

Laurent Series and z-Transform

- Geometric Series

Applications

(A)

20210203 Wed

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Unshifted Geometric Sequences

Causal

$$\frac{1}{1 - az} \rightarrow u(n)$$

Anti-causal

$$\frac{1}{1 - az^{-1}} \rightarrow u(-n)$$

Positive Exponent

$$az, az^{-1} \rightarrow a^n$$

Negative Exponent

$$a^{-1}z, az^{-1} \rightarrow a^{-n}$$

Positive Exponent

unshifted (1)

$$\frac{1}{1 - az} \quad |z| < a^{-1} \quad a^n u(n)$$

complementary (7)

$$\frac{az}{1 - az} \quad |z| < a^{-1} \quad a^n u(n-1)$$

unshifted (5)

$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1)$$

complementary (3)

$$-\frac{1}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n)$$

Negative Exponent

unshifted (2)

$$\frac{1}{1 - a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n)$$

complementary (8)

$$\frac{a^{-1}z}{1 - a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n-1)$$

unshifted (6)

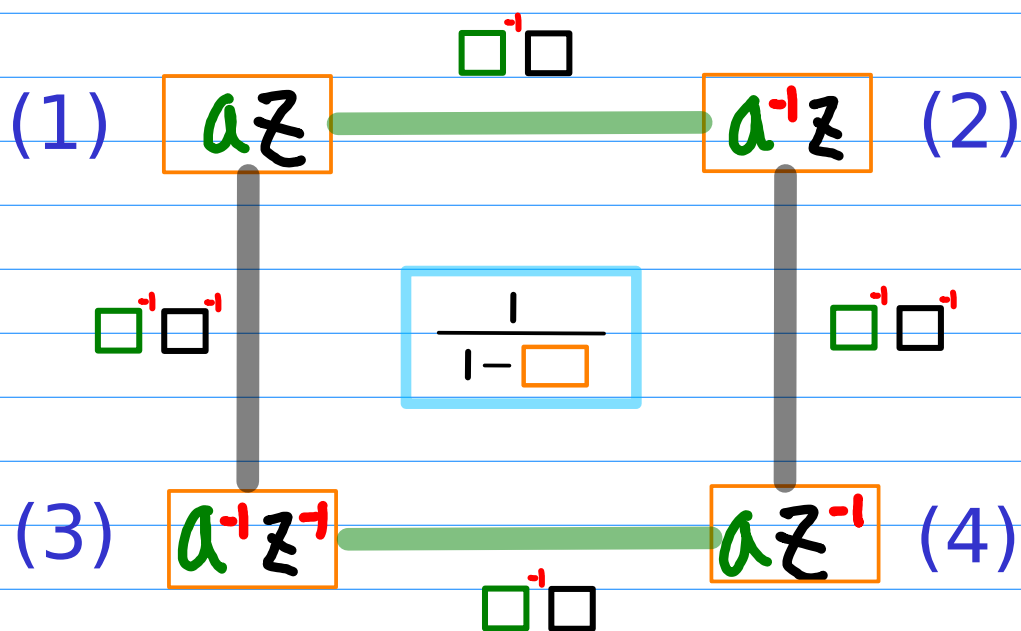
$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n-1)$$

complementary (4)

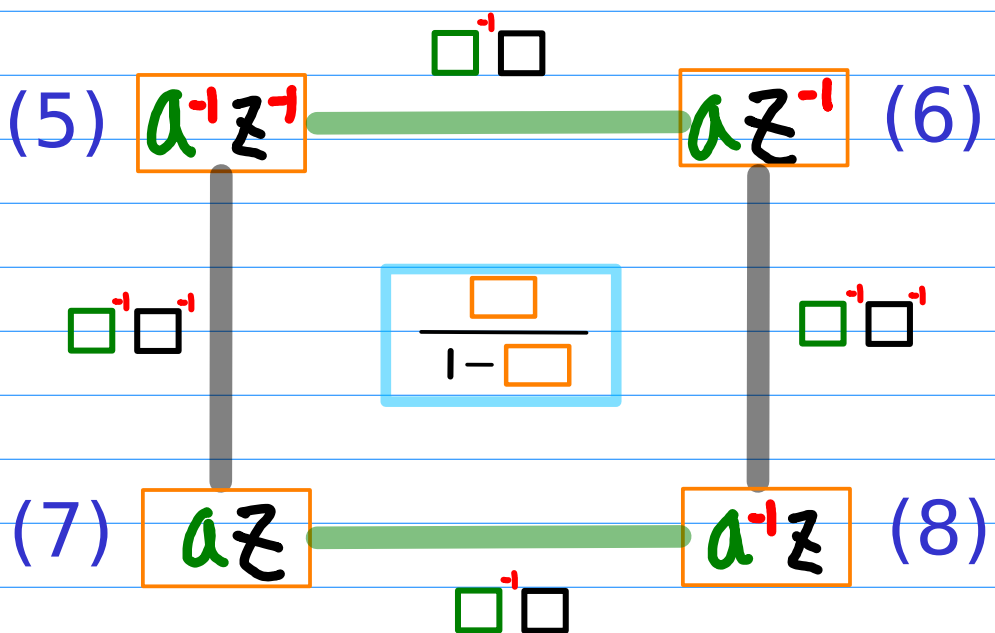
$$-\frac{1}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n)$$

Numbering the basic elements - (1) CR

unshifted geometric sequences

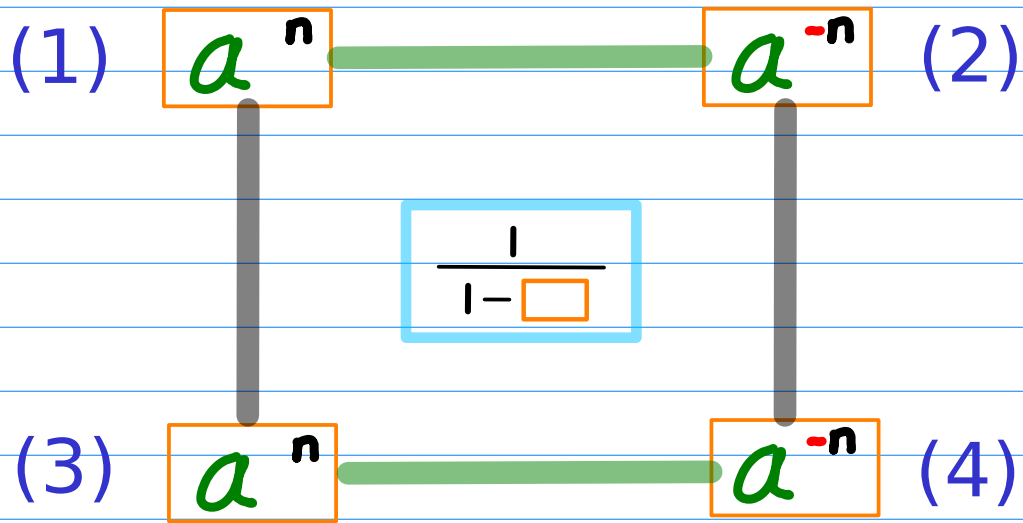


complementary geometric sequences

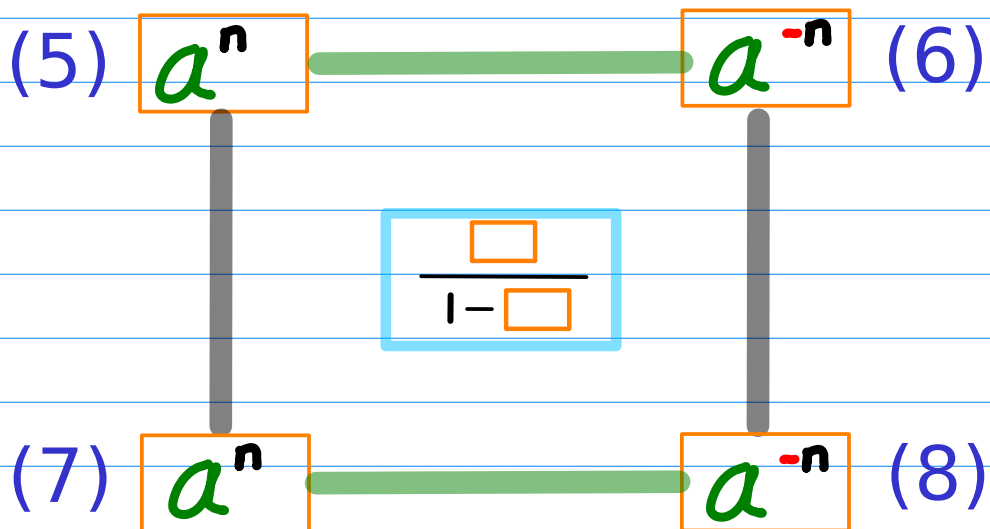


Numbering the basic elements - (2) Power

unshifted geometric sequences

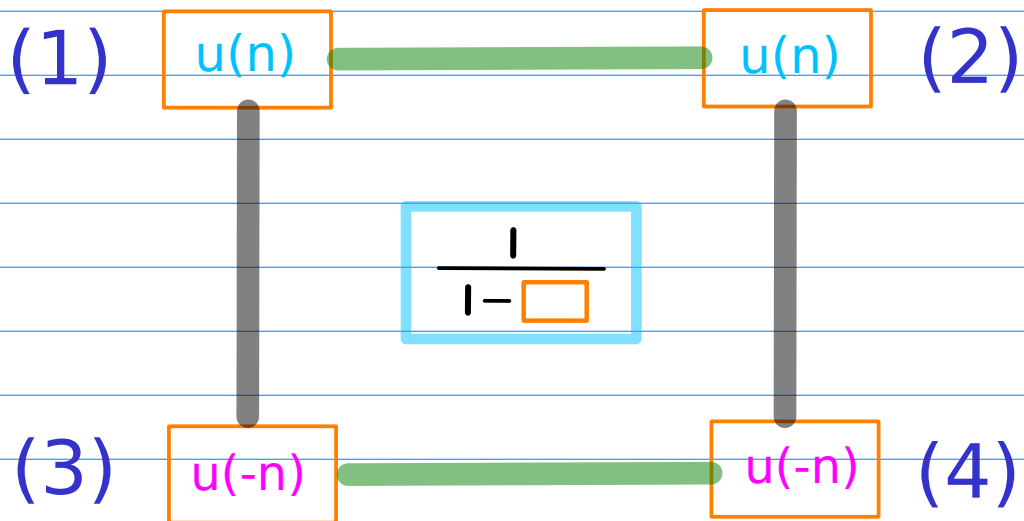


complementary geometric sequences

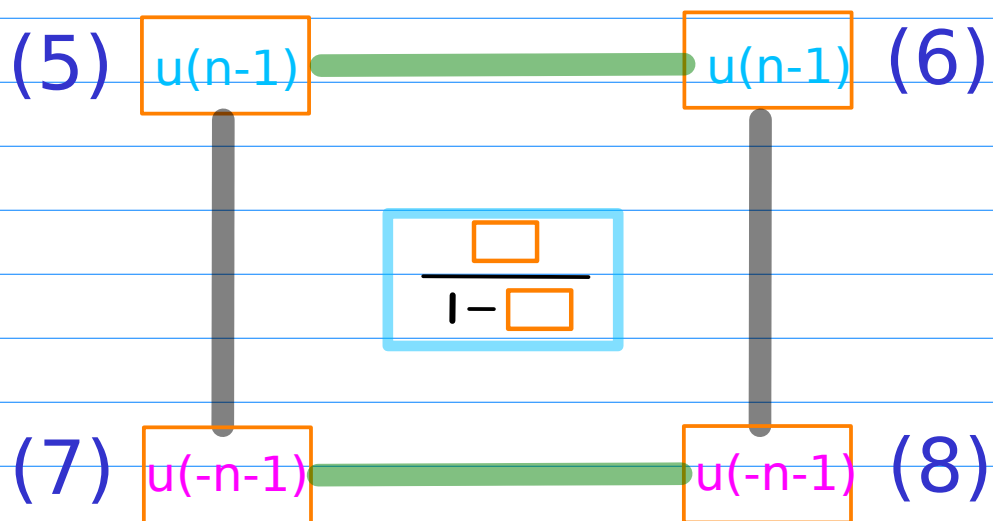


Numbering the basic elements - (3) Range

unshifted geometric sequences

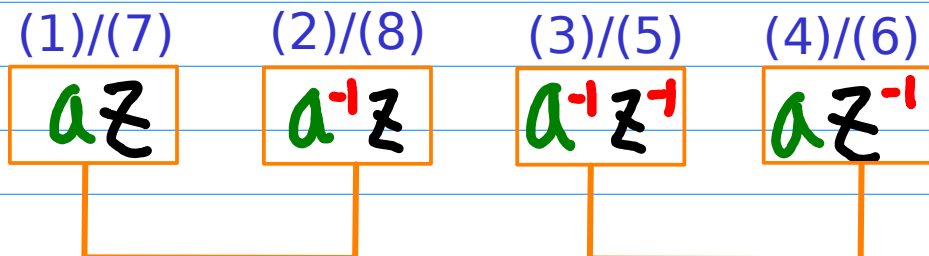


complementary geometric sequences

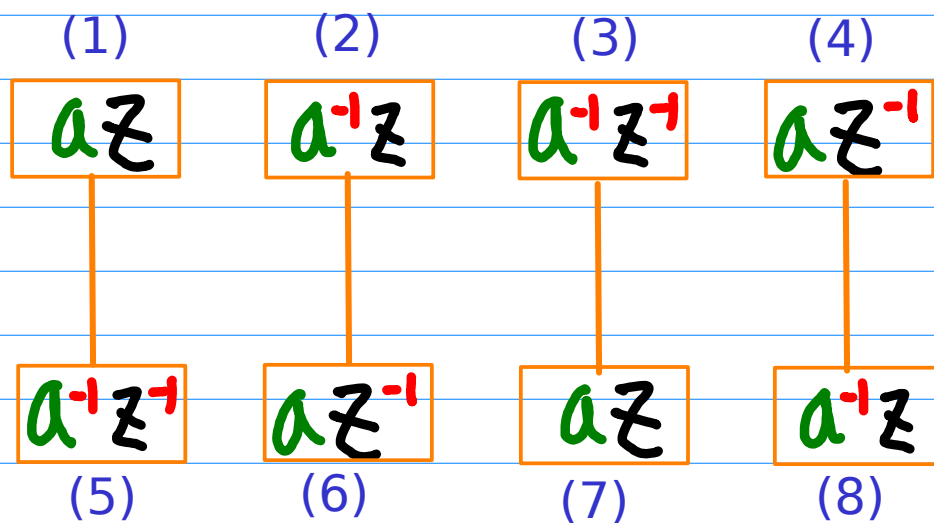


Inverse Relations

inverse power $\square^{\cdot} \square$



complementary $\square^{\cdot} \square^{\cdot}$



Geometric Series Form Combinations with a unit start term unshifted

(1)
$$+ \frac{1}{1 - az}$$

$a^n u(n)$

(2)
$$+ \frac{1}{1 - a^{-1}z}$$

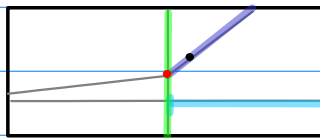
$a^{-n} u(n)$

(3)
$$- \frac{1}{1 - a^{-1}z^{-1}}$$

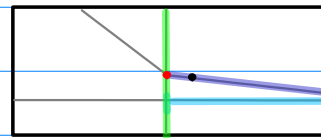
$-a^n u(-n)$

(4)
$$- \frac{1}{1 - az^{-1}}$$

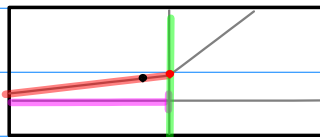
$-a^{-n} u(-n)$



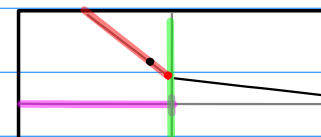
$a^n u(n)$



$a^{-n} u(n)$



$a^n u(-n)$



$a^{-n} u(-n)$

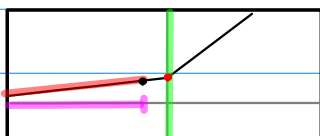
Geometric Series Form Combinations with a common-ratio start term **unshifted, complementary**

(5) $\frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} - a^n u(-n-1)$

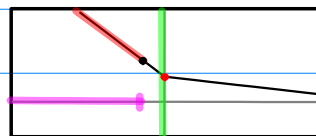
(6) $\frac{az^{-1}}{1 - az^{-1}} - a^{-n} u(-n-1)$

(7) $\frac{az}{1 - az} + a^n u(n-1)$

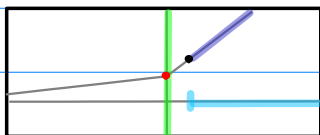
(8) $\frac{a^{-1}z}{1 - a^{-1}z} + a^{-n} u(n-1)$



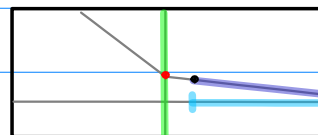
$a^n u(-n-1)$



$a^{-n} u(-n-1)$

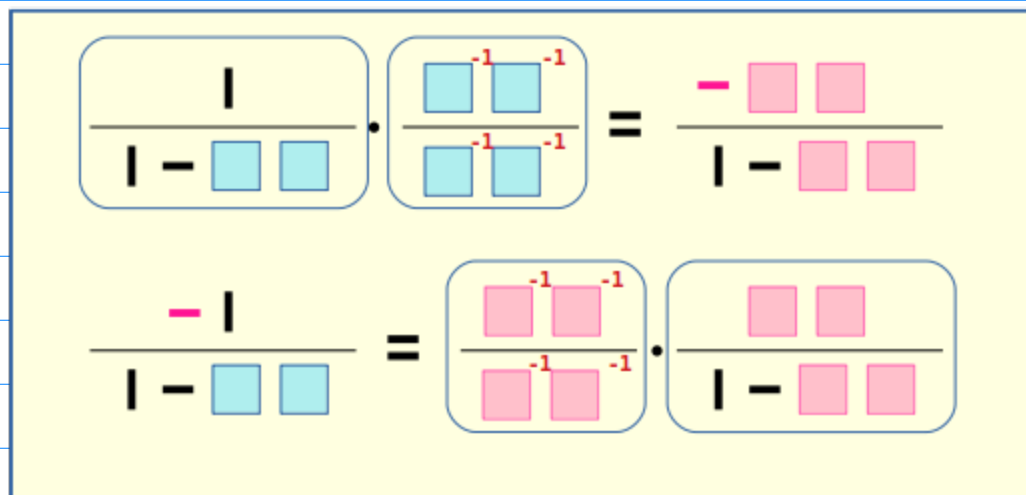
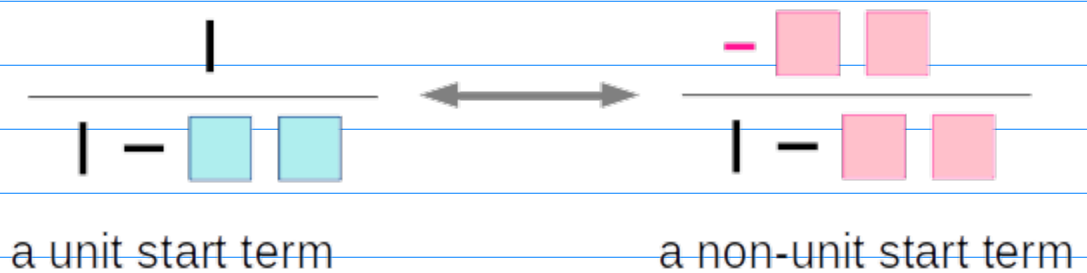
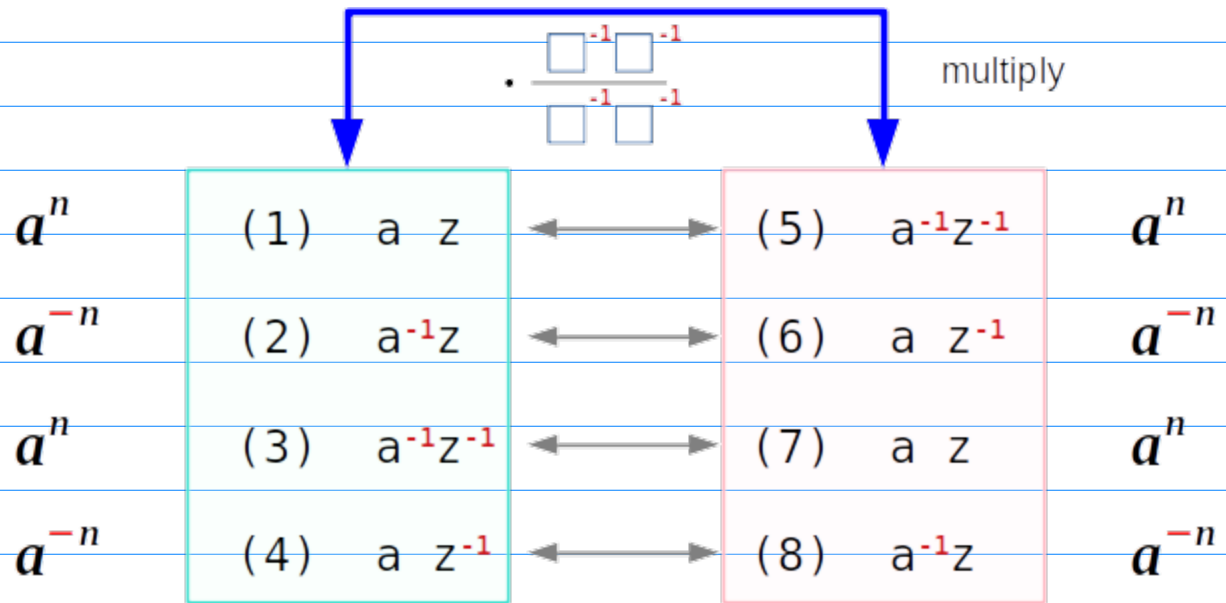


$a^n u(n-1)$

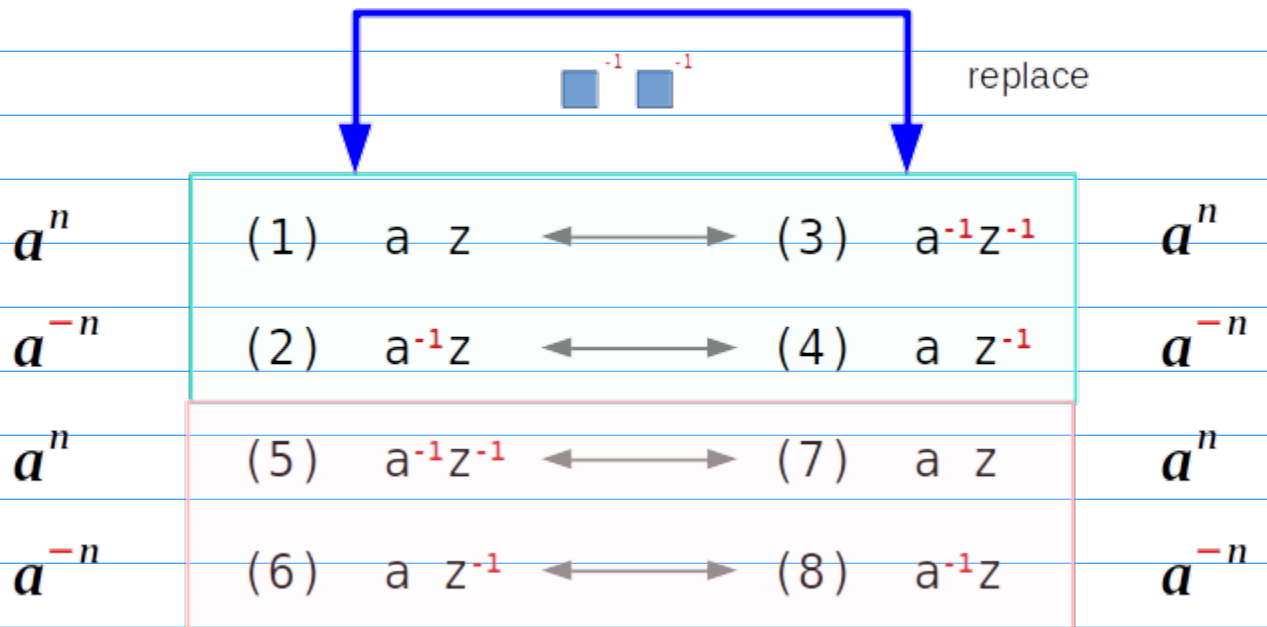


$a^{-n} u(n-1)$

Complementary Range Relations - CR only

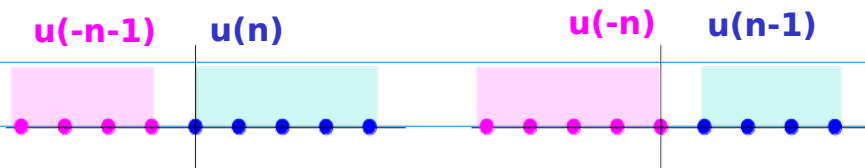
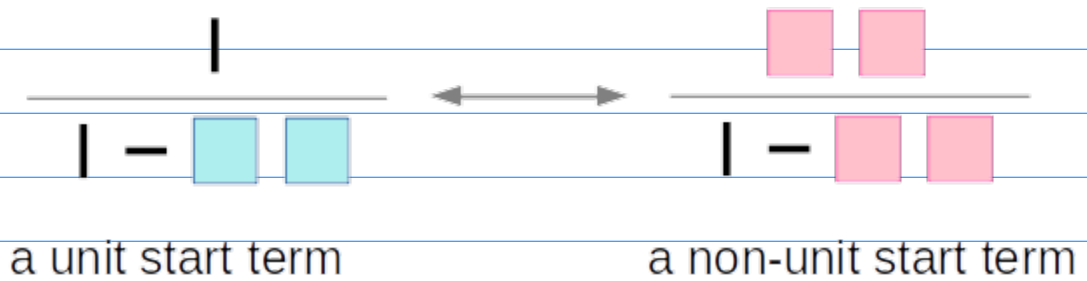


Symmetric Range Relations - CR only



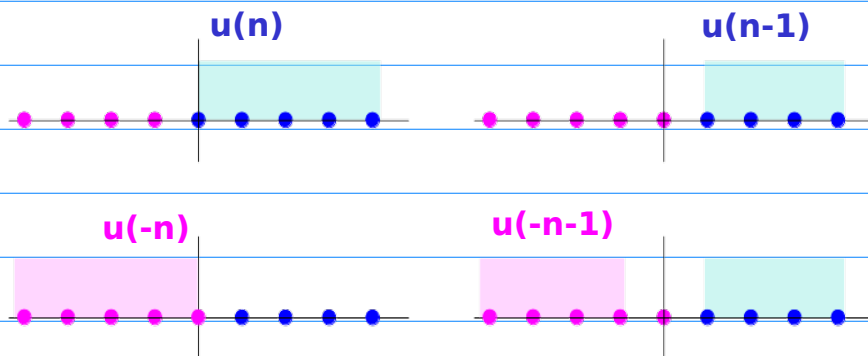
Complementary Range Relations - CR & Range

| | | |
|-----|---|------------------------|
| | $\cdot \frac{\boxed{}^{-1} \boxed{}^{-1}}{\boxed{}^{-1} \boxed{}^{-1}}$ | |
| | | multiply |
| (1) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ |
| (2) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| (3) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ |
| (4) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| (5) | $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ |
| (6) | $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| (7) | $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ |
| (8) | $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |



Symmetric Range Relations - CR & Range

| | |
|---|---|
| | |
| <p>(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$</p> <p>(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$</p> | <p>(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$</p> <p>(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$</p> |
| <p>(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$</p> <p>(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$</p> | <p>(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$</p> <p>(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$</p> |



| Positive Exponent | Negative Exponent |
|-------------------|-------------------|
| (1) | (2) |
| (3) | (4) |
| (5) | (6) |
| (7) | (8) |

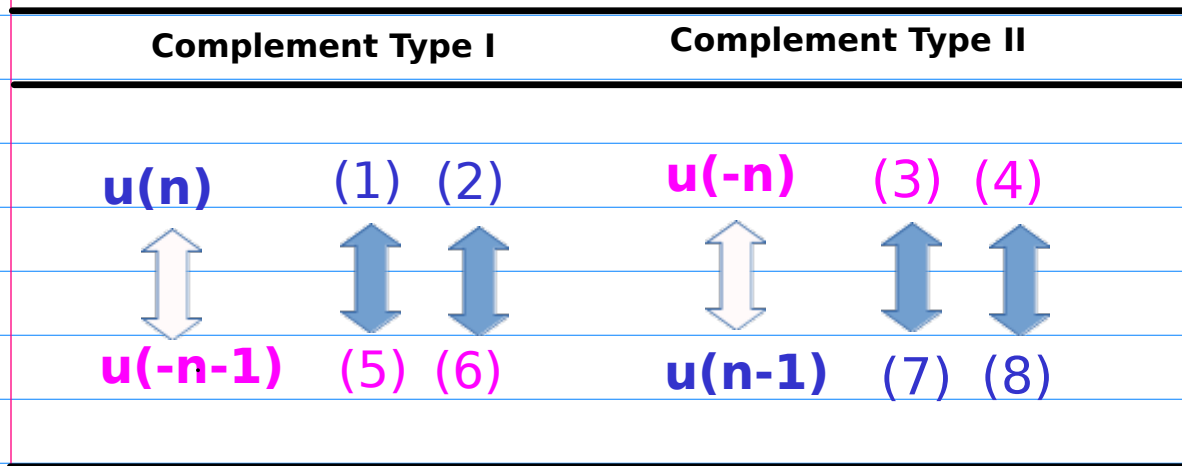
| | | | |
|-----------|-----|-----|-------------|
| $u(n)$ | (1) | (2) | Causal |
| $u(-n)$ | (3) | (4) | Anti-Causal |
| $u(-n-1)$ | (5) | (6) | Anti-Causal |
| $u(n-1)$ | (7) | (8) | Causal |

| | | | |
|---------------------------|-----|-----|-----------|
| ranges include the origin | (1) | (2) | $u(n)$ |
| | (3) | (4) | $u(-n)$ |
| ranges exclude the origin | (5) | (6) | $u(-n-1)$ |
| | (7) | (8) | $u(n-1)$ |

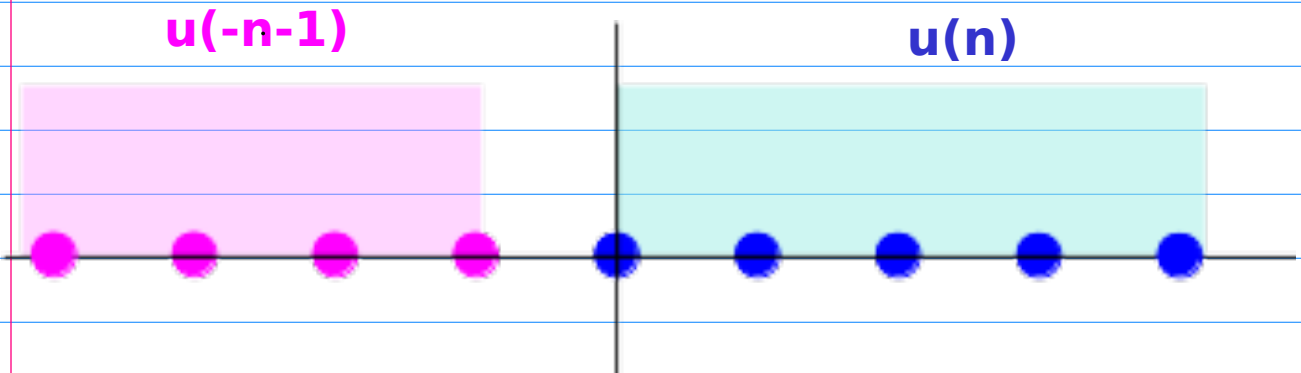
Complement Type I

| | | | |
|-----------|-----|-----|-------------|
| $u(n)$ | (1) | (2) | Causal |
| $u(-n)$ | (3) | (4) | Anti-Causal |
| $u(-n-1)$ | (5) | (6) | Anti-Causal |
| $u(n-1)$ | (7) | (8) | Causal |

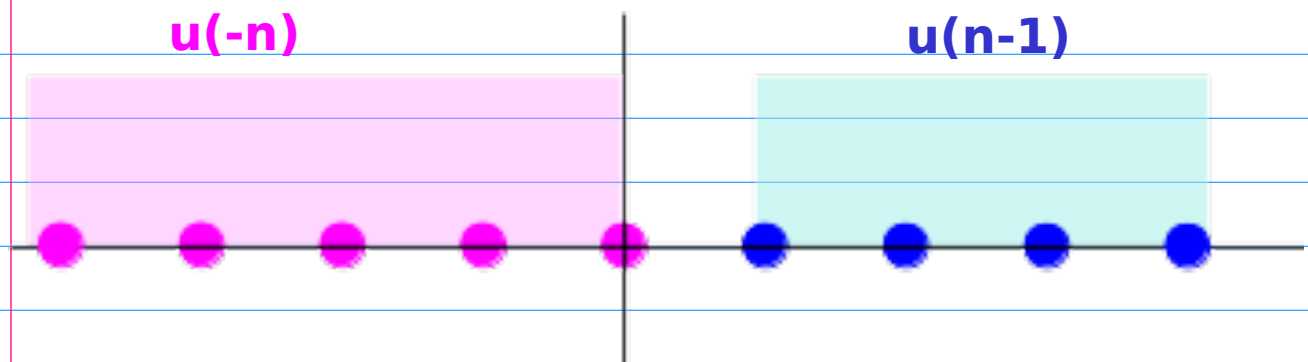
Complement Type II

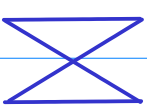

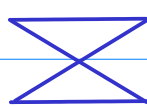
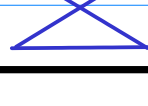


Complement Type I







Complement Type II











| | | | | |
|--------------------|-----------|-----|---|-----|
| Causal | $u(n)$ | (1) |  | (2) |
| | $u(n-1)$ | (7) |  | (8) |
| Anti-Causal | $u(-n-1)$ | (5) |  | (6) |
| | $u(-n)$ | (3) |  | (4) |

butterfly pair ordering

| | | | | |
|---------------------------|-----------|---|-----|-----|
| Complement Type I | $u(n)$ |  | (1) | (2) |
| | $u(-n-1)$ |  | (5) | (6) |
| Complement Type II | $u(-n)$ |  | (3) | (4) |
| | $u(n-1)$ |  | (7) | (8) |

complementary pair ordering

| | | Positive Exponent | Negative Exponent |
|---------------------------|-----------|---|---|
| Complement Type I | $u(n)$ | (1)  | (2)  |
| | $u(-n-1)$ | (5)  | (6)  |
| Complement Type II | $u(-n)$ | (3)  | (4)  |
| | $u(n-1)$ | (7)  | (8)  |

complementary pair ordering

Shifted Geometric Sequences

Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift

Positive Exponent

(1) *a, (7) /z

$$\frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$$

(1) *z, (7) /a

$$\frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$$

(5) *a, (3) /z

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$$

(5) *z, (3) /a

$$-\frac{a^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$$

Negative Exponent

(2) /a, (8) /z

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a \quad a^{-n-1} u(n)$$

(2) *z, (8) *a

$$\frac{z}{1-a^{-1}z} \quad |z| < a \quad a^{-n+1} u(n-1)$$

(6) /a, (4) /z

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n-1} u(-n-1)$$

(6) *z, (4) *a

$$-\frac{a}{1-az^{-1}} \quad |z| > a \quad a^{-n+1} u(-n)$$

Left Shifted

Right Shifted

Left Shifted

Right Shifted

multiplying a or a^{-1}

multiplying z or z^{-1}

Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift

Positive Exponent

unshifted (1)

$$\frac{1}{1-az} \quad |z| < a^{-1} \quad a^n u(n)$$

complementary (7)

$$\frac{az}{1-az} \quad |z| < a^{-1} \quad a^n u(n-1)$$

unshifted (5)

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1)$$

complementary (3)

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n)$$

Negative Exponent

unshifted (2)

$$\frac{1}{1-a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n)$$

complementary (8)

$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n-1)$$

unshifted (6)

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n-1)$$

complementary (4)

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n)$$

Positive Exponent

(1) *a, (7) /z

$$\frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$$

(1) *z, (7) /a

$$\frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$$

(5) *a, (3) /z

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$$

(5) *z, (3) /a

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$$

Negative Exponent

(2) /a, (8) /z

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a^{-1} \quad a^{-n-1} u(n)$$

(2) *z, (8) *a

$$\frac{z}{1-a^{-1}z} \quad |z| < a^{-1} \quad a^{-n+1} u(n-1)$$

(6) /a, (4) /z

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{-n-1} u(-n-1)$$

(6) *z, (4) *a

$$-\frac{a}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{-n+1} u(-n)$$

Left Shifted

Right Shifted

Left Shifted

Right Shifted

Combinations of Shifted Geometric Series (1)

Positive Exponent

/z $n \leftarrow n+1$

*z $n \leftarrow n-1$

(1) $\frac{1}{1-az} \quad |z| < a^{-1}$ $\xrightarrow{*a}$ $\frac{a}{1-az} \quad |z| < a^{-1}$ **Left Shifted**

$a^n u(n)$ $\xrightarrow{/z}$ $a^{n+1} u(n)$

(7) $\frac{az}{1-az} \quad |z| < a^{-1}$ $\xrightarrow{/a}$ $\frac{z}{1-az} \quad |z| < a^{-1}$ **Right Shifted**

$a^n u(n-1)$ $\xrightarrow{*z}$ $a^{n-1} u(n-1)$

(5) $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\xrightarrow{*a}$ $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ **Left Shifted**

$a^n u(-n-1)$ $\xrightarrow{/z}$ $a^{n+1} u(-n-1)$

(3) $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ $\xrightarrow{/a}$ $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$ **Right Shifted**

$a^n u(-n)$ $\xrightarrow{*z}$ $a^{n-1} u(-n)$

| | | | | |
|-------------|-----------|-----|-----|-------------------------|
| Causal | $u(n)$ | (1) | (2) | butterfly pair ordering |
| | $u(n-1)$ | (7) | (8) | |
| Anti-Causal | $u(-n-1)$ | (5) | (6) | |
| | $u(-n)$ | (3) | (4) | |

Combinations of Shifted Geometric Series (2)

Negative Exponent

/z $n \leftarrow n+1$

*z $n \leftarrow n-1$

(2) $\frac{1}{1-a^1z} \quad |z| < a$ $\xrightarrow{/a}$ $\frac{a^{-1}}{1-a^1z} \quad |z| < a$ **Left Shifted**
 $a^{-n} u(n)$ $\xrightarrow{/z}$ $a^{-n-1} u(n)$

(8) $\frac{a^1z}{1-a^1z} \quad |z| < a$ $\xrightarrow{*a}$ $\frac{z}{1-a^1z} \quad |z| < a$ **Right Shifted**
 $a^{-n} u(n-1)$ $\xrightarrow{*z}$ $a^{-n+1} u(n-1)$

(6) $-\frac{a^1z^{-1}}{1-a^1z^{-1}} \quad |z| > a$ $\xrightarrow{/a}$ $-\frac{z^{-1}}{1-a^1z^{-1}} \quad |z| > a$ **Left Shifted**
 $a^{-n} u(-n-1)$ $\xrightarrow{/z}$ $a^{-n-1} u(-n-1)$

(4) $-\frac{1}{1-a^1z^{-1}} \quad |z| > a$ $\xrightarrow{*a}$ $-\frac{a}{1-a^1z^{-1}} \quad |z| > a$ **Right Shifted**
 $a^{-n} u(-n)$ $\xrightarrow{*z}$ $a^{-n+1} u(-n)$

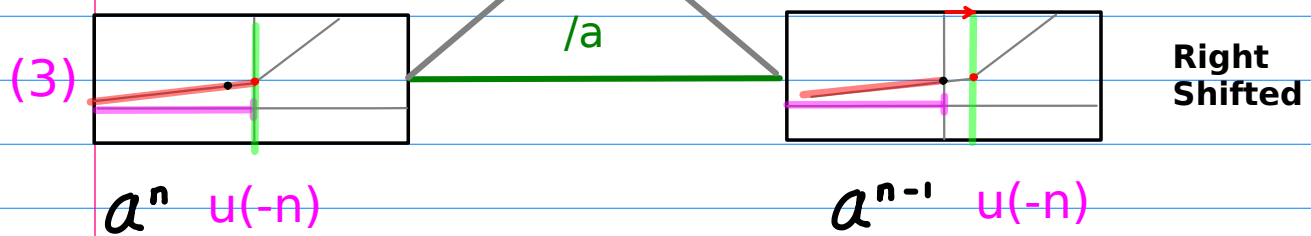
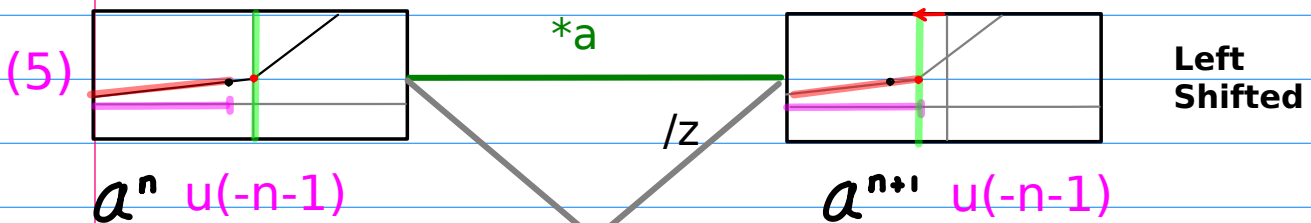
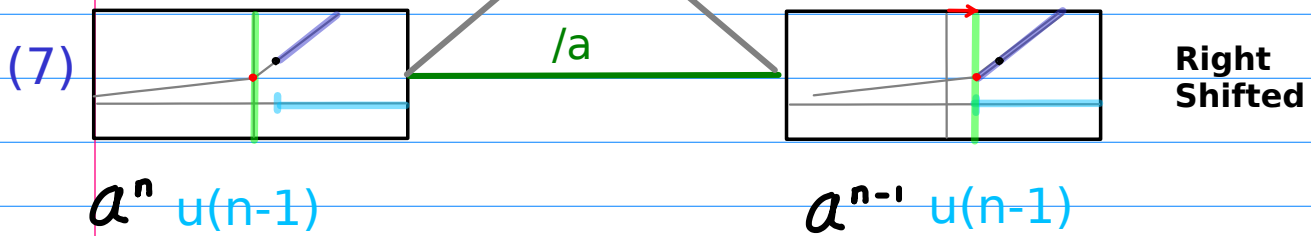
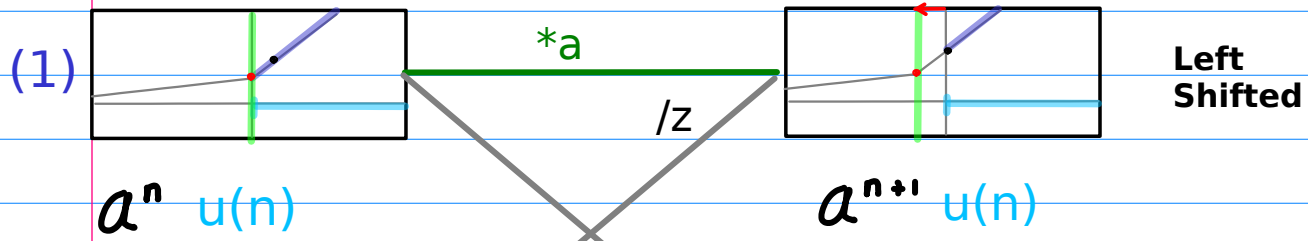
| | | | | |
|-------------|-----------|-----|-----|-------------------------|
| Causal | $u(n)$ | (1) | (2) | butterfly pair ordering |
| | $u(n-1)$ | (7) | (8) | |
| Anti-Causal | $u(-n-1)$ | (5) | (6) | |
| | $u(-n)$ | (3) | (4) | |

Graphs of Shifted Geometric Series (1)

Positive Exponent

$/z \quad n \leftarrow n+1$

$*z \quad n \leftarrow n-1$



| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

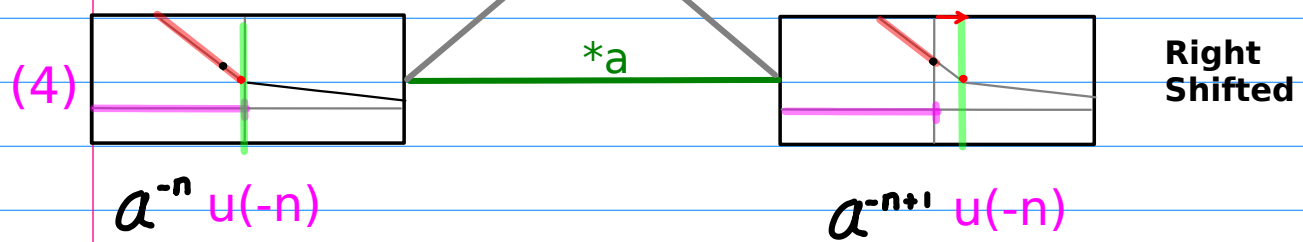
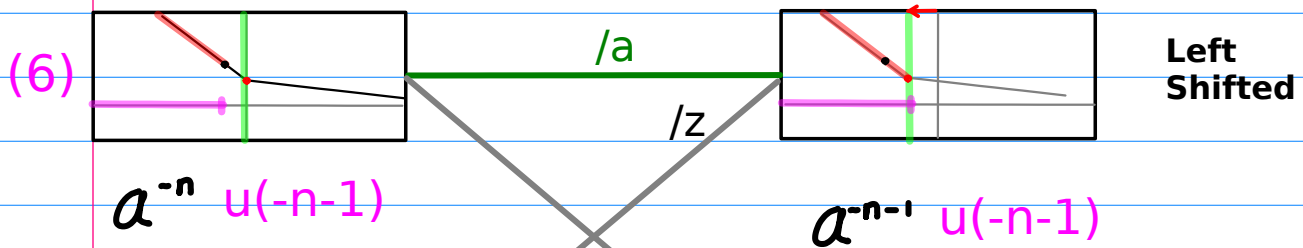
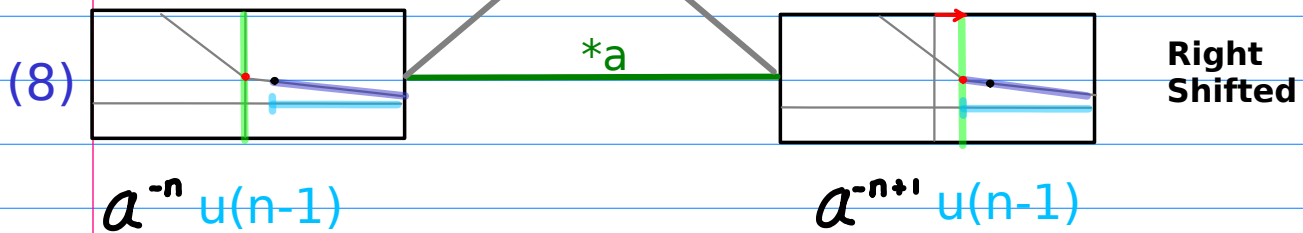
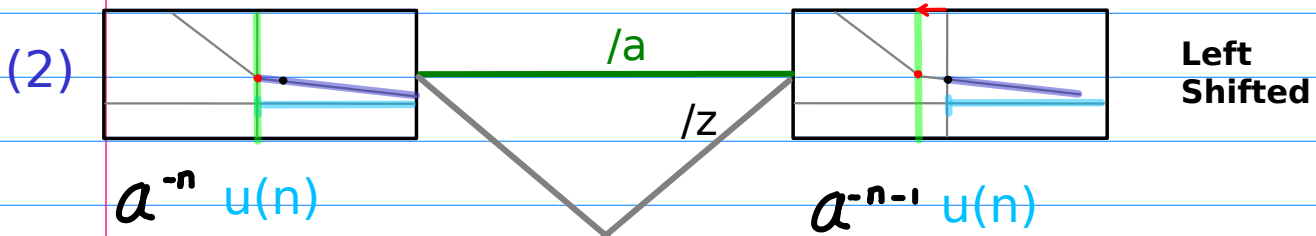
butterfly pair ordering

Graphs of Shifted Geometric Series (2)

Negative Exponent

/z $n \leftarrow n+1$

*z $n \leftarrow n-1$



| | | | | |
|-------------|-----------|-----|-----|-------------------------------|
| Causal | $u(n)$ | (1) | (2) | butterfly pair ordering |
| | $u(n-1)$ | (7) | (8) | |
| Anti-Causal | $u(-n-1)$ | (5) | (6) | |
| | $u(-n)$ | (3) | (4) | |

Shifting Geometric Series by $*a$ or $/a$



(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$
 \leftarrow $* a$ $a^{+n+1} \cdot u(n)$

(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$
 \leftarrow $/ a$ $a^{-n-1} \cdot u(n)$

(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$
 \leftarrow $* a$ $a^{+n+1} \cdot u(-n-1)$

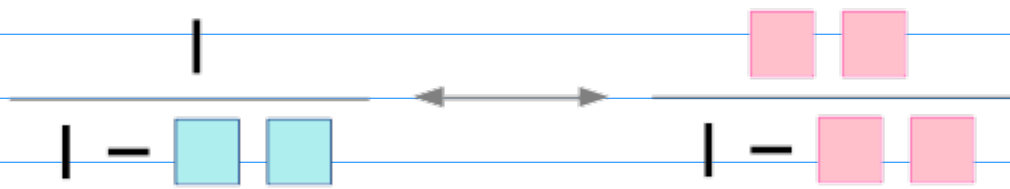
(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$
 \leftarrow $/ a$ $a^{-n-1} \cdot u(-n-1)$

(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$
 \rightarrow $/ a$ $a^{+n-1} \cdot u(-n)$

(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$
 \rightarrow $* a$ $a^{-n+1} \cdot u(-n)$

(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$
 \rightarrow $/ a$ $a^{+n-1} \cdot u(n-1)$

(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$
 \rightarrow $* a$ $a^{-n+1} \cdot u(n-1)$



a unit start term

a non-unit start term

Shifting Geometric Series by $*z$ or $/z$



(1) $a^{+1} z^{+1} \quad a^{+n} \cdot u(n)$
 \Rightarrow $* z \quad a^{+n-1} \cdot u(n-1)$

(2) $a^{-1} z^{+1} \quad a^{-n} \cdot u(n)$
 \Rightarrow $* z \quad a^{-n+1} \cdot u(n-1)$

(5) $a^{-1} z^{-1} \quad a^{+n} \cdot u(-n-1)$
 \Rightarrow $* z \quad a^{+n-1} \cdot u(-n)$

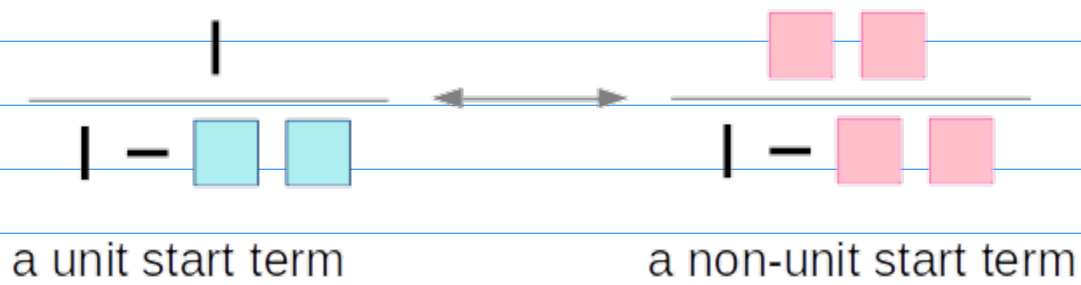
(6) $a^{+1} z^{-1} \quad a^{-n} \cdot u(-n-1)$
 \Rightarrow $* z \quad a^{-n+1} \cdot u(-n)$

(3) $a^{-1} z^{-1} \quad a^{+n} \cdot u(-n)$
 \Leftarrow $/ z \quad a^{+n+1} \cdot u(-n-1)$

(4) $a^{+1} z^{-1} \quad a^{-n} \cdot u(-n)$
 \Leftarrow $/ z \quad a^{-n-1} \cdot u(-n-1)$

(7) $a^{+1} z^{+1} \quad a^{+n} \cdot u(n-1)$
 \Leftarrow $/ z \quad a^{+n+1} \cdot u(n)$

(8) $a^{-1} z^{+1} \quad a^{-n} \cdot u(n-1)$
 \Leftarrow $/ z \quad a^{-n-1} \cdot u(n)$



| | | |
|-----------|-----|-----|
| $u(n)$ | (1) | (2) |
| $u(-n)$ | (3) | (4) |
| $u(-n-1)$ | (5) | (6) |
| $u(n-1)$ | (7) | (8) |

row
major
ordering

| | |
|---|---|
| (1) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n)$ ← $* a$ $a^{+n+1} \cdot u(n)$ | (2) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n)$ ← $/ a$ $a^{-n-1} \cdot u(n)$ |
| (3) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n)$ → $/ a$ $a^{+n-1} \cdot u(-n)$ | (4) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n)$ → $* a$ $a^{-n+1} \cdot u(-n)$ |
| (5) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n-1)$ ← $* a$ $a^{+n+1} \cdot u(-n-1)$ | (6) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n-1)$ ← $/ a$ $a^{-n-1} \cdot u(-n-1)$ |
| (7) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n-1)$ → $/ a$ $a^{+n-1} \cdot u(n-1)$ | (8) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n-1)$ → $* a$ $a^{-n+1} \cdot u(n-1)$ |

| | |
|---|---|
| (1) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n)$ ⇒ $* z$ $a^{+n-1} \cdot u(n-1)$ | (2) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n)$ ⇒ $* z$ $a^{-n+1} \cdot u(n-1)$ |
| (3) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n)$ ⇐ $/ z$ $a^{+n+1} \cdot u(-n-1)$ | (4) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n)$ ⇐ $/ z$ $a^{-n-1} \cdot u(-n-1)$ |
| (5) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n-1)$ ⇒ $* z$ $a^{+n-1} \cdot u(-n)$ | (6) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n-1)$ ⇒ $* z$ $a^{-n+1} \cdot u(-n)$ |
| (7) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n-1)$ ⇐ $/ z$ $a^{+n+1} \cdot u(n)$ | (8) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n-1)$ ⇐ $/ z$ $a^{-n-1} \cdot u(n)$ |

| | | | |
|-------------|-----------|-----|-----|
| Causal | $u(n)$ | (1) | (2) |
| | $u(n-1)$ | (7) | (8) |
| Anti-Causal | $u(-n-1)$ | (5) | (6) |
| | $u(-n)$ | (3) | (4) |

butterfly pair ordering

| | | | |
|--------------------|--------------------------|--------------------|--------------------------|
| (1) $a^{+1}z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1}z^{+1}$ | $a^{-n} \cdot u(n)$ |
| ← $* a$ | $a^{+n+1} \cdot u(n)$ | ← $/ a$ | $a^{-n-1} \cdot u(n)$ |
| (7) $a^{+1}z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1}z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| → $/ a$ | $a^{+n-1} \cdot u(n-1)$ | → $* a$ | $a^{-n+1} \cdot u(n-1)$ |
| (5) $a^{-1}z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1}z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| ← $* a$ | $a^{+n+1} \cdot u(-n-1)$ | ← $/ a$ | $a^{-n-1} \cdot u(-n-1)$ |
| (3) $a^{-1}z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1}z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| → $/ a$ | $a^{+n-1} \cdot u(-n)$ | → $* a$ | $a^{-n+1} \cdot u(-n)$ |

| | | | |
|--------------------|--------------------------|--------------------|--------------------------|
| (1) $a^{+1}z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1}z^{+1}$ | $a^{-n} \cdot u(n)$ |
| ⇒ $* z$ | $a^{+n-1} \cdot u(n-1)$ | ⇒ $* z$ | $a^{-n+1} \cdot u(n-1)$ |
| (7) $a^{+1}z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1}z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| ⇐ $/ z$ | $a^{+n+1} \cdot u(n)$ | ⇐ $/ z$ | $a^{-n-1} \cdot u(n)$ |
| (5) $a^{-1}z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1}z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| ⇒ $* z$ | $a^{+n-1} \cdot u(-n)$ | ⇒ $* z$ | $a^{-n+1} \cdot u(-n)$ |
| (3) $a^{-1}z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1}z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| ⇐ $/ z$ | $a^{+n+1} \cdot u(-n-1)$ | ⇐ $/ z$ | $a^{-n-1} \cdot u(-n-1)$ |

| | | | |
|---------------------------|-----------|-----|-----|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

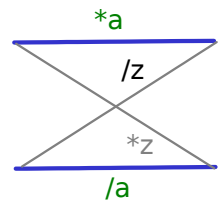
complementary pair ordering

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| ← $* a$ | $a^{+n+1} \cdot u(n)$ | ← $/ a$ | $a^{-n-1} \cdot u(n)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| ← $* a$ | $a^{+n+1} \cdot u(-n-1)$ | ← $/ a$ | $a^{-n-1} \cdot u(-n-1)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| → $/ a$ | $a^{+n-1} \cdot u(-n)$ | → $* a$ | $a^{-n+1} \cdot u(-n)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| → $/ a$ | $a^{+n-1} \cdot u(n-1)$ | → $* a$ | $a^{-n+1} \cdot u(n-1)$ |

| | | | |
|---------------------|--------------------------|---------------------|--------------------------|
| (1) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n)$ | (2) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n)$ |
| ⇒ $* z$ | $a^{+n-1} \cdot u(n-1)$ | ⇒ $* z$ | $a^{-n+1} \cdot u(n-1)$ |
| (5) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n-1)$ | (6) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n-1)$ |
| ⇒ $* z$ | $a^{+n-1} \cdot u(-n)$ | ⇒ $* z$ | $a^{-n+1} \cdot u(-n)$ |
| (3) $a^{-1} z^{-1}$ | $a^{+n} \cdot u(-n)$ | (4) $a^{+1} z^{-1}$ | $a^{-n} \cdot u(-n)$ |
| ⇐ $/ z$ | $a^{+n+1} \cdot u(-n-1)$ | ⇐ $/ z$ | $a^{-n-1} \cdot u(-n-1)$ |
| (7) $a^{+1} z^{+1}$ | $a^{+n} \cdot u(n-1)$ | (8) $a^{-1} z^{+1}$ | $a^{-n} \cdot u(n-1)$ |
| ⇐ $/ z$ | $a^{+n+1} \cdot u(n)$ | ⇐ $/ z$ | $a^{-n-1} \cdot u(n)$ |

Shifted Geometric Series (1)

by multiplying a or a^{-1}



Positive Exponent

(1) ← $\frac{1}{1-az} \quad |z| < a^{-1} \quad a^n u(n) \times a \quad \frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$

(7) → $\frac{az}{1-az} \quad |z| < a^{-1} \quad a^n u(n-1) \times a^{-1} \quad \frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$

(5) ← $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1) \times a \quad -\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$

(3) → $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n) \times a^{-1} \quad -\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$

Negative Exponent

(2) ← $\frac{1}{1-a^{-1}z} \quad |z| < a \quad a^{-n} u(n) \times a^{-1} \quad \frac{a^{-1}}{1-a^{-1}z} \quad |z| < a \quad a^{-n-1} u(n)$

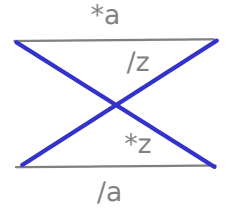
(8) → $\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a \quad a^{-n} u(n-1) \times a \quad \frac{z}{1-a^{-1}z} \quad |z| < a \quad a^{-n+1} u(n-1)$

(6) ← $-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n} u(-n-1) \times a^{-1} \quad -\frac{z^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n-1} u(-n-1)$

(4) → $-\frac{1}{1-az^{-1}} \quad |z| > a \quad a^{-n} u(-n) \times a \quad -\frac{a}{1-az^{-1}} \quad |z| > a \quad a^{-n+1} u(-n)$

Shifted Geometric Series (2)

by multiplying z or z^{-1}



Positive Exponent

$$(1) \quad \leftarrow \quad \frac{az}{1-az} \quad |z| < a^{-1} \quad \begin{matrix} n \leftarrow n+1 \\ a^n u(n-1) \end{matrix} \times z^{-1} \quad \boxed{\frac{a}{1-az} \quad |z| < a^{-1}} \quad a^{n+1} u(n)$$

$$(7) \quad \Rightarrow \quad \frac{1}{1-az} \quad |z| < a^{-1} \quad \begin{matrix} n \leftarrow n-1 \\ a^n u(n) \end{matrix} \times z \quad \boxed{\frac{z}{1-az} \quad |z| < a^{-1}} \quad a^{n-1} u(n-1)$$

$$(5) \quad \leftarrow \quad -\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad \begin{matrix} n \leftarrow n+1 \\ a^n u(-n) \end{matrix} \times z^{-1} \quad \boxed{-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}} \quad a^{n+1} u(-n-1)$$

$$(3) \quad \Rightarrow \quad -\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad \begin{matrix} n \leftarrow n-1 \\ a^n u(-n-1) \end{matrix} \times z \quad \boxed{-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}} \quad a^{n-1} u(-n)$$

Negative Exponent

$$(2) \quad \leftarrow \quad \frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a \quad \begin{matrix} n \leftarrow n+1 \\ a^{-n} u(n-1) \end{matrix} \times z^{-1} \quad \boxed{\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a} \quad a^{-n-1} u(n)$$

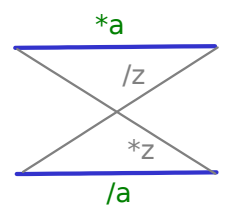
$$(8) \quad \Rightarrow \quad \frac{1}{1-a^{-1}z} \quad |z| < a \quad \begin{matrix} n \leftarrow n-1 \\ a^{-n} u(n) \end{matrix} \times z \quad \boxed{\frac{z}{1-a^{-1}z} \quad |z| < a} \quad a^{-n+1} u(n-1)$$

$$(6) \quad \leftarrow \quad -\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a \quad \begin{matrix} n \leftarrow n+1 \\ a^{-n} u(-n) \end{matrix} \times z^{-1} \quad \boxed{-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a} \quad a^{-n-1} u(-n-1)$$

$$(4) \quad \Rightarrow \quad -\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a \quad \begin{matrix} n \leftarrow n-1 \\ a^{-n} u(-n-1) \end{matrix} \times z \quad \boxed{-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a} \quad a^{-n+1} u(-n)$$

Shifted Geometric Series (3)

by multiplying a or a^{-1} Assume $a > 1$



- (1) $a^n u(n)$ $*a \leftarrow$ $a^{n+1} u(n)$
- (2) $a^{-n} u(n)$ $/a \leftarrow$ $a^{-n-1} u(n)$
- (3) $a^n u(-n)$ $/a \rightarrow$ $a^{n-1} u(-n)$
- (4) $a^{-n} u(-n)$ $*a \rightarrow$ $a^{-n+1} u(-n)$
- (5) $a^n u(-n-1)$ $*a \leftarrow$ $a^{n+1} u(-n-1)$
- (6) $a^{-n} u(-n-1)$ $/a \leftarrow$ $a^{-n-1} u(-n-1)$
- (7) $a^n u(n-1)$ $/a \rightarrow$ $a^{n-1} u(n-1)$
- (8) $a^{-n} u(n-1)$ $*a \rightarrow$ $a^{-n+1} u(n-1)$

row major ordering

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (3) | (4) | $/a$ | $*a$ | \rightarrow | \rightarrow |
| (5) | (6) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (7) | (8) | $/a$ | $*a$ | \rightarrow | \rightarrow |

complementary pair ordering

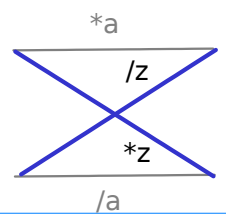
| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (5) | (6) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (3) | (4) | $/a$ | $*a$ | \rightarrow | \rightarrow |
| (7) | (8) | $/a$ | $*a$ | \rightarrow | \rightarrow |

butterfly pair ordering

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (7) | (8) | $/a$ | $*a$ | \rightarrow | \rightarrow |
| (5) | (6) | $*a$ | $/a$ | \leftarrow | \leftarrow |
| (3) | (4) | $/a$ | $*a$ | \rightarrow | \rightarrow |

Shifted Geometric Series (4)

by multiplying z or z^{-1} Assume $a > 1$



- (1) $a^n u(n)$ $*z \Rightarrow a^{n-1} u(n-1)$
- (2) $a^{-n} u(n)$ $*z \Rightarrow a^{-n+1} u(n-1)$
- (3) $a^n u(-n)$ $/z \Leftarrow a^{n+1} u(-n-1)$
- (4) $a^{-n} u(-n)$ $/z \Leftarrow a^{-n-1} u(-n-1)$
- (5) $a^n u(-n-1)$ $*z \Rightarrow a^{n-1} u(-n)$
- (6) $a^{-n} u(-n-1)$ $*z \Rightarrow a^{-n+1} u(-n)$
- (7) $a^n u(n-1)$ $/z \Leftarrow a^{n+1} u(n)$
- (8) $a^{-n} u(n-1)$ $/z \Leftarrow a^{-n-1} u(n)$

row major ordering

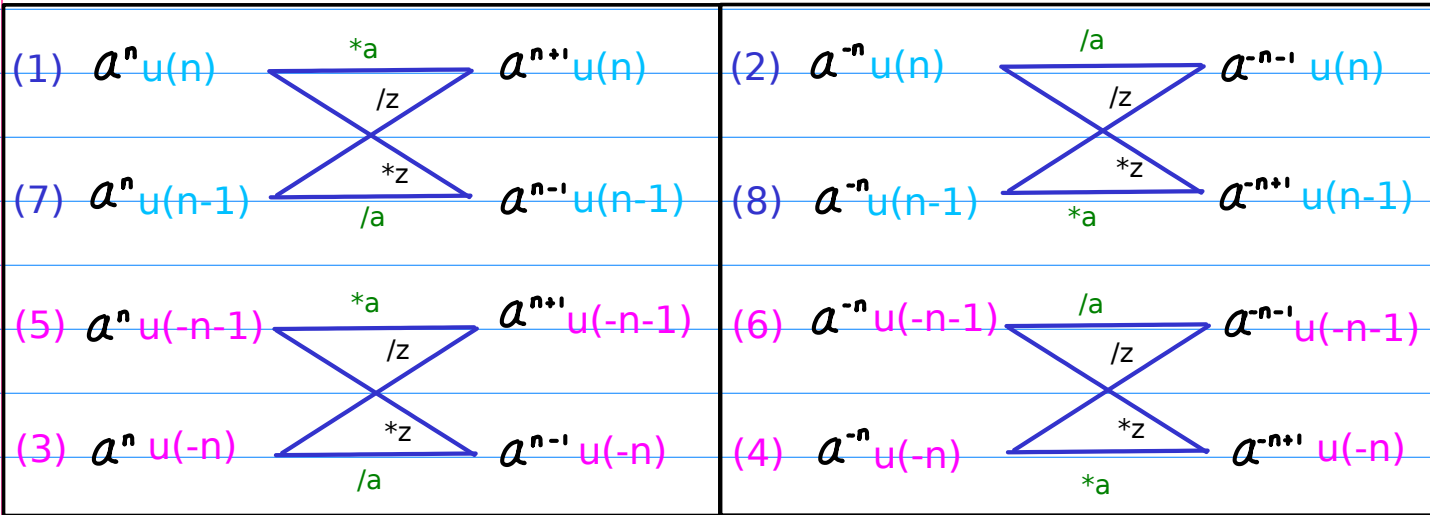
| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (3) | (4) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |
| (5) | (6) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (7) | (8) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |

complementary pair ordering

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (5) | (6) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (3) | (4) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |
| (7) | (8) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |

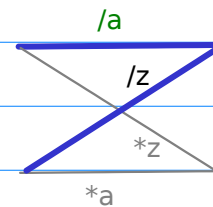
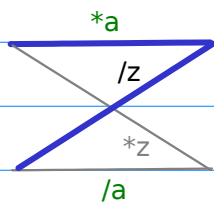
butterfly pair ordering

| | | | | | |
|-----|-----|------|------|---------------|---------------|
| (1) | (2) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (7) | (8) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |
| (5) | (6) | $*z$ | $*z$ | \Rightarrow | \Rightarrow |
| (3) | (4) | $/z$ | $/z$ | \Leftarrow | \Leftarrow |



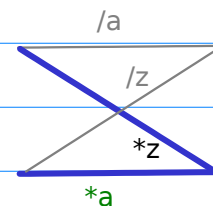
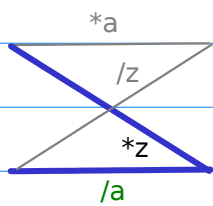
$$\leftarrow (1) *a = (7) /z \leftarrow$$

$$\leftarrow (2) /a = (8) /z \leftarrow$$



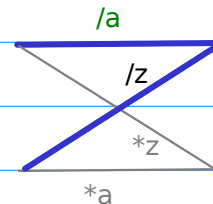
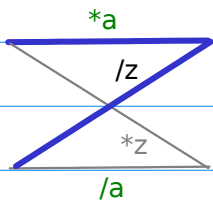
$$\Rightarrow (1) *z = (7) /a \Rightarrow$$

$$\Rightarrow (2) *z = (8) *a \Rightarrow$$



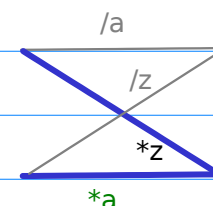
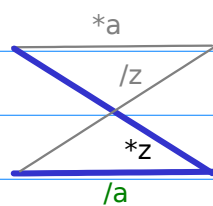
$$\leftarrow (5) *a = (3) /z \leftarrow$$

$$\leftarrow (6) /a = (4) /z \leftarrow$$



$$\Rightarrow (5) *z = (3) /a \Rightarrow$$

$$\Rightarrow (6) *z = (4) *a \Rightarrow$$



$\leftarrow (1) *a = (7) /z \rightleftarrows$
 $\Rightarrow (1) *z = (7) /a \rightarrow$
 $\rightleftarrows (3) /z = (5) *a \leftarrow$
 $\rightarrow (3) /a = (5) *z \Rightarrow$

$\leftarrow (2) /a = (8) /z \rightleftarrows$
 $\Rightarrow (2) *z = (8) *a \rightarrow$
 $\rightleftarrows (4) /z = (6) /a \leftarrow$
 $\rightarrow (4) *a = (6) *z \Rightarrow$

row major ordering

$\leftarrow (1) *a \quad *z \Rightarrow$
 $\rightarrow (3) /a \quad /z \rightleftarrows$
 $\leftarrow (5) *a \quad *z \Rightarrow$
 $\rightarrow (7) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$
 $\rightarrow (4) *a \quad /z \rightleftarrows$
 $\leftarrow (6) /a \quad *z \Rightarrow$
 $\rightarrow (8) *a \quad /z \rightleftarrows$

complementary pair ordering

$\leftarrow (1) *a \quad *z \Rightarrow$
 $\leftarrow (5) *a \quad *z \Rightarrow$
 $\rightarrow (3) /a \quad /z \rightleftarrows$
 $\rightarrow (7) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$
 $\leftarrow (6) /a \quad *z \Rightarrow$
 $\rightarrow (4) *a \quad /z \rightleftarrows$
 $\rightarrow (8) *a \quad /z \rightleftarrows$

butterfly pair ordering

$\leftarrow (1) *a \quad *z \Rightarrow$
 $\rightarrow (7) /a \quad /z \rightleftarrows$
 $\leftarrow (5) *a \quad *z \Rightarrow$
 $\rightarrow (3) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$
 $\rightarrow (8) *a \quad /z \rightleftarrows$
 $\leftarrow (6) /a \quad *z \Rightarrow$
 $\rightarrow (4) *a \quad /z \rightleftarrows$

Assume $a > 1$

- (1) $a^n u(n)$ $*a \leftarrow a^{n+1} u(n)$
 (7) $a^n u(n-1)$ $/a \rightarrow a^{n-1} u(n-1)$
 (5) $a^n u(-n-1)$ $*a \leftarrow a^{n+1} u(-n-1)$
 (3) $a^n u(-n)$ $/a \rightarrow a^{n-1} u(-n)$
 (2) $a^{-n} u(n)$ $/a \leftarrow a^{-n-1} u(n)$
 (8) $a^{-n} u(n-1)$ $*a \rightarrow a^{-n+1} u(n-1)$
 (6) $a^{-n} u(-n-1)$ $/a \leftarrow a^{-n-1} u(-n-1)$
 (4) $a^{-n} u(-n)$ $*a \rightarrow a^{-n+1} u(-n)$

| | |
|---------------|-------------------|
| $2^n u(n)$ | $2^{n+1} u(n)$ |
| $2^n u(n-1)$ | $2^{n-1} u(n-1)$ |
| $2^n u(-n-1)$ | $2^{n+1} u(-n-1)$ |
| $2^n u(-n)$ | $2^{n-1} u(-n)$ |

| | |
|------------------------------|--------------------------------|
| $(\frac{1}{2})^{-n} u(n)$ | $(\frac{1}{2})^{-n-1} u(n)$ |
| $(\frac{1}{2})^{-n} u(n-1)$ | $(\frac{1}{2})^{-n+1} u(n-1)$ |
| $(\frac{1}{2})^{-n} u(-n-1)$ | $(\frac{1}{2})^{-n-1} u(-n-1)$ |
| $(\frac{1}{2})^{-n} u(-n)$ | $(\frac{1}{2})^{-n+1} u(-n)$ |

| | |
|---------------------------|-------------------------------|
| $(\frac{1}{2})^n u(n)$ | $(\frac{1}{2})^{n+1} u(n)$ |
| $(\frac{1}{2})^n u(n-1)$ | $(\frac{1}{2})^{n-1} u(n-1)$ |
| $(\frac{1}{2})^n u(-n-1)$ | $(\frac{1}{2})^{n+1} u(-n-1)$ |
| $(\frac{1}{2})^n u(-n)$ | $(\frac{1}{2})^{n-1} u(-n)$ |

| | |
|------------------|--------------------|
| $2^{-n} u(n)$ | $2^{-n-1} u(n)$ |
| $2^{-n} u(n-1)$ | $2^{-n+1} u(n-1)$ |
| $2^{-n} u(-n-1)$ | $2^{-n-1} u(-n-1)$ |
| $2^{-n} u(-n)$ | $2^{-n+1} u(-n)$ |

| | | | |
|--------------------|-----------|-----|-----|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

complementary pair ordering

$$\begin{array}{l} (1) \quad a z \quad \mathbf{a}^n \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{a}^{-n} \\ (6) \quad a z^{-1} \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{a}^n \\ (7) \quad a z \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{a}^{-n} \\ (8) \quad a^{-1}z \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (1) \quad a z \quad \mathbf{*a} \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{/a} \\ (6) \quad a z^{-1} \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{/a} \\ (7) \quad a z \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{*a} \\ (8) \quad a^{-1}z \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (1) \quad a z \quad \mathbf{*z} \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{*z} \\ (6) \quad a z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{/z} \\ (7) \quad a z \quad \mathbf{/z} \end{array}$$

$$\begin{array}{l} (4) \quad a z^{-1} \quad \mathbf{/z} \\ (8) \quad a^{-1}z \quad \mathbf{/z} \end{array}$$

Geometric Series Combinations

complementary pair ordering

(1)

| | | |
|----------|---|----------------|
| unit | $\frac{1}{1-az}$ $ z < a^{-1}$ | $a^n u(n)$ |
| non-unit | $-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z > a^{-1}$ | $-a^n u(-n-1)$ |

(2)

| | | |
|----------|--|----------------------------|
| unit | $\frac{1}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^n u(n)$ |
| non-unit | $-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z > a$ | $-(\frac{1}{a})^n u(-n-1)$ |

(5)

(3)

| | | |
|----------|--|--------------|
| unit | $-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^n u(-n)$ |
| non-unit | $\frac{az}{1-az}$ $ z < a^{-1}$ | $a^n u(n-1)$ |

(6)

(4)

| | | |
|----------|----------------------------------|--------------------------|
| unit | $-\frac{1}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^n u(-n)$ |
| non-unit | $\frac{a^nz}{1-a^nz}$ $ z < a$ | $(\frac{1}{a})^n u(n-1)$ |

(7)

(8)

| | | Positive Exponent | Negative Exponent |
|--------------------|-----------|-------------------|-------------------|
| Complement Type I | $u(n)$ | (1) | (2) |
| | $u(-n-1)$ | (5) | (6) |
| Complement Type II | $u(-n)$ | (3) | (4) |
| | $u(n-1)$ | (7) | (8) |

Shifted Combinations (I) by scaling $*a$ $/a$

complementary
pair
ordering

(1) $*a$

| | |
|---|--------------------|
| $\frac{a}{1-az}$ $ z < a^{-1}$ | $a^{n+1} u(n)$ |
| $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n+1} u(-n-1)$ |

(2) $/a$

| | |
|---------------------------------------|--------------------------------|
| $\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n+1} u(n)$ |
| $-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |

(5) $*a$

(3) $/a$

| | |
|---|------------------|
| $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n-1} u(-n)$ |
| $\frac{z}{1-az}$ $ z < a^{-1}$ | $a^{n-1} u(n-1)$ |

(6) $/a$

(4) $*a$

| | |
|----------------------------------|------------------------------|
| $-\frac{a}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n-1} u(-n)$ |
| $\frac{z}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n-1} u(n-1)$ |

(7) $/a$

(8) $*a$

| | Positive Exponent | Negative Exponent | | |
|--------------------|-------------------|-------------------|----------------|----------------|
| Complement Type I | (1) $u(n)$ | (2) $u(n)$ | $*a$ $u(n)$ | $/a$ $u(n)$ |
| | (5) $u(-n-1)$ | (6) $u(-n-1)$ | $*a$ $u(-n-1)$ | $/a$ $u(-n-1)$ |
| Complement Type II | (3) $u(-n)$ | (4) $u(-n)$ | $/a$ $u(-n)$ | $*a$ $u(-n)$ |
| | (7) $u(n-1)$ | (8) $u(n-1)$ | $/a$ $u(n-1)$ | $*a$ $u(n-1)$ |

Shifted Combinations (II) by scaling $*z$ $/z$

complementary
pair
ordering

(1) $*z$

| | |
|--|------------------|
| $\frac{z}{1-az}$ $ z < a^{-1}$ | $a^{n-1} u(n-1)$ |
| $-\frac{a}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n-1} u(-n)$ |

(2) $*z$

| | |
|----------------------------------|------------------------------|
| $\frac{z}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n-1} u(n-1)$ |
| $-\frac{a}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n-1} u(-n)$ |

(5) $*z$

(3) $/z$

| | |
|---|--------------------|
| $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$ | $-a^{n+1} u(-n-1)$ |
| $\frac{a}{1-az}$ $ z < a^{-1}$ | $a^{n+1} u(n)$ |

(7) $/z$

(6) $*z$

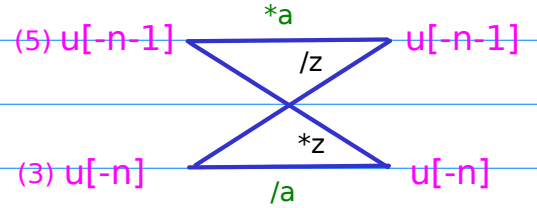
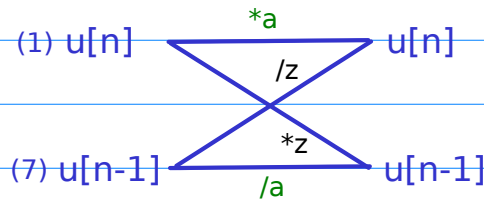
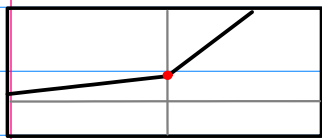
(4) $/z$

| | |
|---------------------------------------|--------------------------------|
| $-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |
| $\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$ | $(\frac{1}{a})^{n+1} u(n)$ |

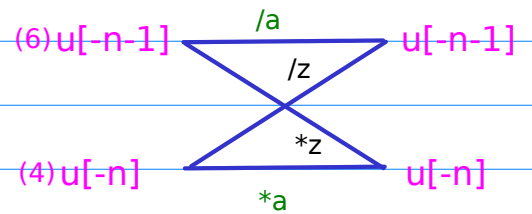
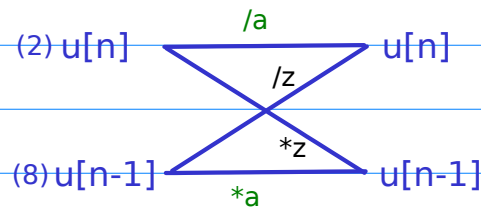
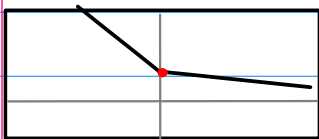
(8) $/z$

| | Positive Exponent | Negative Exponent | | |
|--------------------|-------------------|-------------------|----------------|----------------|
| Complement Type I | (1) $u(n)$ | (2) $u(n)$ | $*z$ $u(n-1)$ | $*z$ $u(n-1)$ |
| | (5) $u(-n-1)$ | (6) $u(-n-1)$ | $*z$ $u(-n)$ | $*z$ $u(-n)$ |
| Complement Type II | (3) $u(-n)$ | (4) $u(-n)$ | $/z$ $u(-n-1)$ | $/z$ $u(-n-1)$ |
| | (7) $u(n-1)$ | (8) $u(n-1)$ | $/z$ $u(n)$ | $/z$ $u(n)$ |

a^n



a^{-n}



| | | | | |
|--------------------|-----------|-----|-----|--------------------------------|
| Causal | $u(n)$ | (1) | (2) | butterfly pair ordering |
| | $u(n-1)$ | (7) | (8) | |
| Anti-Causal | $u(-n-1)$ | (5) | (6) | |
| | $u(-n)$ | (3) | (4) | |

| Positive Exponent | Negative Exponent |
|-------------------|-------------------|
| (1) | (2) |
| (3) | (4) |
| (5) | (6) |
| (7) | (8) |

row major ordering

$$(1) a^n u(n) \xrightarrow[*z]{*a} a^{n+1} u(n)$$

$$(7) a^n u(n-1) \xrightarrow[*a]{/z} a^{n-1} u(n-1)$$

$$(5) a^n u(-n-1) \xrightarrow[*a]{/z} a^{n+1} u(-n-1)$$

$$(3) a^n u(-n) \xrightarrow[*z]{/a} a^{n-1} u(-n)$$

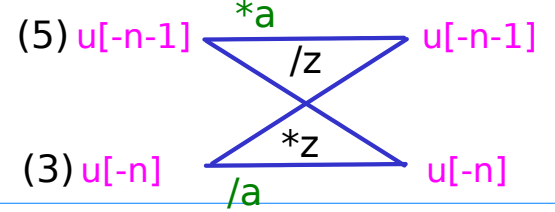
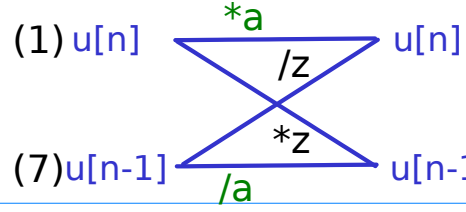
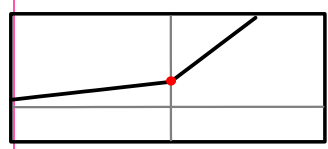
$$(2) a^{-n} u(n) \xrightarrow{/a} a^{-n-1} u(n)$$

$$(8) a^{-n} u(n-1) \xrightarrow[*a]{*z} a^{-n+1} u(n-1)$$

$$(6) a^{-n} u(-n-1) \xrightarrow{/a} a^{-n-1} u(-n-1)$$

$$(4) a^{-n} u(-n) \xrightarrow[*z]{*a} a^{-n+1} u(-n)$$

a^n



(1) $*a$

| | |
|--------------------------------------|-------------------------------------|
| $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ |
| $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ |

(7) $/a$

(1) $*a$

| | |
|--|--|
| $a^n u(n)$ (a^0, a^1, a^2, \dots) | $a^{n+1} u(n)$ (a^1, a^2, a^3, \dots) |
| $a^n u(n-1)$ (a^1, a^2, a^3, \dots) | $a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots) |

(7) $/a$

(1) $*z$

| | |
|--------------------------------------|-------------------------------------|
| $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ |
| $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ |

(7) $/z$

(1) $*z$

| | |
|--|--|
| $a^n u(n)$ (a^0, a^1, a^2, \dots) | $a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots) |
| $a^n u(n-1)$ (a^1, a^2, a^3, \dots) | $a^{n+1} u(n)$ (a^1, a^2, a^3, \dots) |

(7) $/z$

(5) $*a$

| | |
|---|---|
| $-\frac{a'z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ |
| $-\frac{1}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{a'}{1-a'z^{-1}} \quad z > a^{-1}$ |

(3) $/a$

(5) $*a$

| | |
|---|---|
| $-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ | $-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |
| $-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ |

(3) $/a$

(5) $*z$

| | |
|---|---|
| $-\frac{a'z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{a'}{1-a'z^{-1}} \quad z > a^{-1}$ |
| $-\frac{1}{1-a'z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a'z^{-1}} \quad z > a^{-1}$ |

(3) $/z$

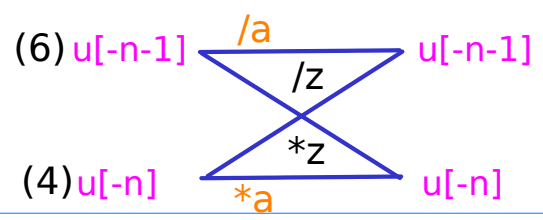
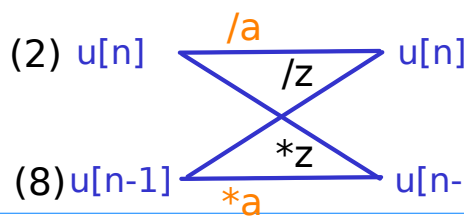
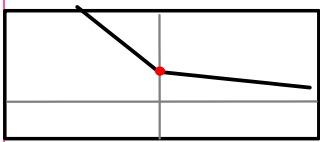
(5) $*z$

| | |
|---|---|
| $-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ | $-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ |
| $-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |

(3) $/z$

butterfly pair ordering

a^{-n}



(2) $/a$

| | |
|---|--|
| $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |
| $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |

(2) $/a$

| | |
|--|--|
| $(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ |

(8) $*a$

(8) $*a$

(2) $*z$

| | |
|---|--|
| $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |
| $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |

(2) $*z$

| | |
|--|--|
| $(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^3}, \dots)$ |
| $(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |

(8) $/z$

(8) $/z$

(6) $/a$

| | |
|--|---|
| $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |
| $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |

(6) $/a$

| | |
|---|--|
| $-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$ |
| $-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$ | $-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^1, a^0, a^{-1})$ |

(4) $*a$

(4) $*a$

(6) $*z$

| | |
|--|---|
| $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |
| $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |

(6) $*z$

| | |
|---|---|
| $-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$ | $-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$ |
| $-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$ |

(4) $/z$

(4) $/z$

butterfly pair ordering

Scale by **a**

1. Geometric Series

complementary
pair
ordering

(1)

***a**

(2)

/a

| | | | | |
|----------|---|---|--|---|
| | $\frac{1}{1-az} \quad z < a^{-1}$ | $\frac{a}{1-az} \quad z < a^{-1}$ | $\frac{1}{1-a^{-1}z} \quad z < a$ | $\frac{a^{-1}}{1-a^{-1}z} \quad z < a$ |
| Comp.ROC | $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{az^{-1}}{1-az^{-1}} \quad z > a$ | $-\frac{z^{-1}}{1-az^{-1}} \quad z > a$ |

(5)

***a**

(6)

/a

(3)

/a

(4)

***a**

| | | | | |
|----------|--|---|---|--------------------------------------|
| | $-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$ | $-\frac{1}{1-az^{-1}} \quad z > a$ | $-\frac{a}{1-az^{-1}} \quad z > a$ |
| Comp.ROC | $\frac{az}{1-az} \quad z < a^{-1}$ | $\frac{z}{1-az} \quad z < a^{-1}$ | $\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$ | $\frac{z}{1-a^{-1}z} \quad z < a$ |

(7)

/a

(8)

***a**

| | | | | | | | |
|-----|---------------|---|-------------------|-----|------------------|---|--------------------|
| (1) | $a^n u(n)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n+1} u(n)$ | (2) | $a^{-n} u(n)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n-1} u(n)$ |
| (7) | $a^n u(n-1)$ | $\begin{array}{c} *z \\ /a \end{array}$ | $a^{n-1} u(n-1)$ | (8) | $a^{-n} u(n-1)$ | $\begin{array}{c} *a \end{array}$ | $a^{-n+1} u(n-1)$ |
| (5) | $a^n u(-n-1)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n+1} u(-n-1)$ | (6) | $a^{-n} u(-n-1)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n-1} u(-n-1)$ |
| (3) | $a^n u(-n)$ | $\begin{array}{c} *z \\ /a \end{array}$ | $a^{n-1} u(-n)$ | (4) | $a^{-n} u(-n)$ | $\begin{array}{c} *a \end{array}$ | $a^{-n+1} u(-n)$ |

Scale by **a**

2. Sequences

complementary
pair
ordering

(1)

***a**

| |
|-----------------|
| $a^n u(n)$ |
| $- a^n u(-n-1)$ |

| |
|---------------------|
| $a^{n+1} u(n)$ |
| $- a^{n+1} u(-n-1)$ |

(2)

/a

| |
|----------------------------|
| $(\frac{1}{a})^n u(n)$ |
| $-(\frac{1}{a})^n u(-n-1)$ |

| |
|--------------------------------|
| $(\frac{1}{a})^{n+1} u(n)$ |
| $-(\frac{1}{a})^{n+1} u(-n-1)$ |

Comp.ROC

(5)

***a**

| |
|---------------|
| $- a^n u(-n)$ |
| $a^n u(n-1)$ |

| |
|-------------------|
| $- a^{n-1} u(-n)$ |
| $a^{n-1} u(n-1)$ |

(6)

/a

| |
|--------------------------|
| $-(\frac{1}{a})^n u(-n)$ |
| $(\frac{1}{a})^n u(n-1)$ |

| |
|------------------------------|
| $-(\frac{1}{a})^{n-1} u(-n)$ |
| $(\frac{1}{a})^{n-1} u(n-1)$ |

Comp.ROC

(3)

/a

(4)

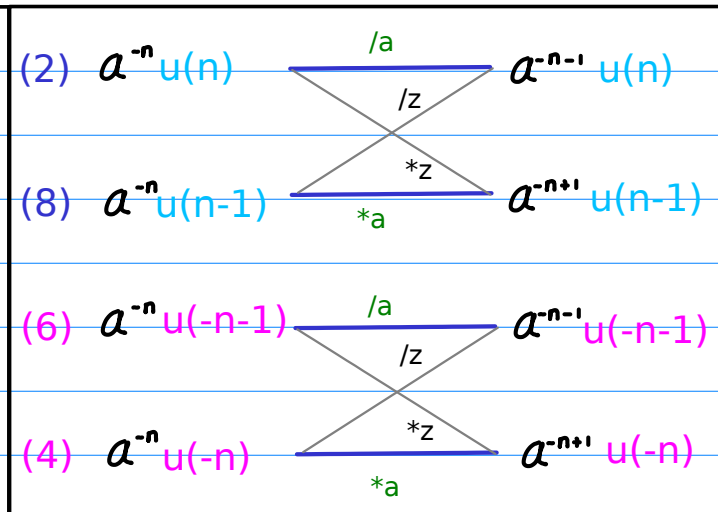
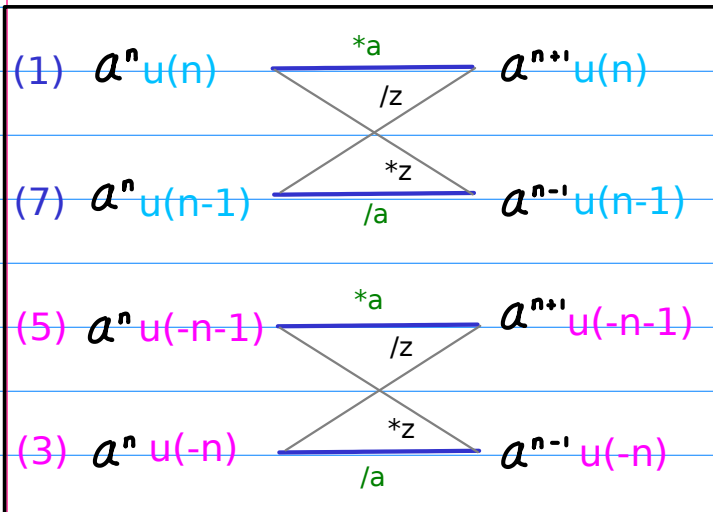
***a**

(7)

/a

(8)

***a**



Scale by **a**

3. Sequence values

complementary
pair
ordering

(1)

***a**

| | |
|---|---|
| (a^0, a^1, a^2, \dots) | (a^1, a^2, a^3, \dots) |
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ |

(2)

/a

| | |
|--|--|
| $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $-(\dots, a^3, a^2, a^1)$ | $-(\dots, a^2, a^1, a^0)$ |

Comp.ROC

(5)

***a**

| | |
|---|---|
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$ | $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$ |
| (a^1, a^2, a^3, \dots) | (a^0, a^1, a^2, \dots) |

(6)

/a

| | |
|--|--|
| $-(\dots, a^2, a^1, a^0)$ | $-(\dots, a^3, a^2, a^1)$ |
| $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ |

Comp.ROC

(3)

/a

(7)

/a

(4)

***a**

(8)

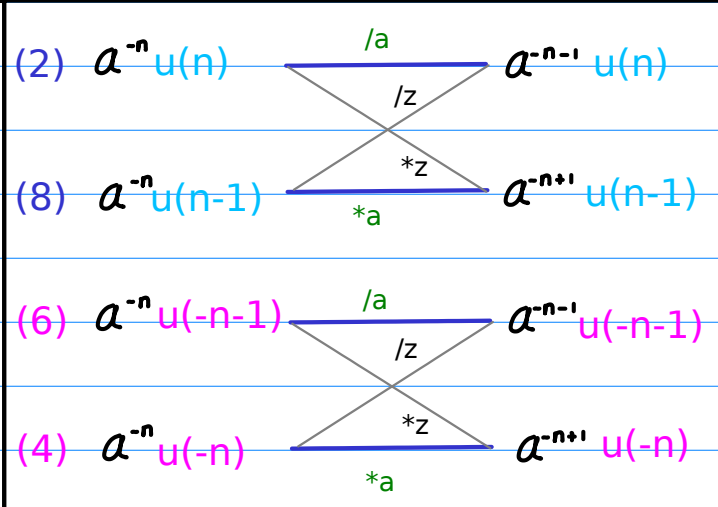
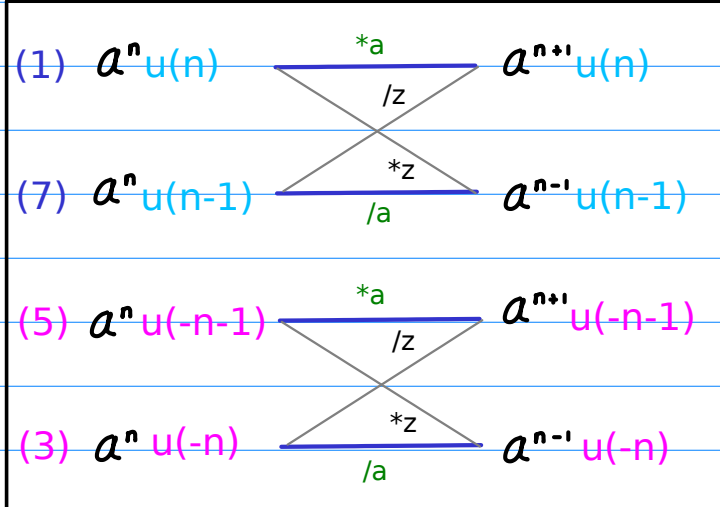
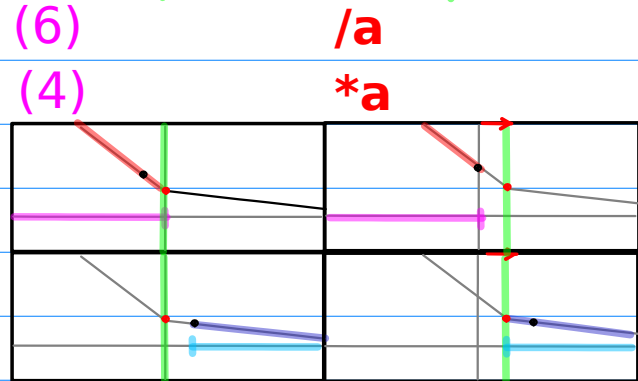
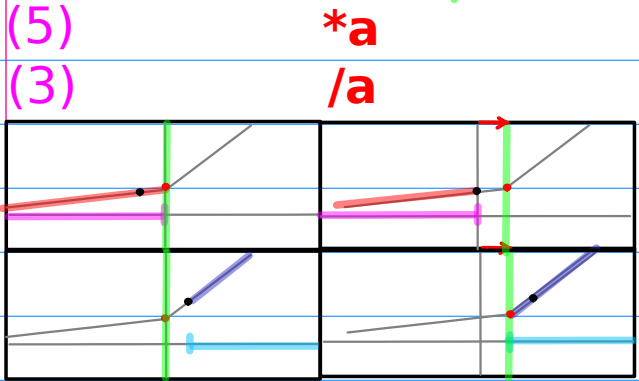
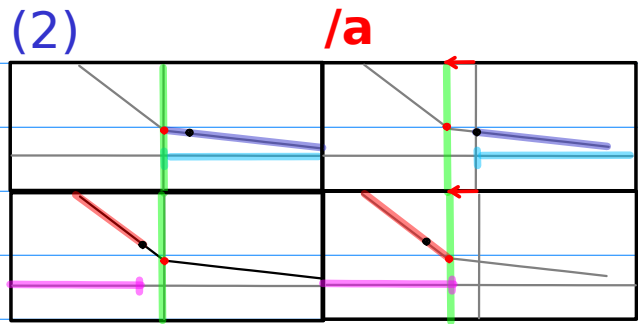
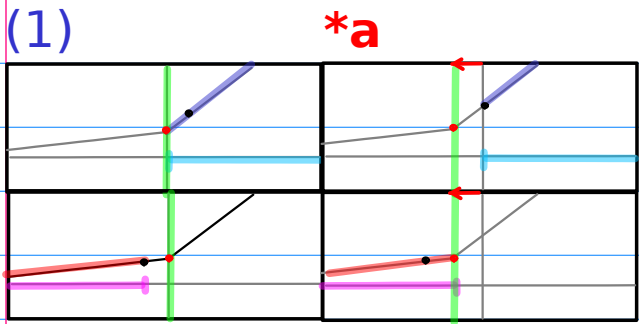
***a**

| | | | | | |
|-------------------|---|-------------------|----------------------|---|--------------------|
| (1) $a^n u(n)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n+1} u(n)$ | (2) $a^{-n} u(n)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n-1} u(n)$ |
| (7) $a^n u(n-1)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n-1} u(n-1)$ | (8) $a^{-n} u(n-1)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n+1} u(n-1)$ |
| (5) $a^n u(-n-1)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n+1} u(-n-1)$ | (6) $a^{-n} u(-n-1)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n-1} u(-n-1)$ |
| (3) $a^n u(-n)$ | $\begin{array}{c} *a \\ /z \\ *z \\ /a \end{array}$ | $a^{n-1} u(-n)$ | (4) $a^{-n} u(-n)$ | $\begin{array}{c} /a \\ /z \\ *z \\ *a \end{array}$ | $a^{-n+1} u(-n)$ |

Scale by **a**

4. Graphs

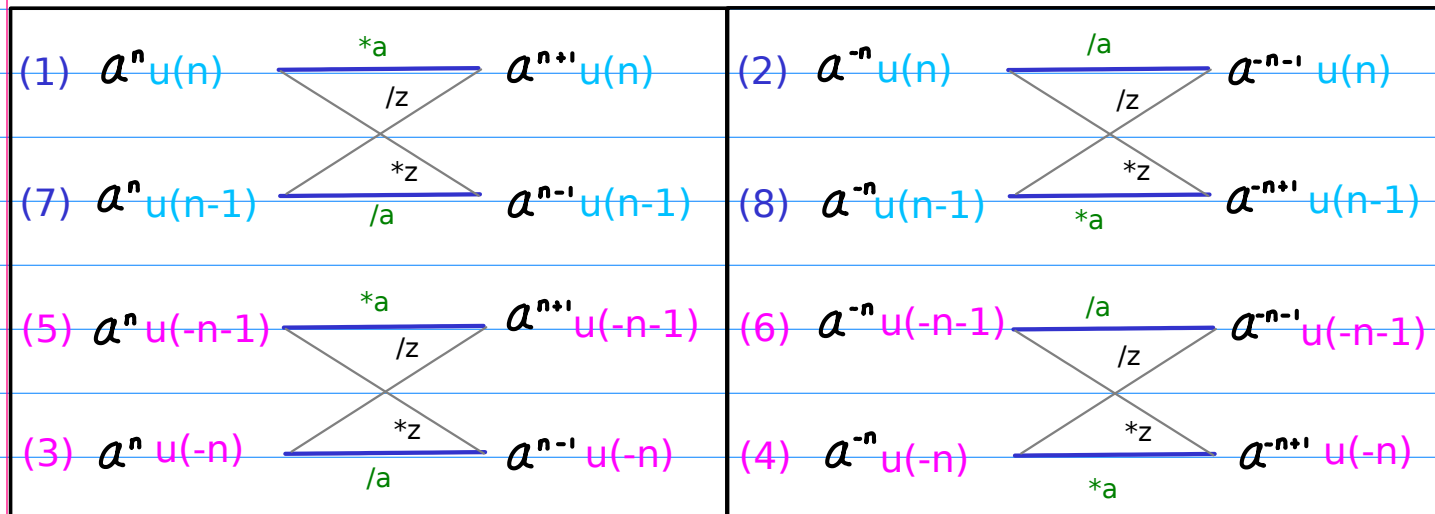
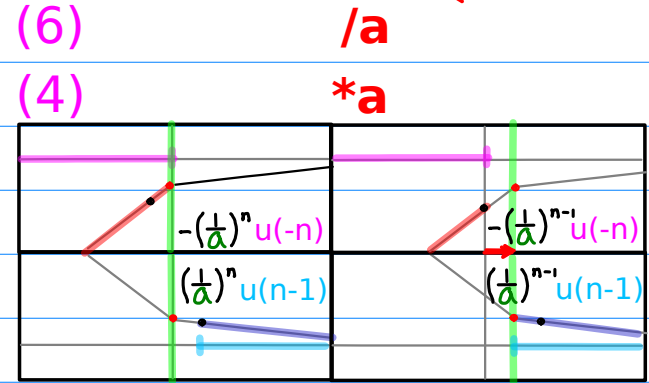
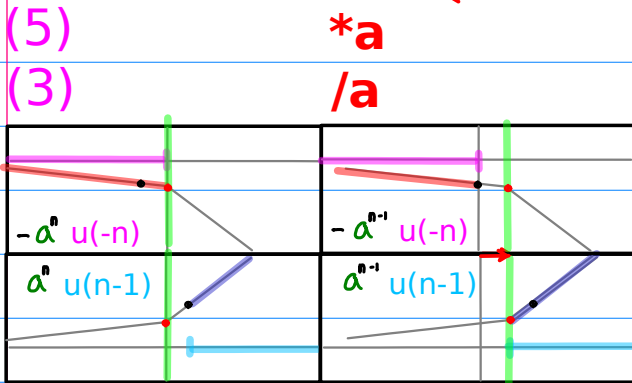
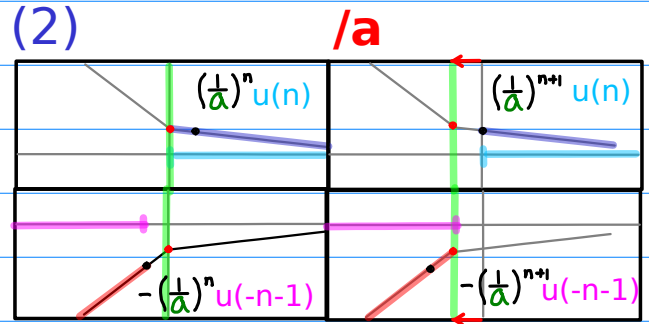
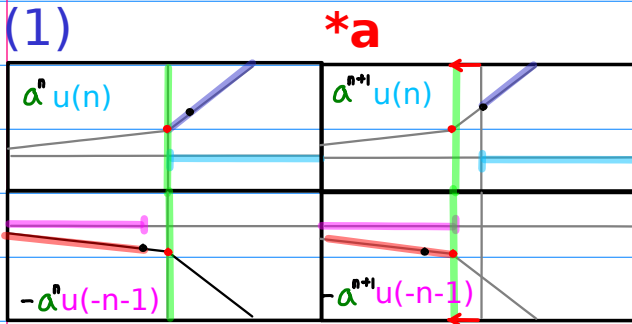
complementary
pair
ordering



Scale by **a**

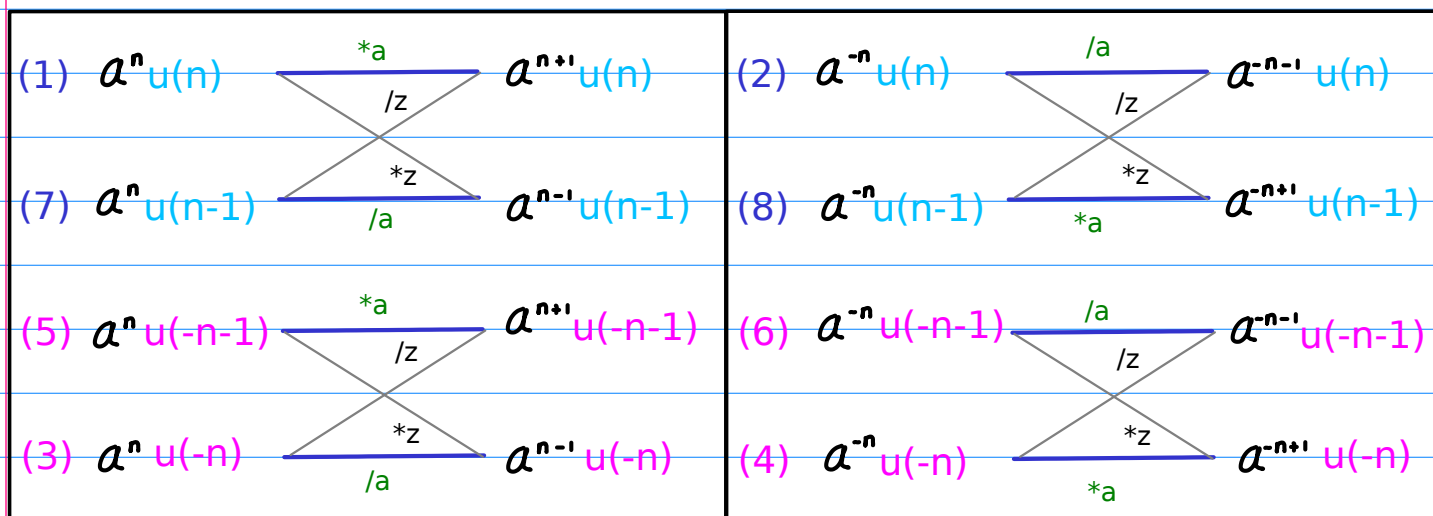
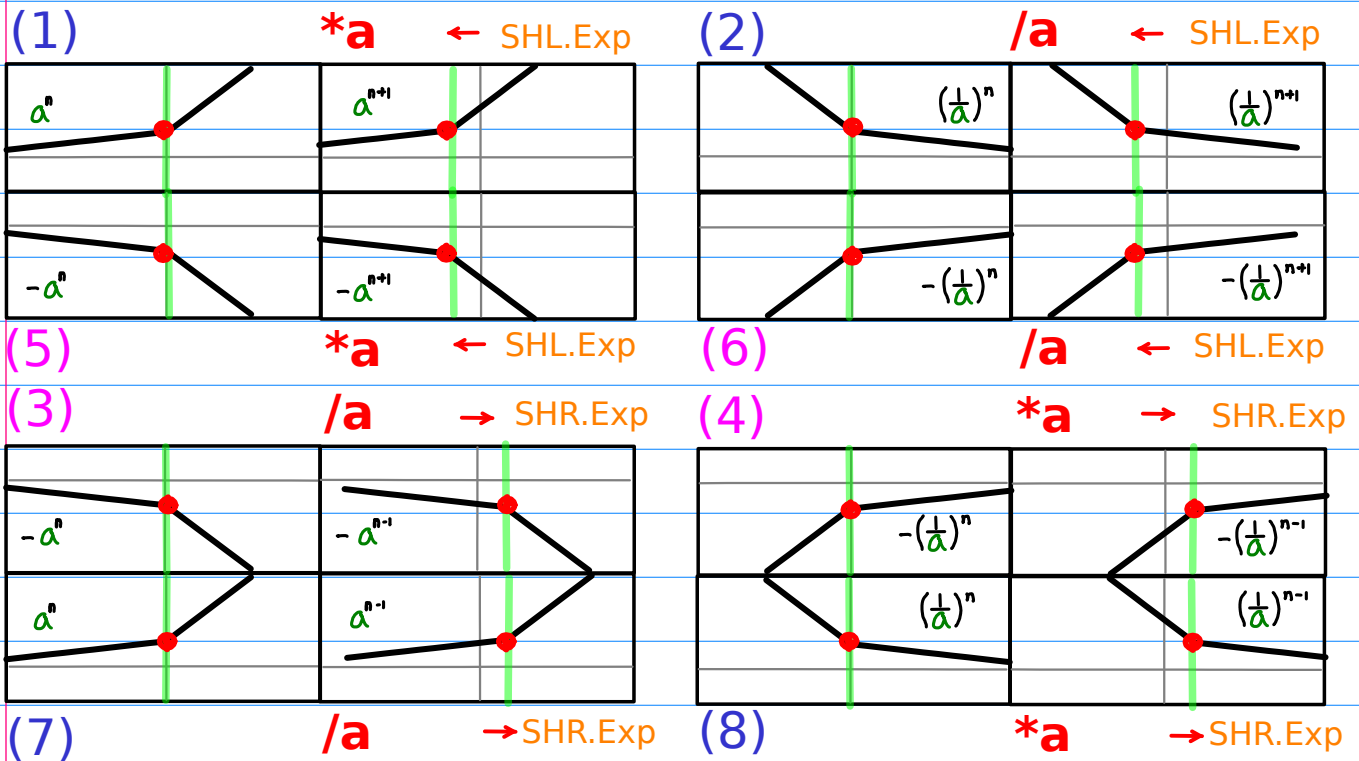
5. Graphs - signs

complementary
pair
ordering



Scale by **a**

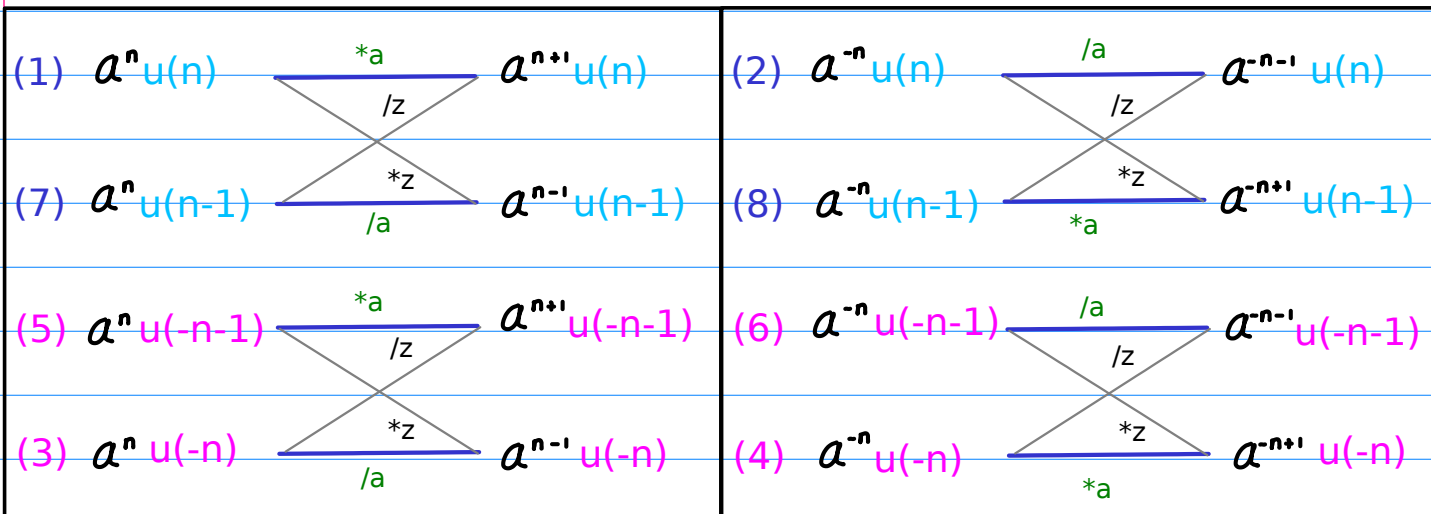
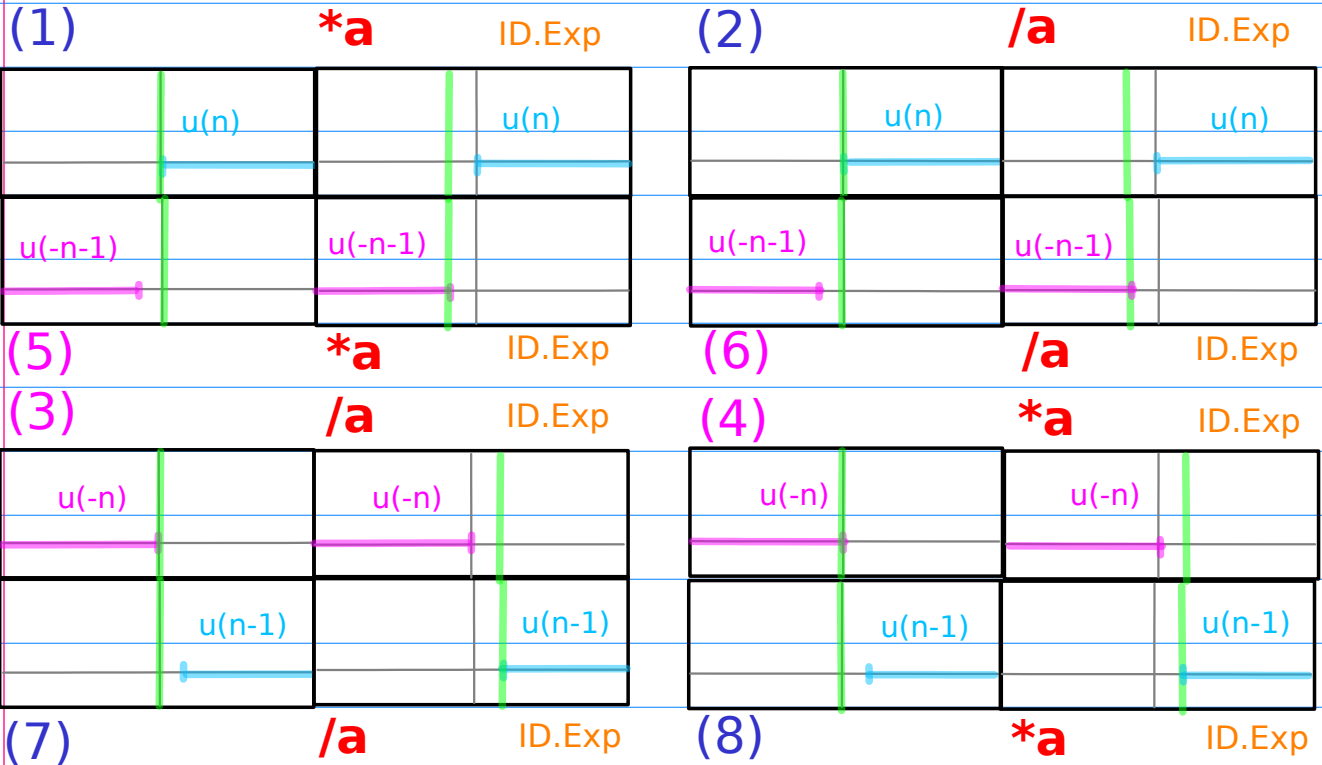
6. Graphs - Exponents



Scale by **a**

7. Graphs - Ranges

complementary
pair
ordering



Scale by z

1. Geometric Series

(1)

$*z$

$$\frac{1}{1-az} \quad |z| < a^{-1}$$

$$\frac{z}{1-az} \quad |z| < a^{-1}$$

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

Comp.ROC

(2)

$*z$

$$\frac{1}{1-a^{-1}z} \quad |z| < a$$

$$\frac{z}{1-a^{-1}z} \quad |z| < a$$

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a$$

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a$$

(5)

$*z$

(3)

$/z$

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$$\frac{az}{1-az} \quad |z| < a^{-1}$$

$$\frac{a}{1-az} \quad |z| < a^{-1}$$

Comp.ROC

(6)

$*z$

(4)

$/z$

$$-\frac{1}{1-az^{-1}} \quad |z| > a$$

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a$$

$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$$

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a$$

(7)

$/z$

(8)

$/z$

(1) $a^n u(n) \xrightarrow[*a]{/z} a^{n+1} u(n)$

(2) $a^{-n} u(n) \xrightarrow{/a}{*z} a^{-n-1} u(n)$

(7) $a^n u(n-1) \xrightarrow[*z]{/a} a^{n-1} u(n-1)$

(8) $a^{-n} u(n-1) \xrightarrow[*a]{/z} a^{-n+1} u(n-1)$

(5) $a^n u(-n-1) \xrightarrow[*a]{/z} a^{n+1} u(-n-1)$

(6) $a^{-n} u(-n-1) \xrightarrow{/a}{*z} a^{-n-1} u(-n-1)$

(3) $a^n u(-n) \xrightarrow[*z]{/a} a^{n-1} u(-n)$

(4) $a^{-n} u(-n) \xrightarrow[*a]{/z} a^{-n+1} u(-n)$

Scale by z

2. Sequences

complementary pair ordering

(1)

$*z$

| | |
|----------------|------------------|
| $a^n u(n)$ | $a^{n-1} u(n-1)$ |
| $-a^n u(-n-1)$ | $-a^{n-1} u(-n)$ |

(2)

$*z$

| | |
|----------------------------|------------------------------|
| $(\frac{1}{a})^n u(n)$ | $(\frac{1}{a})^{n-1} u(n-1)$ |
| $-(\frac{1}{a})^n u(-n-1)$ | $-(\frac{1}{a})^{n-1} u(-n)$ |

Comp.ROC

(5)

$*z$

| | |
|--------------|--------------------|
| $-a^n u(-n)$ | $-a^{n+1} u(-n-1)$ |
| $a^n u(n-1)$ | $a^{n+1} u(n)$ |

(6)

$*z$

| | |
|--------------------------|--------------------------------|
| $-(\frac{1}{a})^n u(-n)$ | $-(\frac{1}{a})^{n+1} u(-n-1)$ |
| $(\frac{1}{a})^n u(n-1)$ | $(\frac{1}{a})^{n+1} u(n)$ |

Comp.ROC

(3)

$/z$

(7)

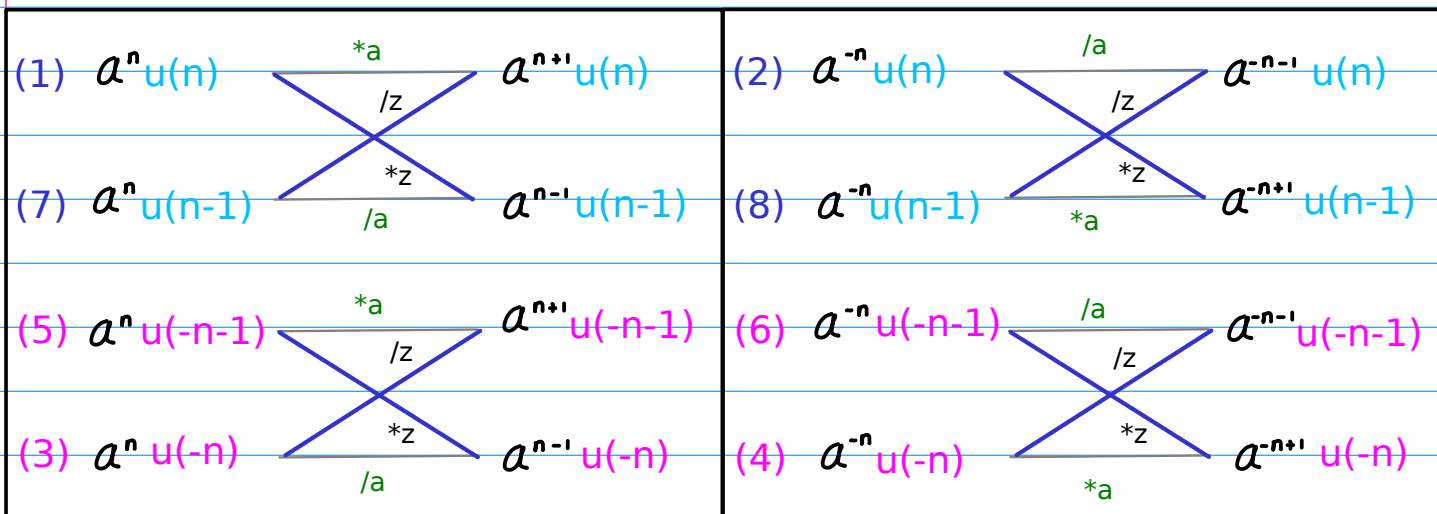
$/z$

(4)

$/z$

(8)

$/z$



Scale by z

3. Sequence values

complementary pair ordering

(1)

$*z$

| | |
|---|---|
| (a^0, a^1, a^2, \dots) | (a^0, a^1, a^2, \dots) |
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$ |

(2)

$*z$

| | |
|--|--|
| $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |
| $-(\dots, a^3, a^2, a^1)$ | $-(\dots, a^3, a^2, a^1)$ |

Comp.ROC

(5)

$*z$

| | |
|---|---|
| $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ | $-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$ |
| (a^1, a^2, a^3, \dots) | (a^1, a^2, a^3, \dots) |

(6)

$*z$

| | |
|--|--|
| $-(\dots, a^2, a^1, a^0)$ | $-(\dots, a^2, a^1, a^0)$ |
| $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ | $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$ |

Comp.ROC

(3)

$/z$

(7)

$/z$

(4)

$/z$

(8)

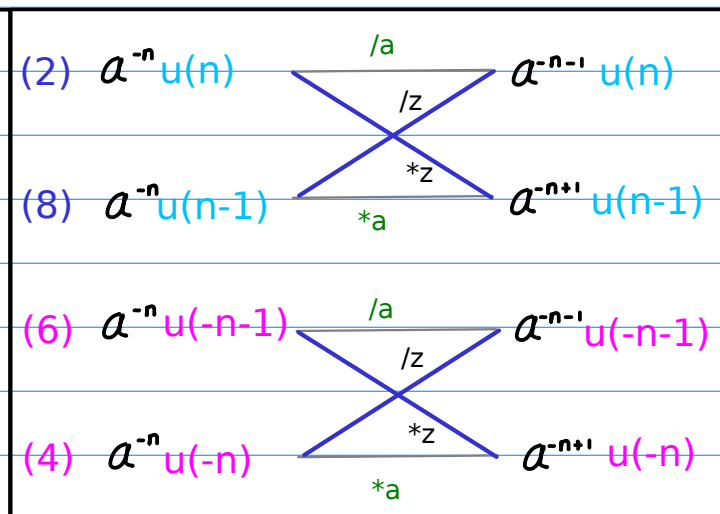
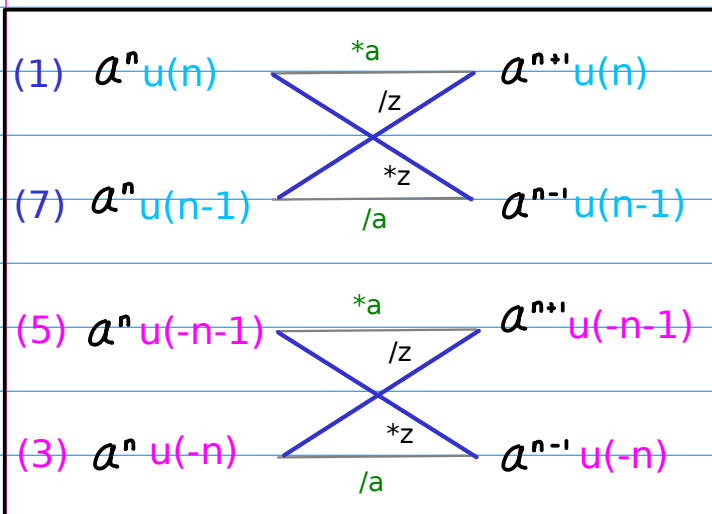
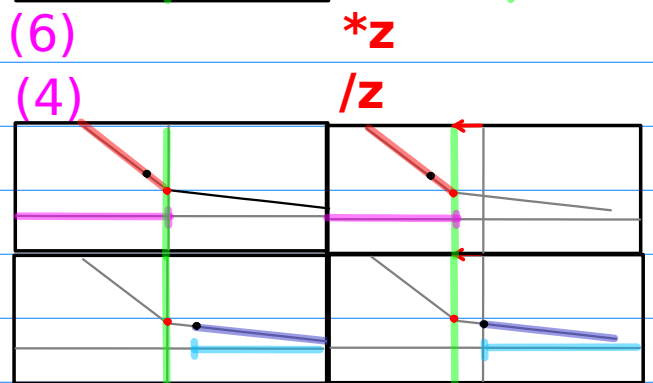
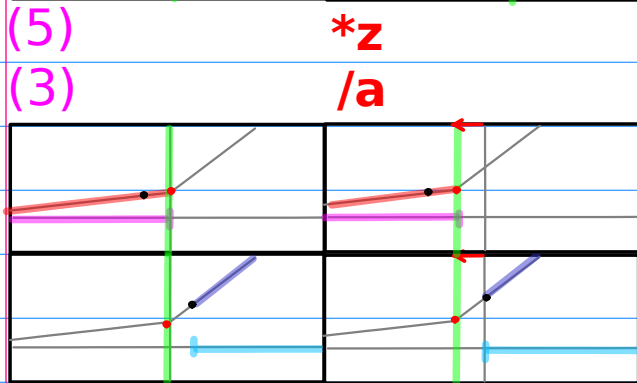
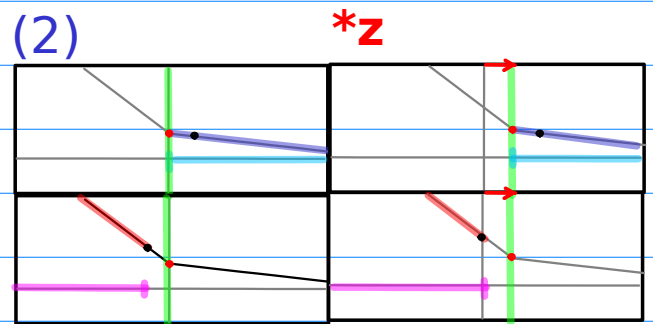
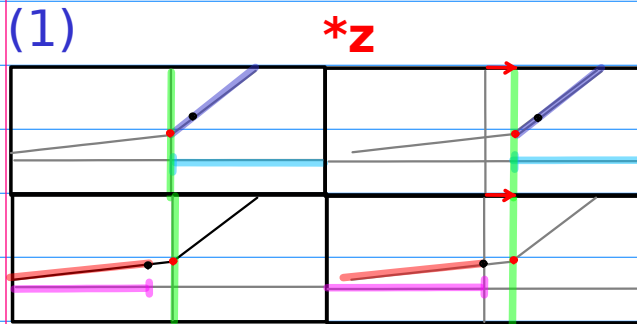
$/z$

| | | | | | |
|-------------------|------|-------------------|----------------------|------|--------------------|
| (1) $a^n u(n)$ | $*a$ | $a^{n+1} u(n)$ | (2) $a^{-n} u(n)$ | $/a$ | $a^{-n-1} u(n)$ |
| (7) $a^n u(n-1)$ | $/z$ | $a^{n-1} u(n-1)$ | (8) $a^{-n} u(n-1)$ | $/z$ | $a^{-n+1} u(n-1)$ |
| (5) $a^n u(-n-1)$ | $*z$ | $a^{n+1} u(-n-1)$ | (6) $a^{-n} u(-n-1)$ | $*a$ | $a^{-n-1} u(-n-1)$ |
| (3) $a^n u(-n)$ | $/a$ | $a^{n-1} u(-n)$ | (4) $a^{-n} u(-n)$ | $/z$ | $a^{-n+1} u(-n)$ |

Scale by z

4. Graphs

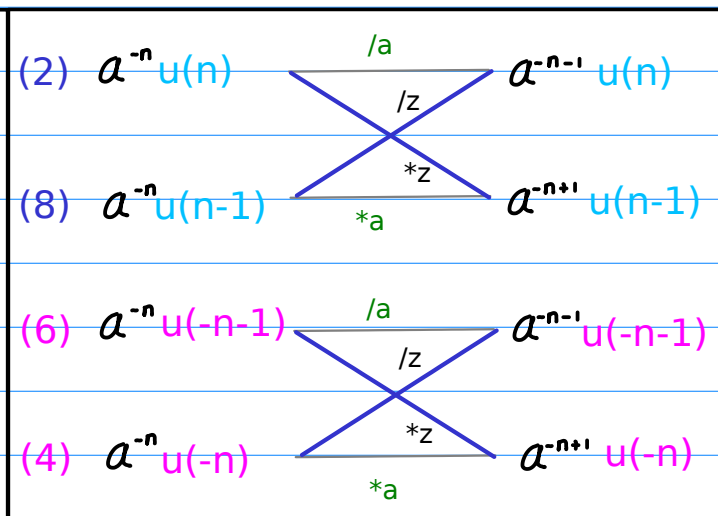
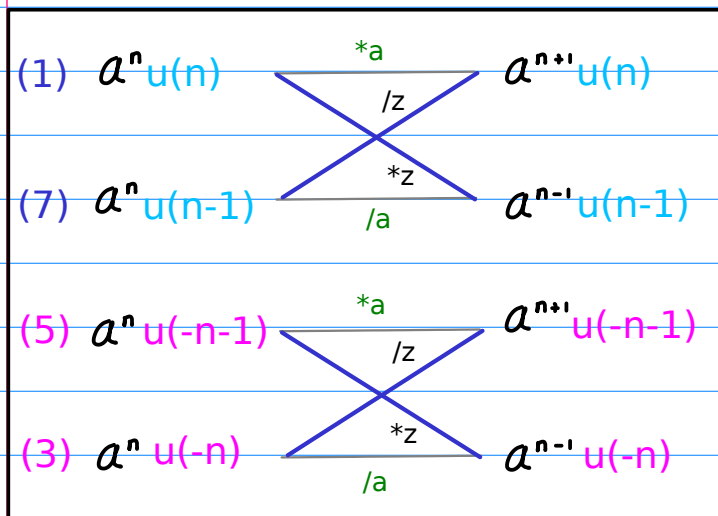
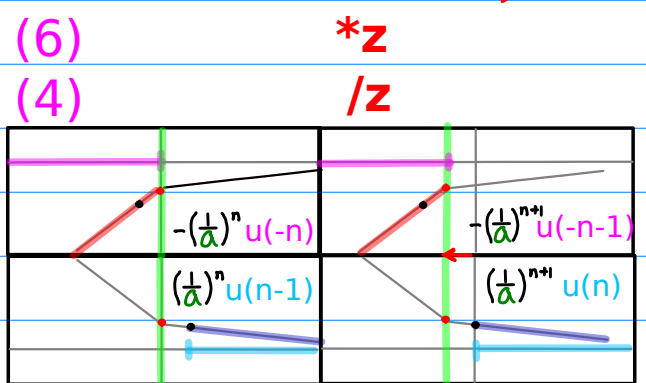
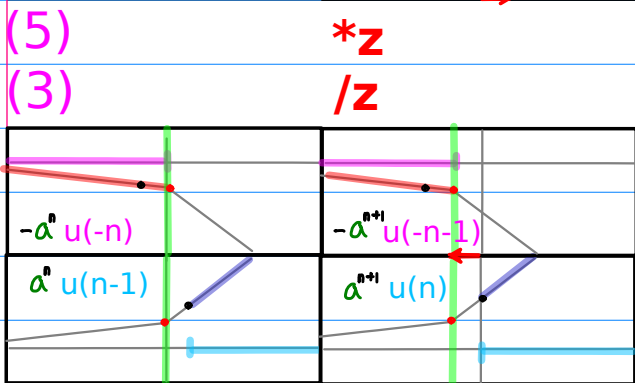
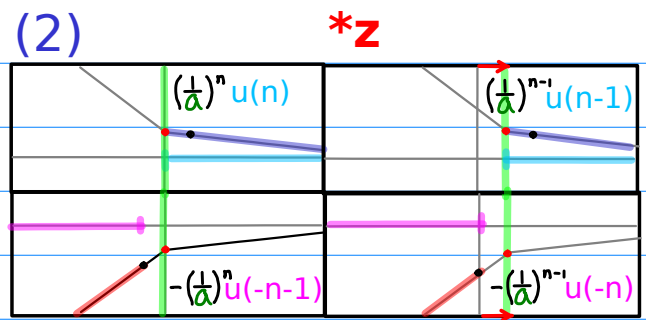
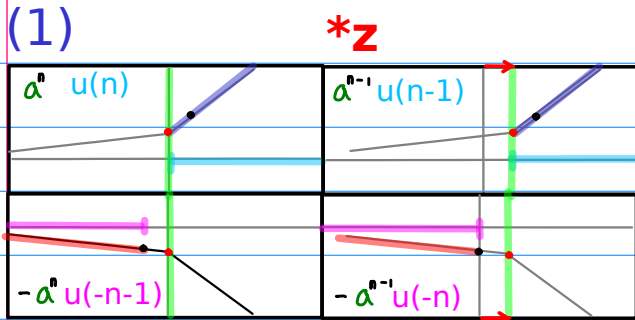
complementary pair ordering



Scale by z

5. Graphs - signs

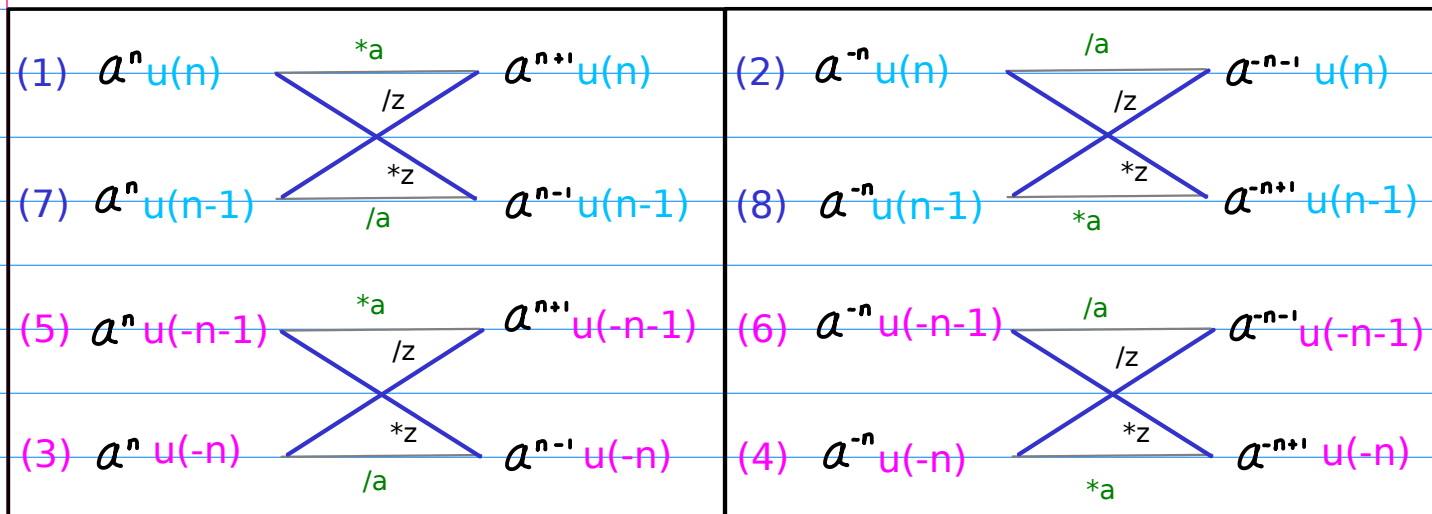
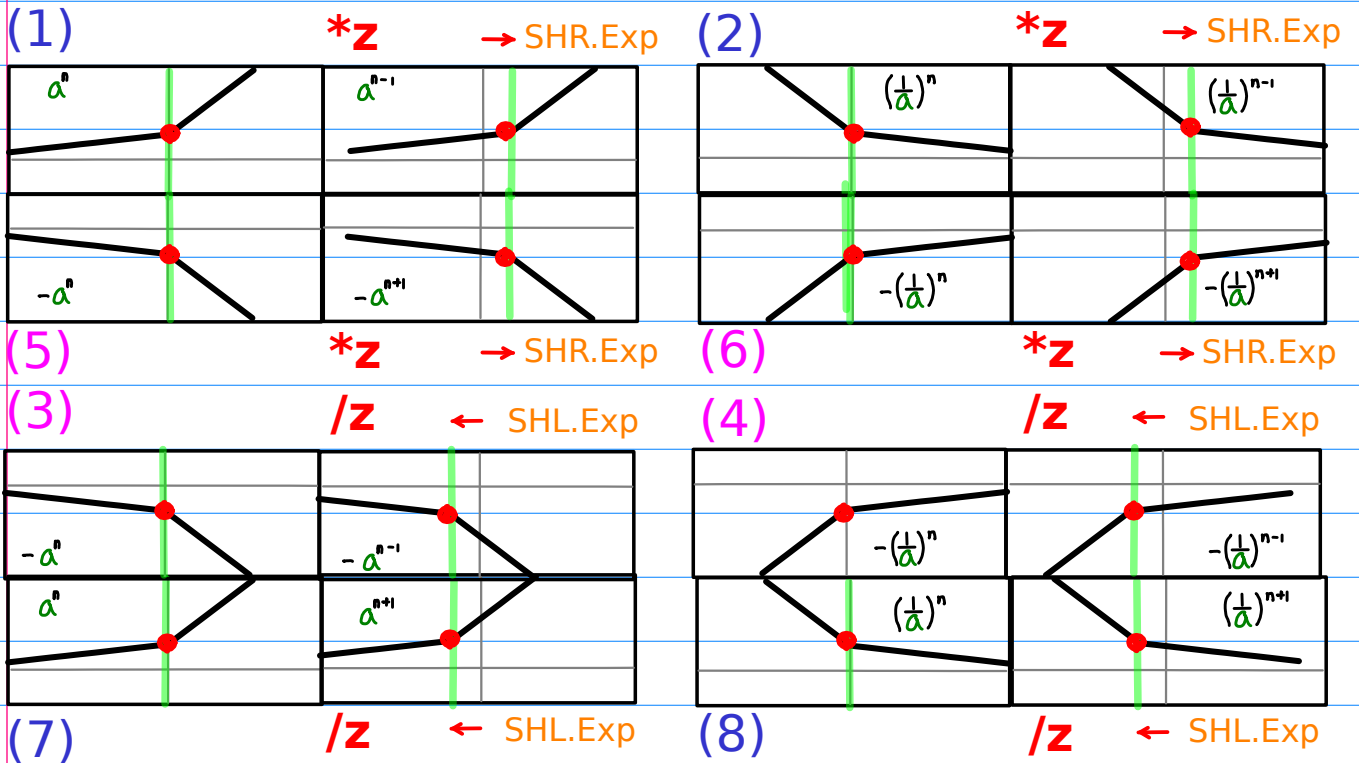
complementary pair ordering



Scale by z

6. Graphs - Exponents

complementary pair ordering

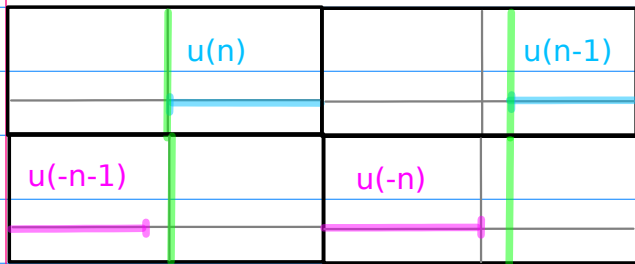


Scale by z

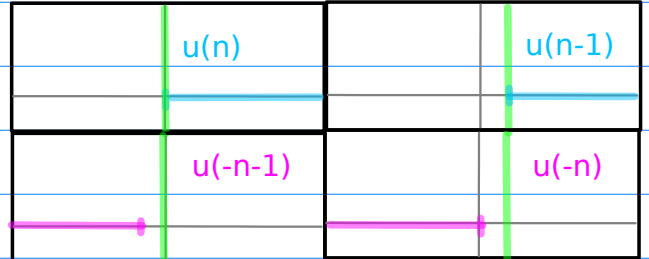
7. Graphs - Ranges

complementary pair ordering

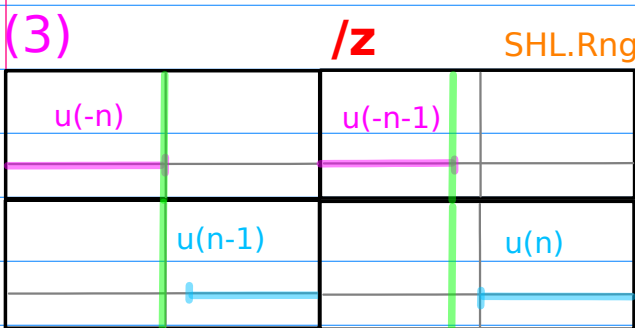
(1) $*z$ SHR.Rng



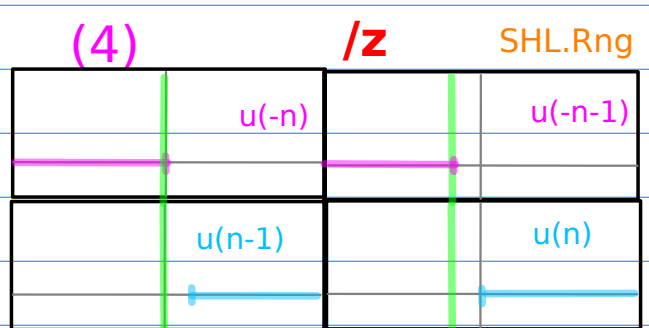
(2) $*z$ SHR.Rng



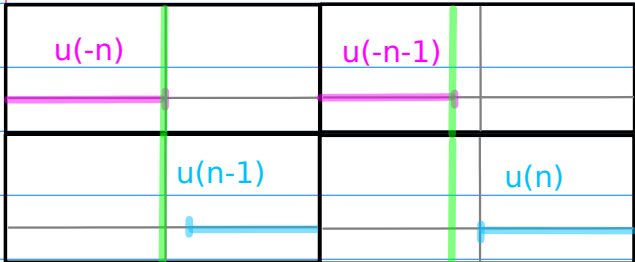
(5) $*z$ SHR.Rng



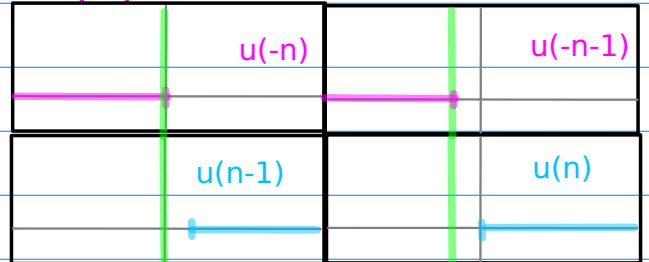
(6) $*z$ SHR.Rng



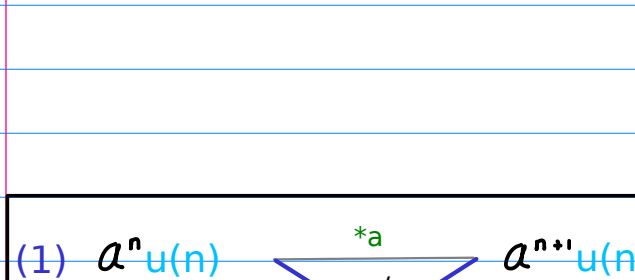
(3) $/z$ SHL.Rng



(4) $/z$ SHL.Rng



(7) $/z$ SHL.Rng



(8) $/z$ SHL.Rng

