function $f(z)$ — non-negative powers

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{1}{n!} f^{(n)}(0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$a_n = \frac{1}{n!} f^{(n)}(0) \quad \Rightarrow \quad f^{(n)}(0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw$$

References

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