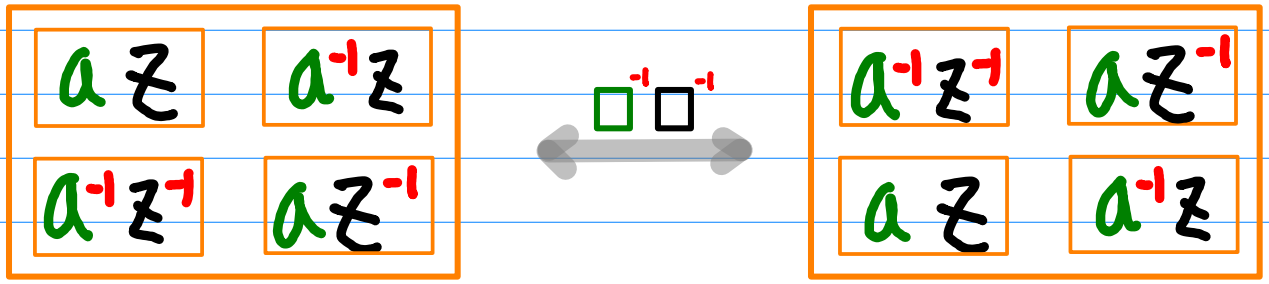
Laurent Series and z-Transform - Geometric Series Applications

A

20220808 Mon

Common ratios in geometric series

Assume $a \geq 1$



considered geometric series forms

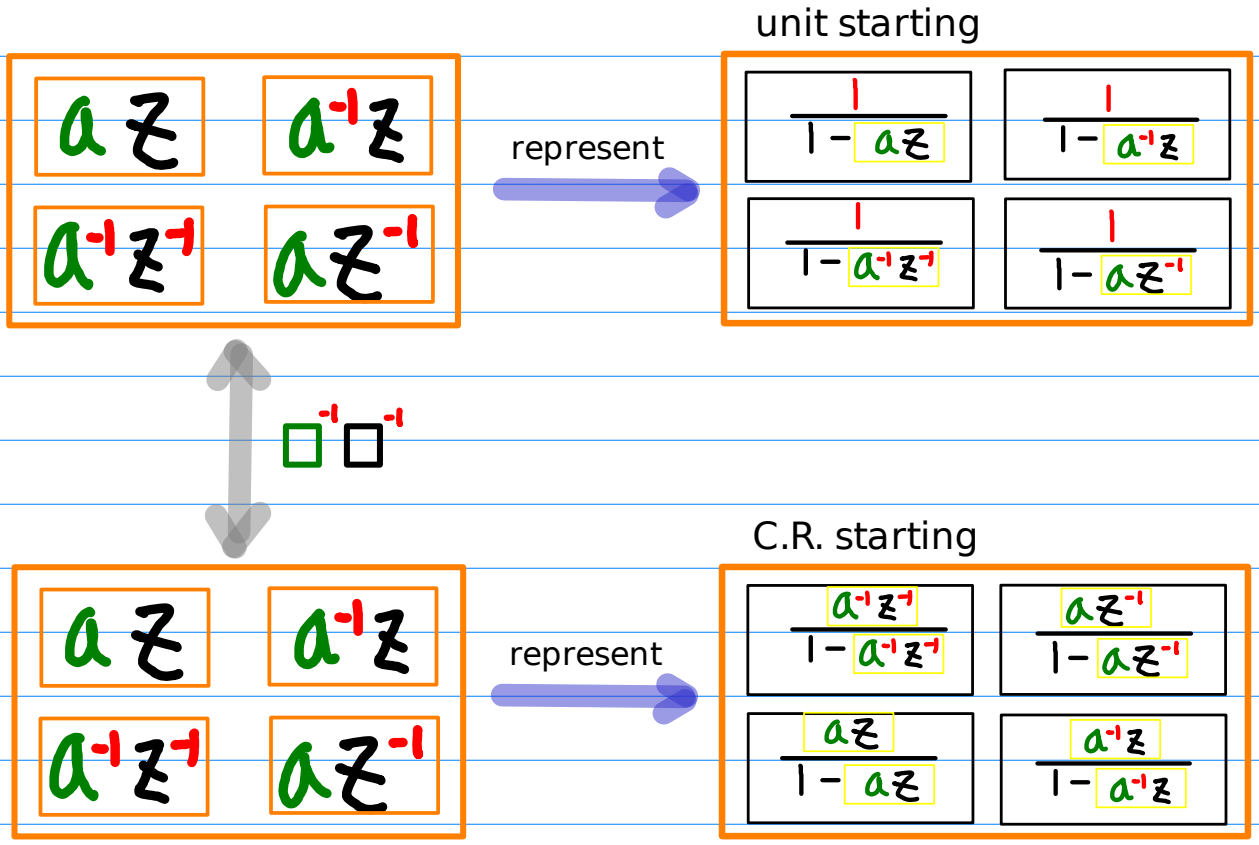
unit starting

$$\frac{1}{1-\text{C.R.}}$$

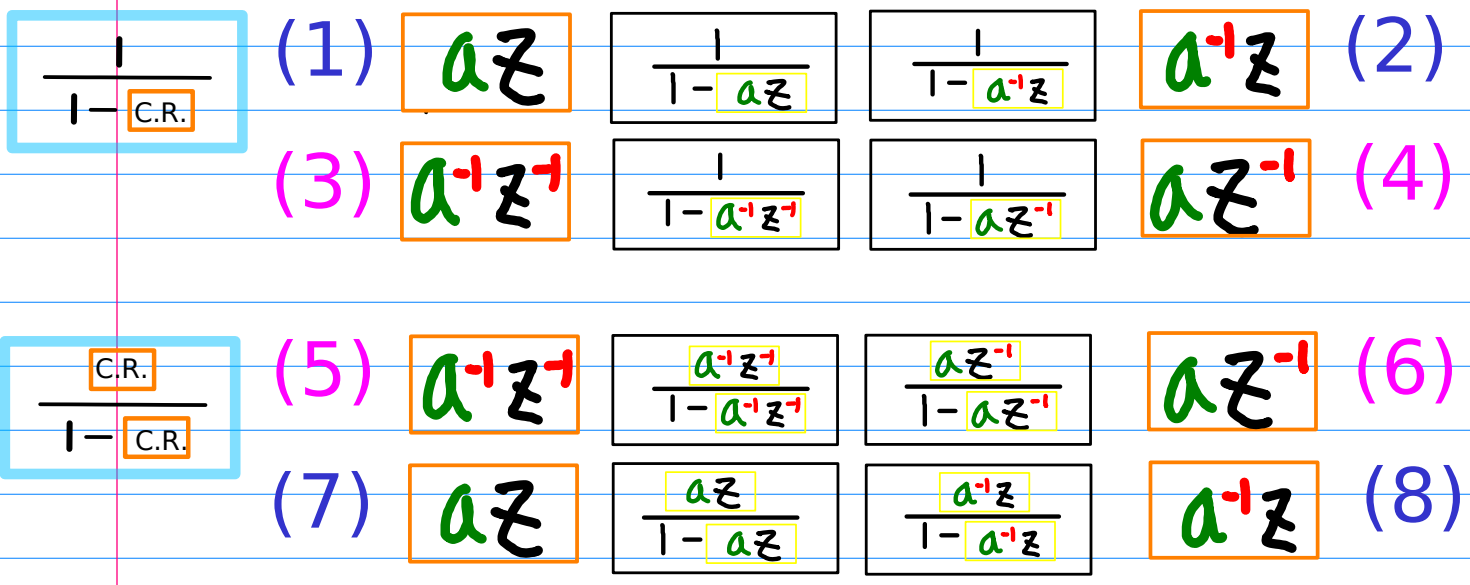
C.R. starting

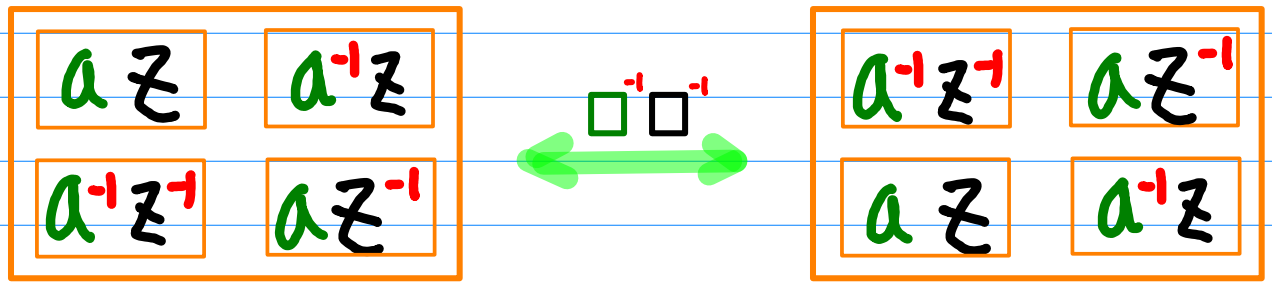
$$\frac{\text{C.R.}}{1-\text{C.R.}}$$

Representing geometric series



Numbering combinations





unit starting

(1), (2)

(3), (4)

C.R. starting

(5), (6)

(7), (8)

Decoding Geometric Series

Positive Exponent

$az, a^{-1}z^{-1} \Rightarrow a^{n}$

even number of -1 exponent

Negative Exponent

$a^{-1}z, az^{-1} \Rightarrow a^{-n}$

odd number of -1 exponent

Causal

$\square z \Rightarrow u(n), u(n-1)$

Anti-causal

$\square z^{-1} \Rightarrow u(-n), u(-n-1)$

$\frac{1}{1-\square z} \Rightarrow u(n)$

$\frac{\square z}{1-\square z} \Rightarrow u(n-1)$

$\frac{1}{1-\square z^{-1}} \Rightarrow u(-n)$

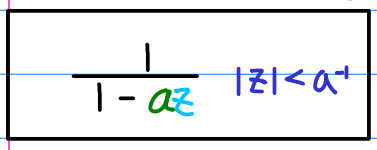
$\frac{\square z^{-1}}{1-\square z^{-1}} \Rightarrow u(-n-1)$

Decoding examples

Positive Exponent

Negative Exponent

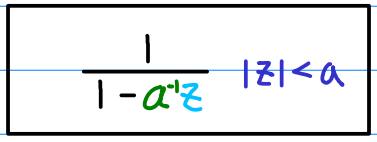
(1) **o-including**



$a z \Rightarrow a^{n}$

$1, a z \Rightarrow u(n)$

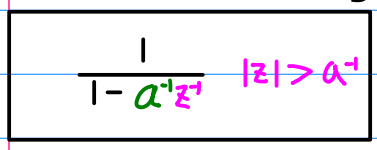
(2) **o-including**



$a^{-1} z \Rightarrow a^{-n}$

$1, a^{-1} z \Rightarrow u(n)$

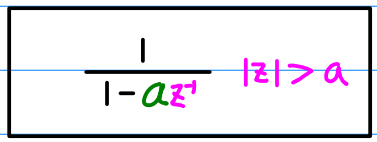
(3) **o-including**



$a^{-1} z^{-1} \Rightarrow a^{n}$

$1, a^{-1} z^{-1} \Rightarrow u(-n)$

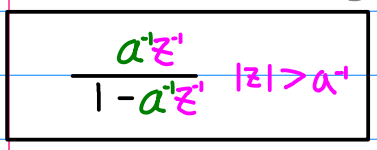
(4) **o-including**



$a z^{-1} \Rightarrow a^{-n}$

$1, a z^{-1} \Rightarrow u(-n)$

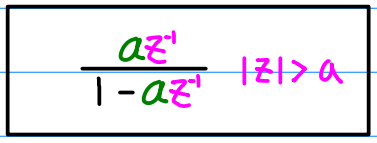
(5) **o-execluding**



$a^{-1} z^{-1} \Rightarrow a^{n}$

$a^{-1} z^{-1}, a^{-1} z^{-1} \Rightarrow u(-n-1)$

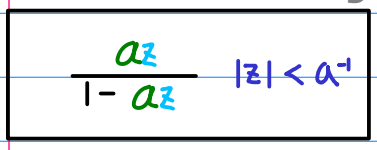
(6) **o-excluding**



$a z^{-1} \Rightarrow a^{-n}$

$a z^{-1}, a z^{-1} \Rightarrow u(-n-1)$

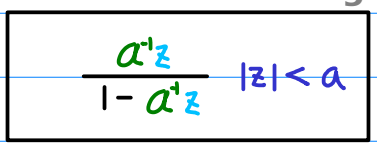
(7) **o-excluding**



$a z \Rightarrow a^{n}$

$a z, a z \Rightarrow u(n-1)$

(8) **o-excluding**



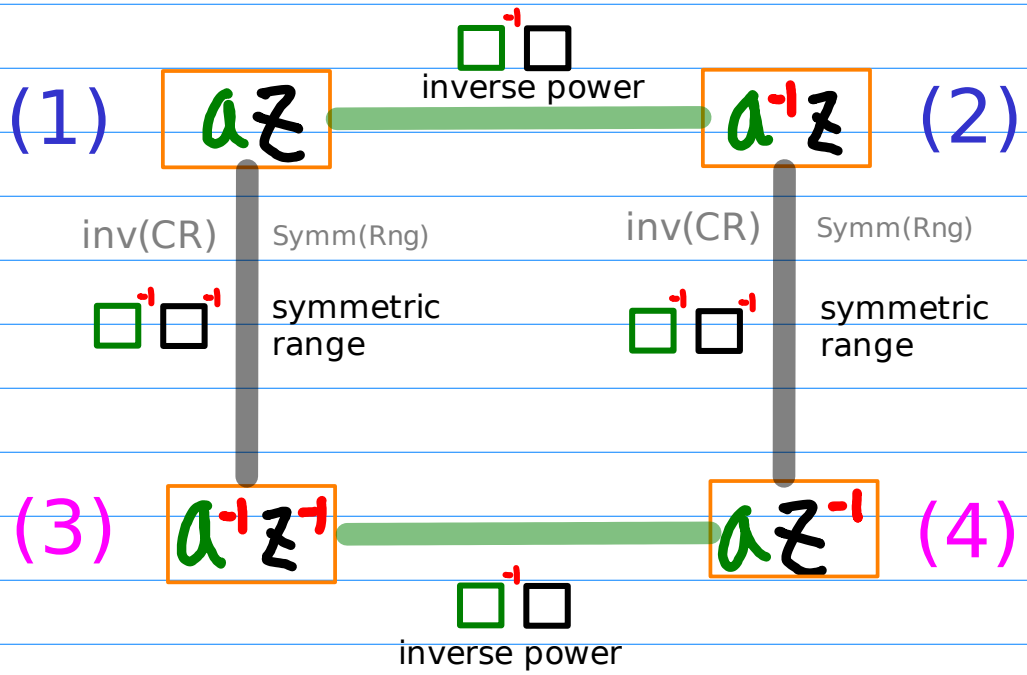
$a^{-1} z \Rightarrow a^{-n}$

$a^{-1} z, a^{-1} z \Rightarrow u(n-1)$

Inverse power, Symmetric range relations

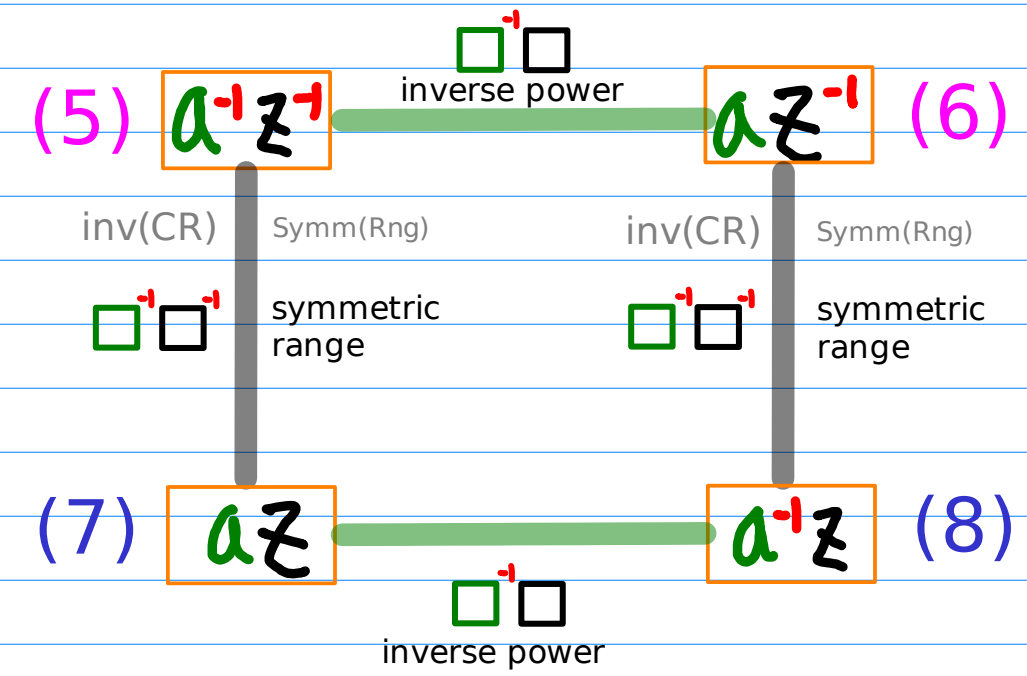
origin including geometric sequences

$$\frac{1}{1-\square}$$



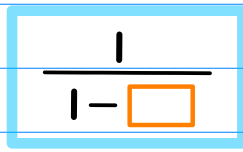
origin excluding geometric sequences

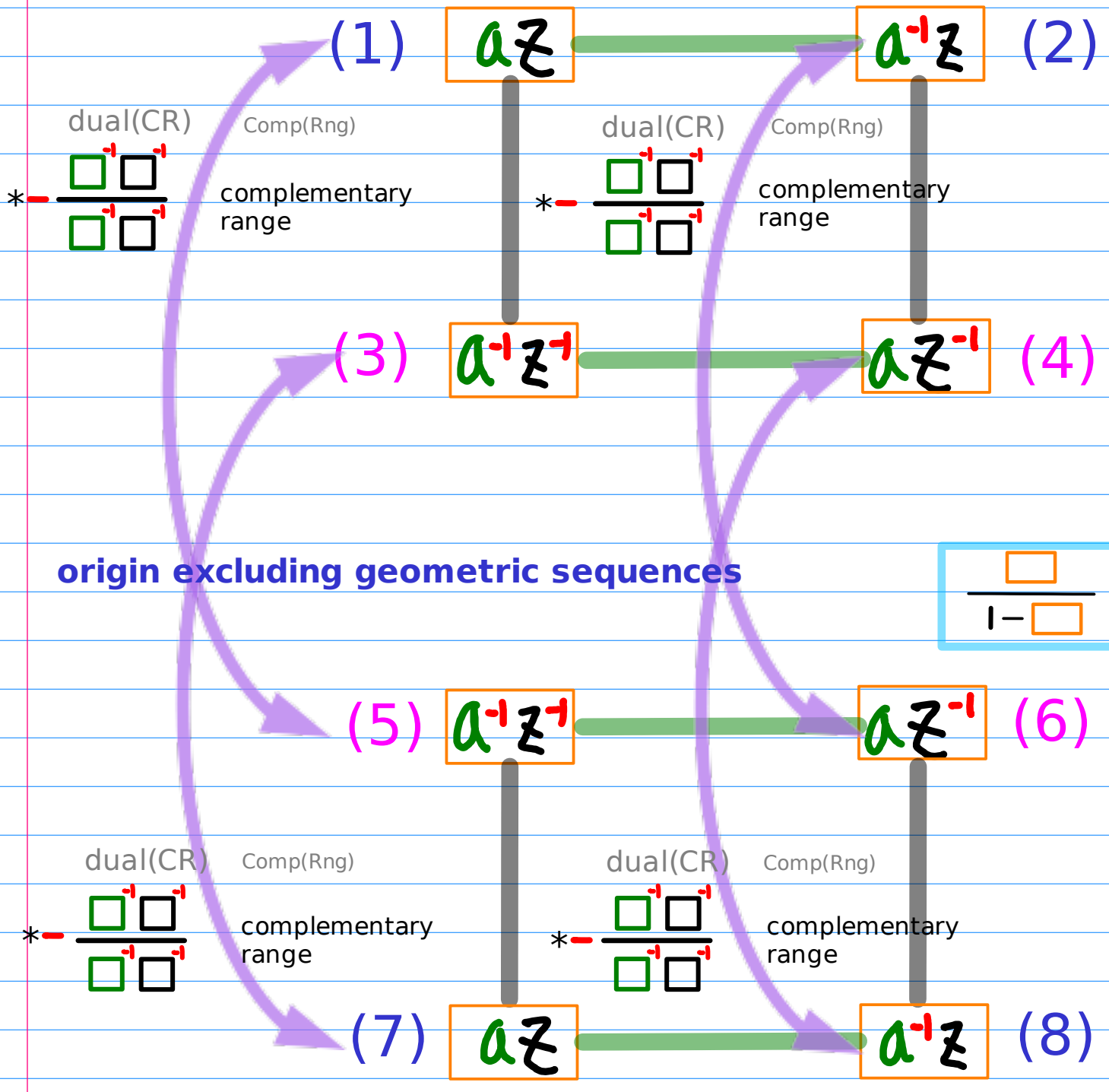
$$\frac{\square}{1-\square}$$



Complementary range relations

origin including geometric sequences



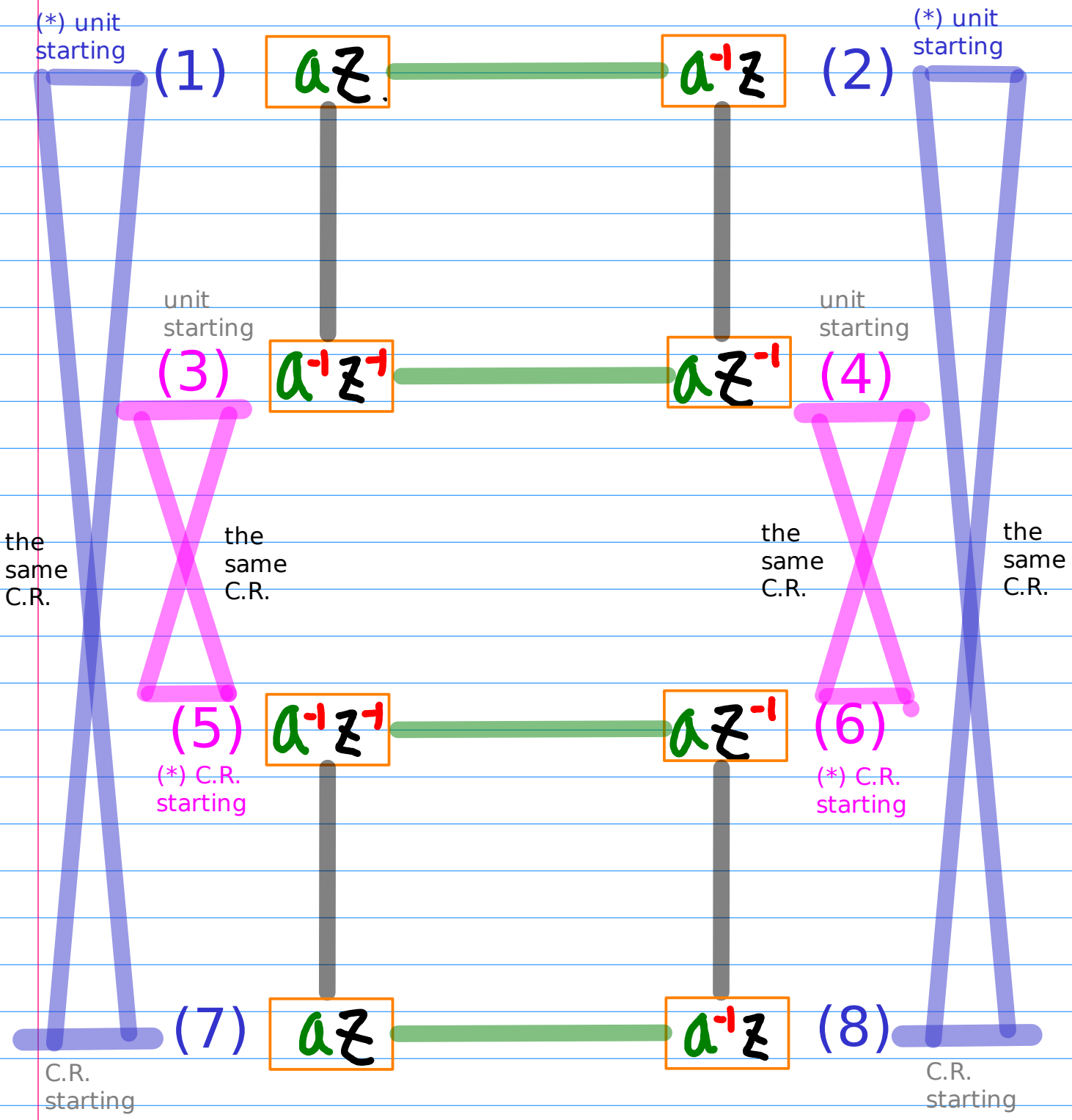


origin excluding geometric sequences

Shifting relations (a)

(1)
(7)
(5)
(3)

(2)
(8)
(6)
(4)

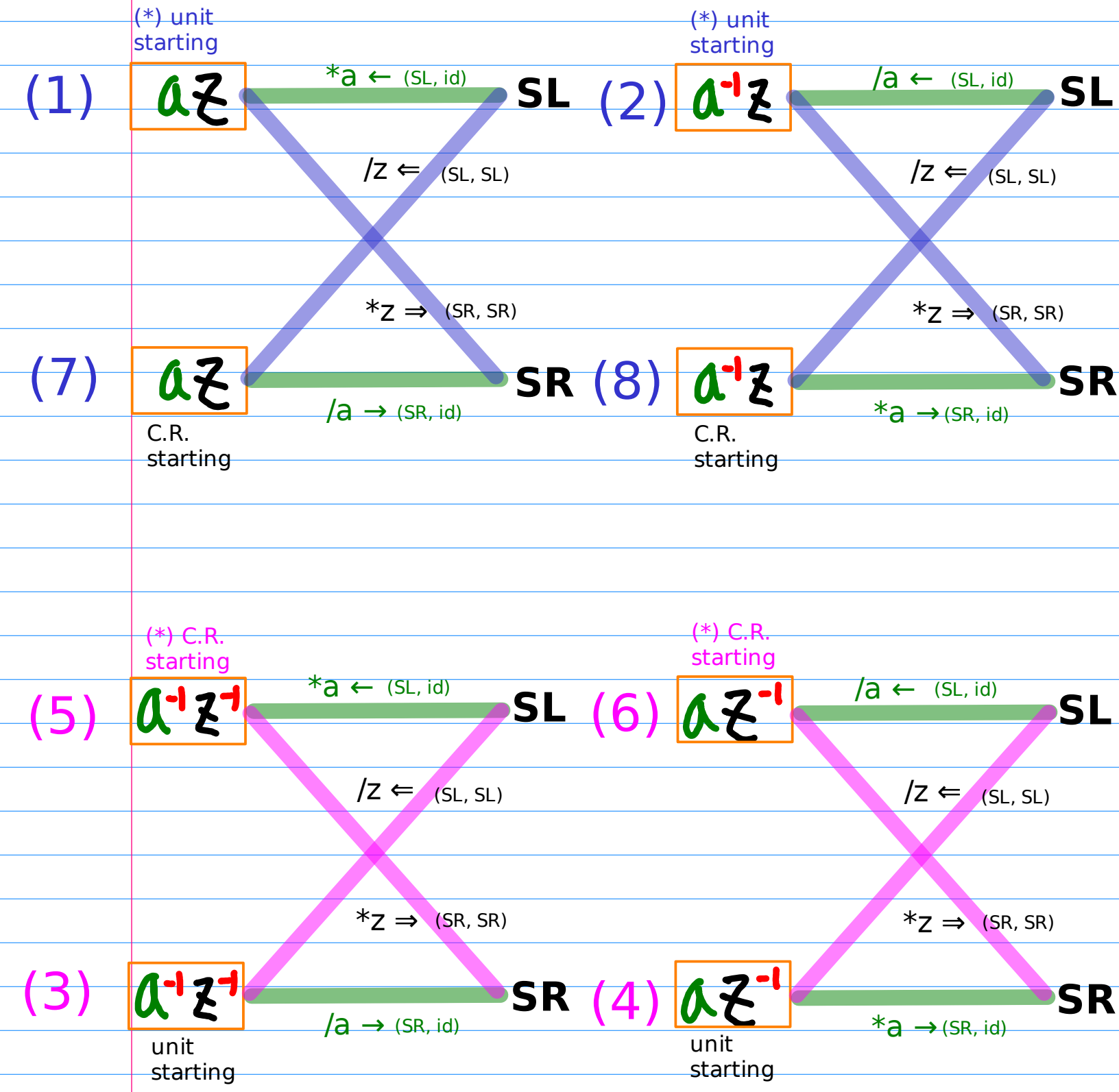


Shifting relations (b)

(exponent, range)

{SL, SR} {SL, SR, id}

(1) (2)
(7) (8)
(5) (6)
(3) (4)



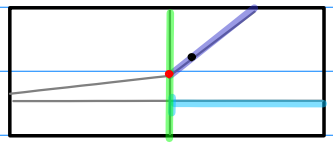
Unit starting **origin including**

(1) $\frac{1}{1-az}$ a^n u(n)

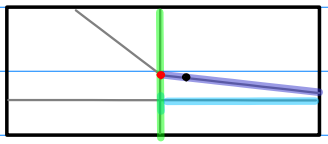
(2) $\frac{1}{1-a^{-1}z}$ a^{-n} u(n)

(3) $\frac{1}{1-a^{-1}z^{-1}}$ $-a^n$ u(-n)

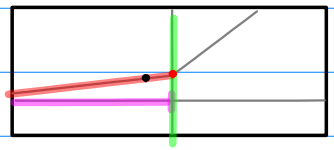
(4) $\frac{1}{1-az^{-1}}$ $-a^{-n}$ u(-n)



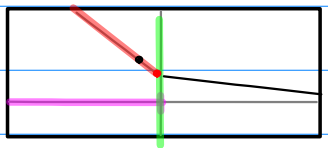
a^n u(n)



a^{-n} u(n)



a^n u(-n)



a^{-n} u(-n)

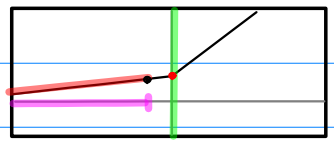
C.R. starting **origin excluding**

(5) $\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $-a^n\,u(-n-1)$

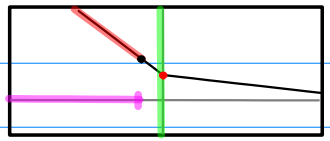
(6) $\dfrac{az^{-1}}{1-az^{-1}}$ $-a^{-n}u(-n-1)$

(7) $\dfrac{az}{1-az}$ $a^n\,u(n-1)$

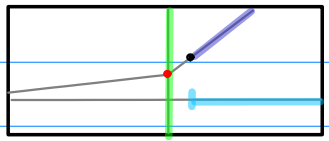
(8) $\dfrac{a^{-1}z}{1-a^{-1}z}$ $a^{-n}\,u(n-1)$



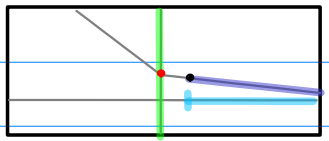
$a^n\,u(-n-1)$



$a^{-n}\,u(-n-1)$



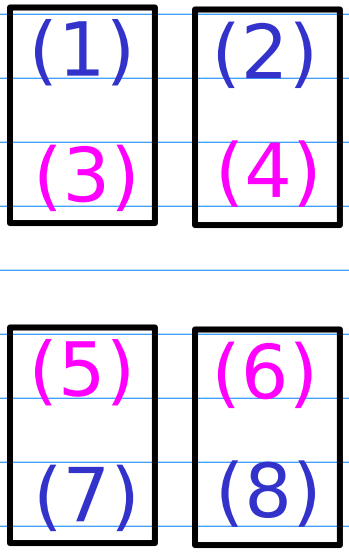
$a^n\,u(n-1)$



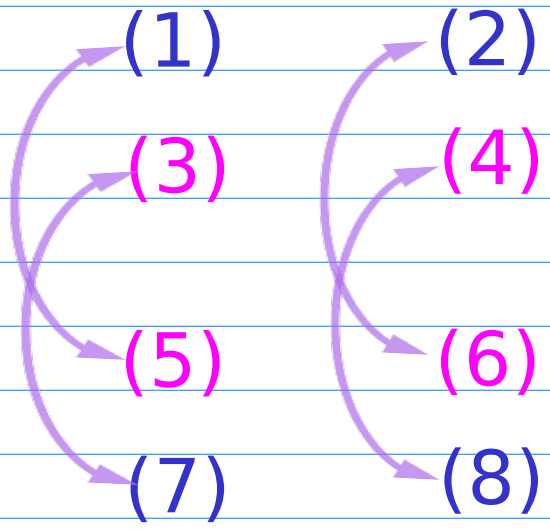
$a^{-n}\,u(n-1)$

Range Combinations (1)

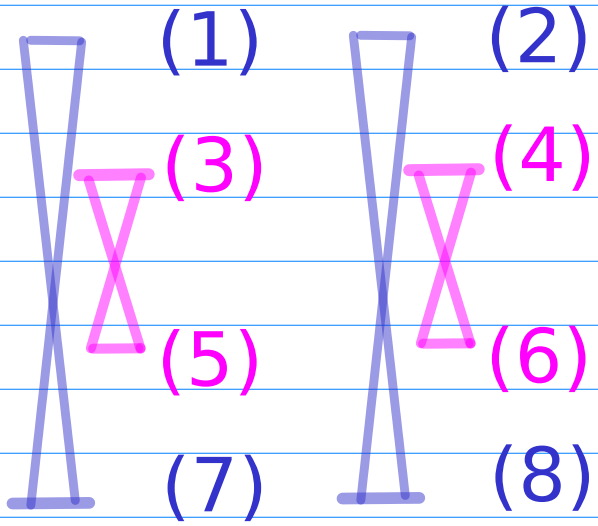
Symmetric Range



Complementary Range

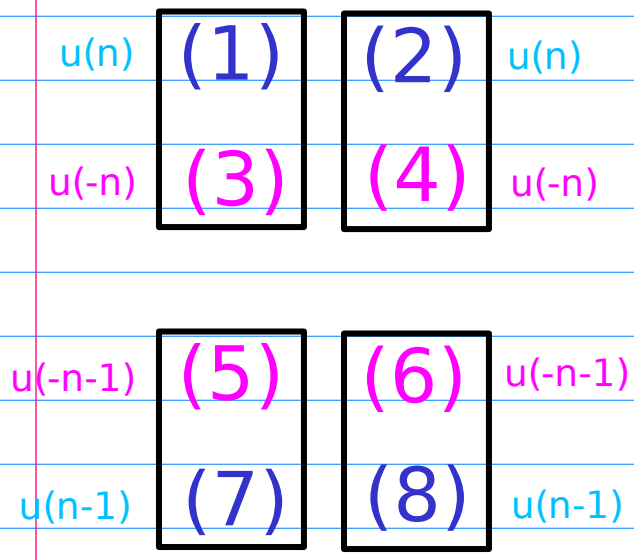


Shifted Range

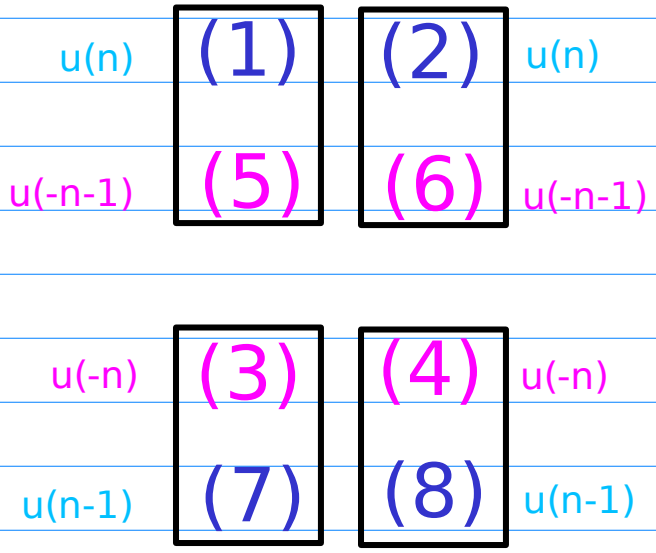


Range Combinations (2)

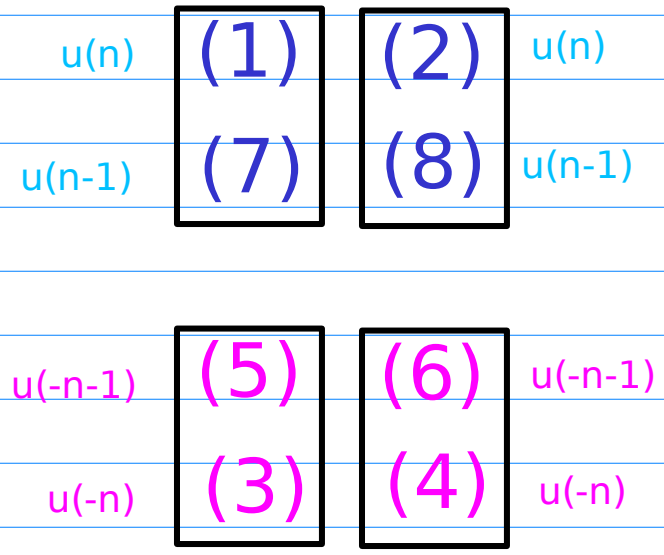
Symmetric Range

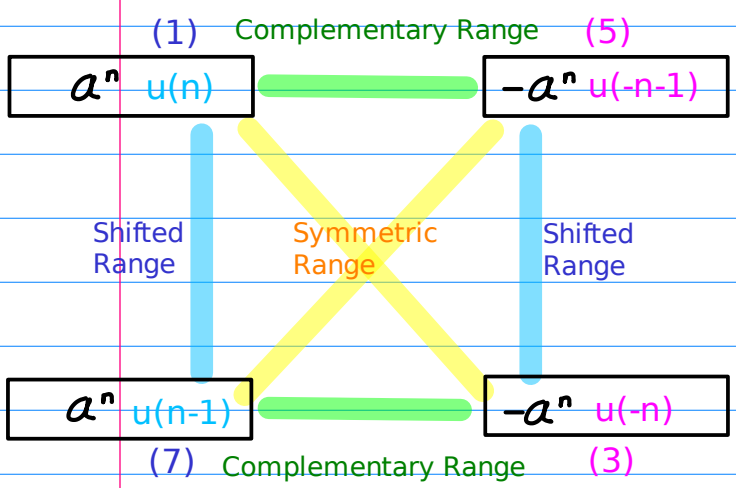


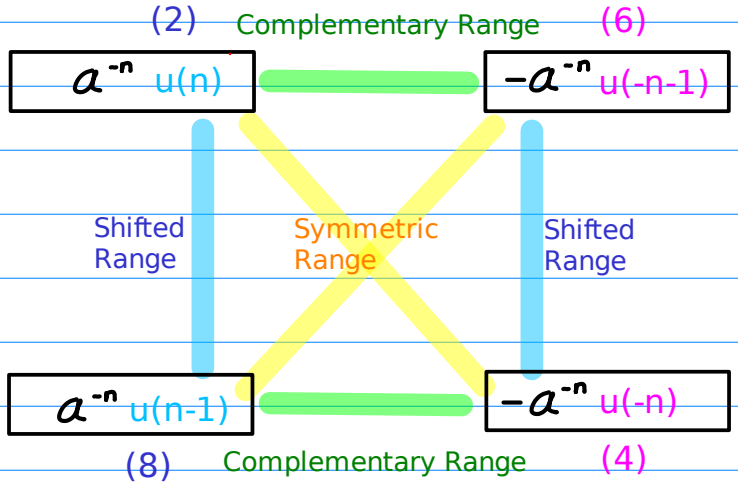
Complementary Range

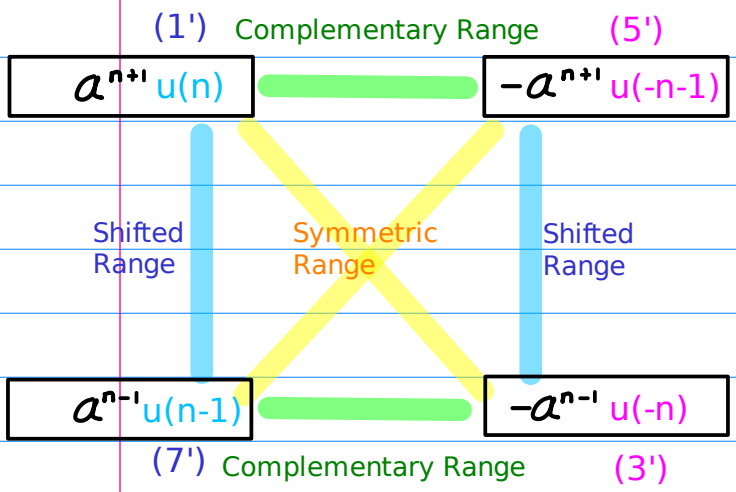


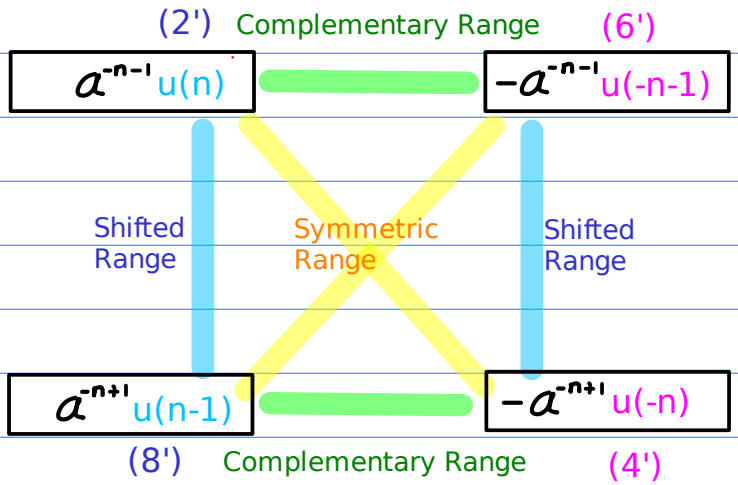
Shifted Range

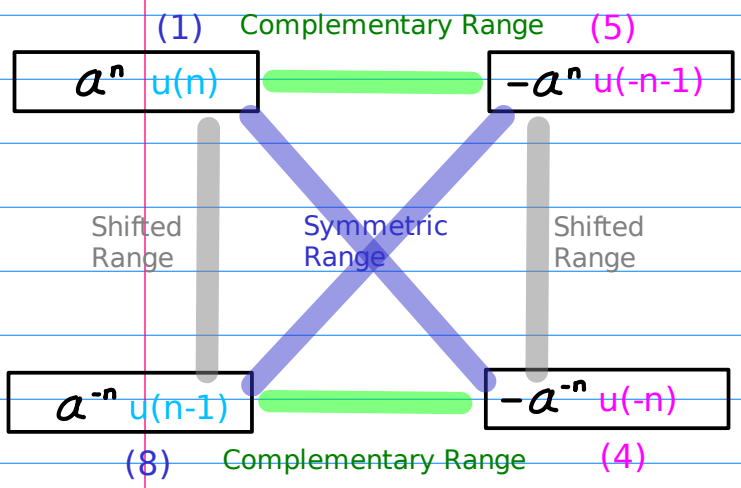


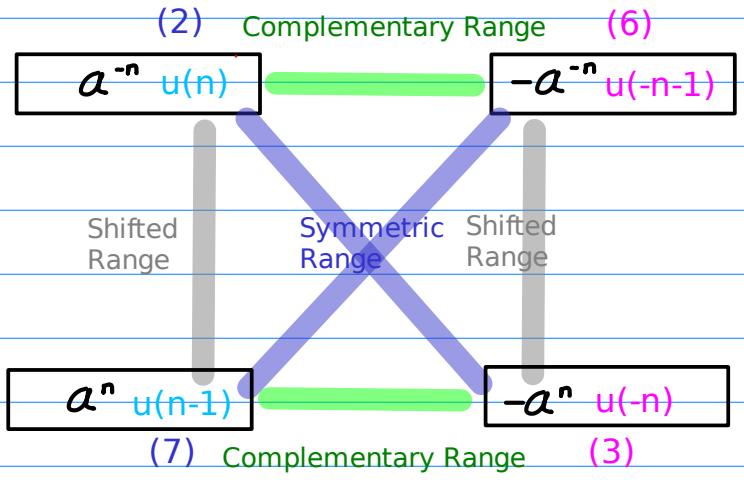

(1)
Complementary Range
(5)
a^n u(n)
$-a^n$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^n u(n-1)
$-a^n$ u(-n)
(7)
Complementary Range
(3)

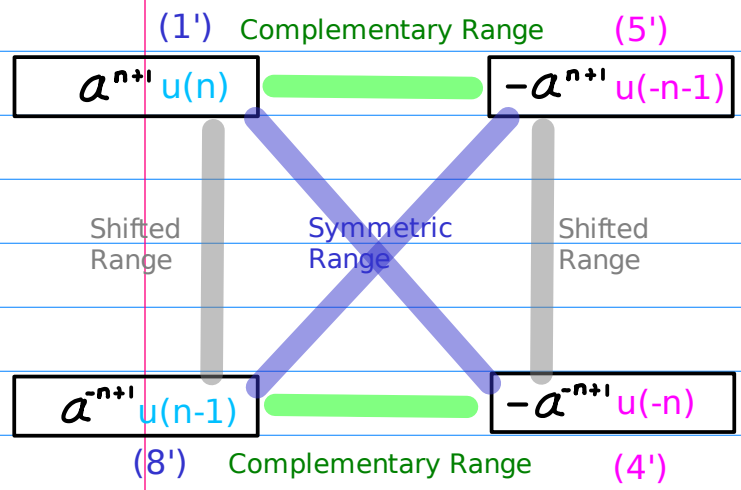

(2)
Complementary Range
(6)
a^{-n} u(n)
$-a^{-n}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{-n} u(n-1)
$-a^{-n}$ u(-n)
(8)
Complementary Range
(4)

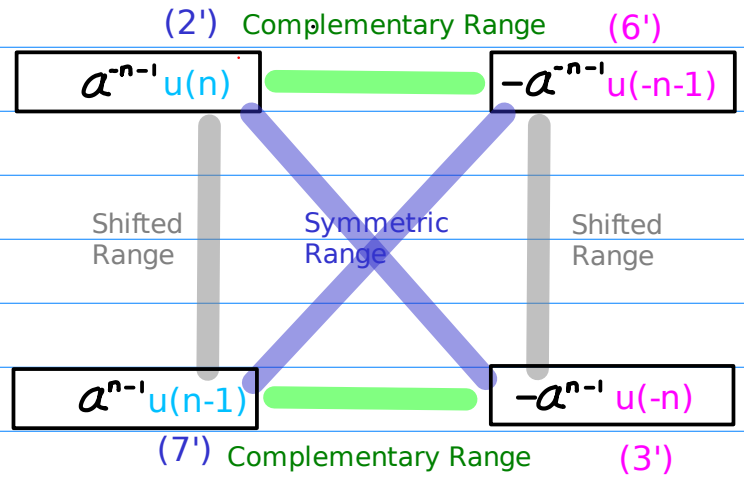

(1')
Complementary Range
(5')
a^{n+1} u(n)
$-a^{n+1}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{n-1} u(n-1)
$-a^{n-1}$ u(-n)
(7')
Complementary Range
(3')


(2')
Complementary Range
(6')
a^{-n-1} u(n)
$-a^{-n-1}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{-n+1} u(n-1)
$-a^{-n+1}$ u(-n)
(8')
Complementary Range
(4')


(1) Complementary Range (5)
a^n u(n)
$-a^n$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{-n} u(n-1)
$-a^{-n}$ u(-n)
(8) Complementary Range (4)


(2) Complementary Range (6)
a^{-n} u(n)
$-a^{-n}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^n u(n-1)
$-a^n$ u(-n)
(7) Complementary Range (3)


(1') Complementary Range (5')
a^{n+1} u(n)
$-a^{n+1}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{-n+1} u(n-1)
$-a^{-n+1}$ u(-n)
(8') Complementary Range (4')


(2') Complementary Range (6')
a^{-n-1} u(n)
$-a^{-n-1}$ u(-n-1)
Shifted Range
Symmetric Range
Shifted Range
a^{n-1} u(n-1)
$-a^{n-1}$ u(-n)
(7') Complementary Range (3')

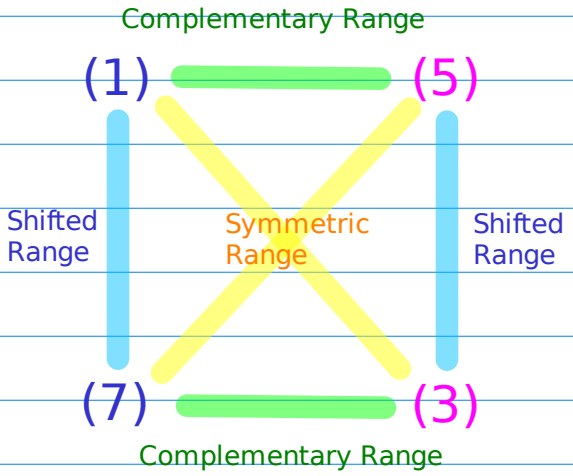
(1) (5)
(7) (3)

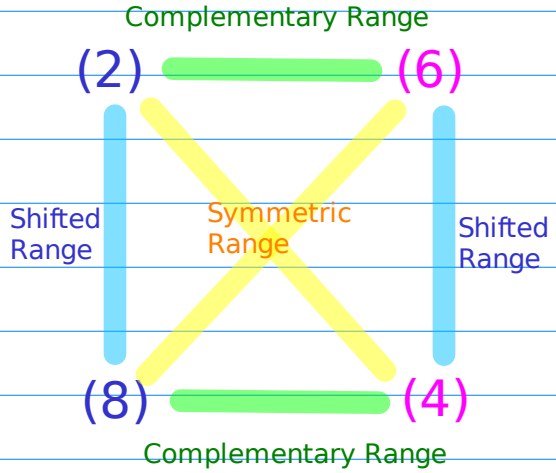
(2) (6)
(8) (4)

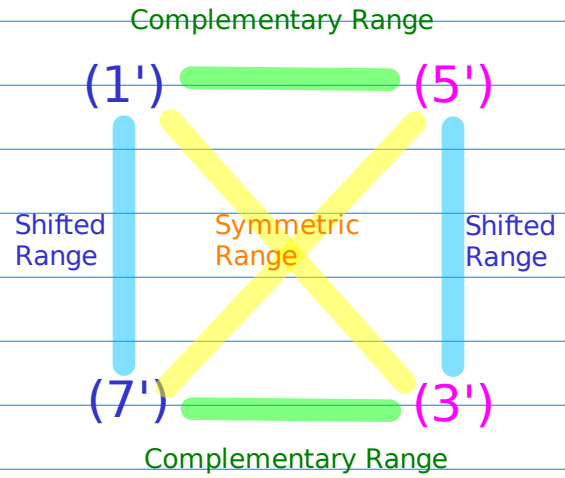
(1') (5')
(7') (3')

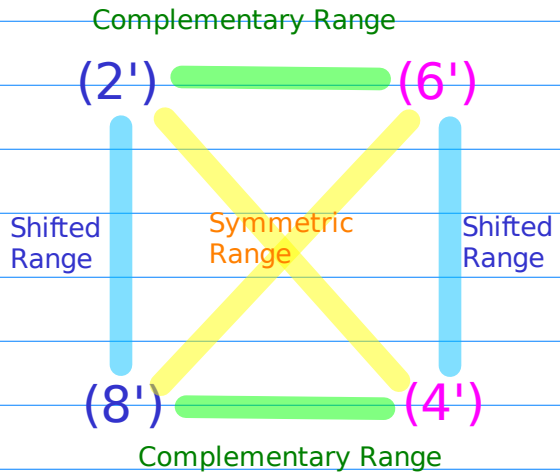
(2')(6')
(8')(4')

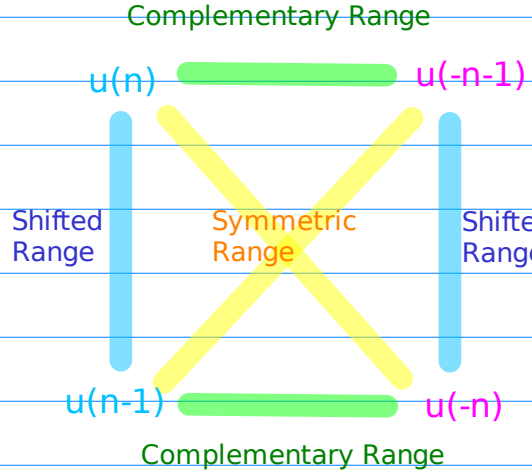
Complementary, Shifted, Symmetric Ranges

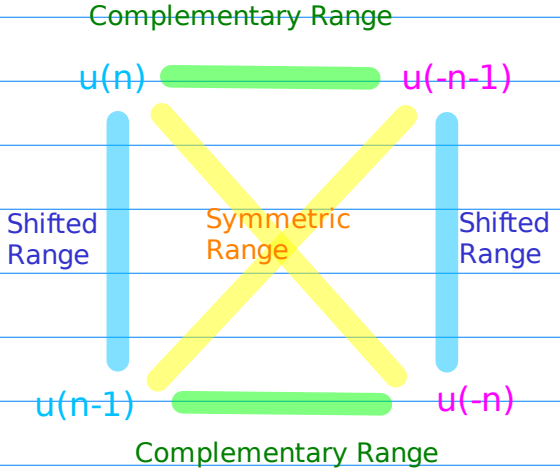












(1) (5)
(7) (3)

(2) (6)
(8) (4)

(1') (5')
(7') (3')

(2')(6')
(8')(4')

Partial fractions and geometric power series

$P^{-1} = 0.5$

$P = 2$

-(1) $-\frac{1}{1-2z}$ $-2^n\ u(n)$

(5) $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^n\ u(-n-1)$

-(2) $-\frac{1}{1-0.5z}$ $-2^{-n}\ u(n)$

(6) $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n}\ u(-n-1)$

-(7) $-\frac{2z}{1-2z}$ $-2^n\ u(n-1)$

(3) $\frac{1}{1-0.5z^{-1}}$ $2^n\ u(-n)$

-(8) $-\frac{0.5z}{1-0.5z}$ $2^{-n}\ u(n-1)$

(4) $\frac{1}{1-2z^{-1}}$ $-2^{-n}\ u(-n)$

-(1') $-\frac{2}{1-2z}$ $-2^{n+1}\ u(n)$

(5') $\frac{z^{-1}}{1-0.5z^{-1}}$ $2^{n+1}\ u(-n-1)$

-(2') $-\frac{0.5}{1-0.5z}$ $-2^{-n-1}\ u(n)$

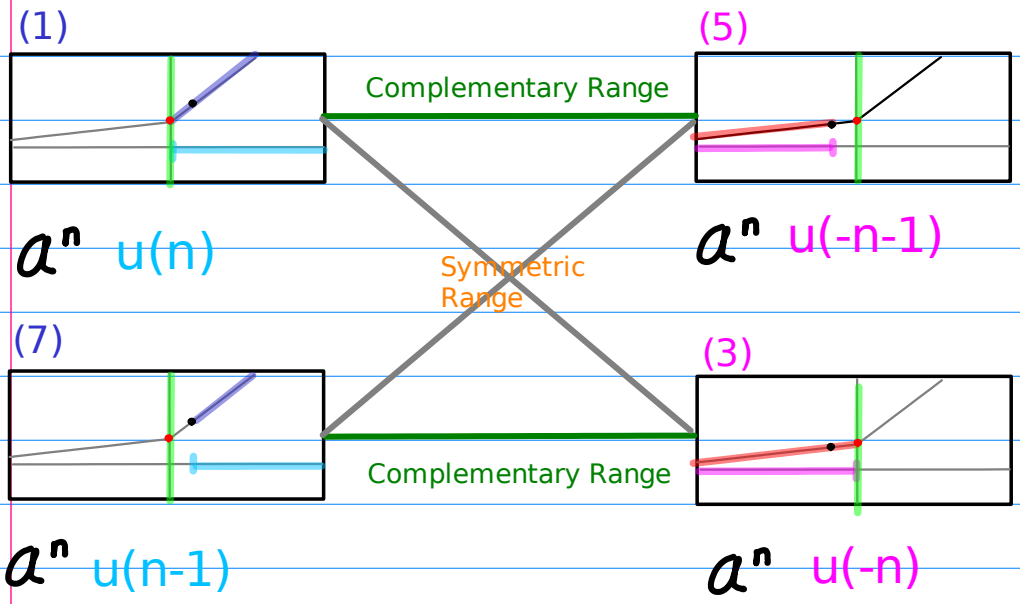
(6') $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1}\ u(-n-1)$

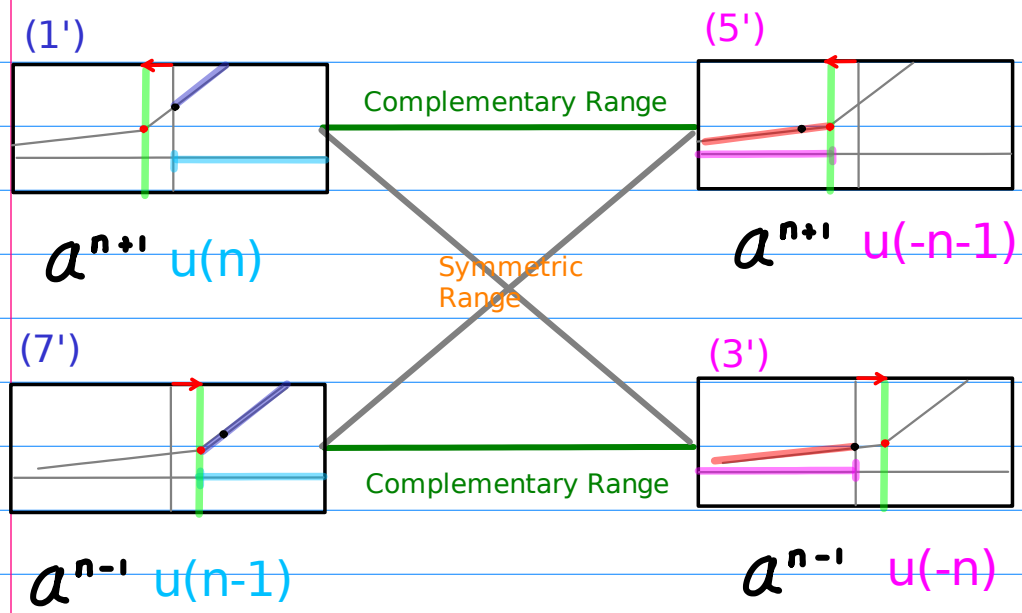
-(7') $-\frac{z}{1-2z}$ $-2^{n-1}\ u(n-1)$

(3') $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1}\ u(-n)$

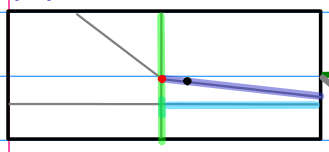
-(8') $-\frac{z}{1-0.5z}$ $-2^{-n+1}\ u(n-1)$

(4') $\frac{2}{1-2z^{-1}}$ $2^{-n+1}\ u(-n)$


(1)
Complementary Range
(5)
a^n u(n)
Symmetric Range
a^n u(-n-1)
(7)
(3)
Complementary Range
a^n u(n-1)
a^n u(-n)

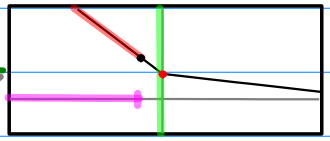

(1')
Complementary Range
(5')
a^{n+1} u(n)
Symmetric Range
a^{n+1} u(-n-1)
(7')
(3')
Complementary Range
a^{n-1} u(n-1)
a^{n-1} u(-n)

(2)



$a^{-n}\ u(n)$

(6)

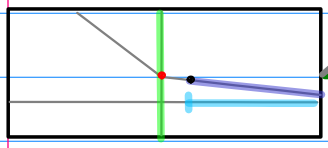


$a^{-n}\ u(-n-1)$

Complementary Range

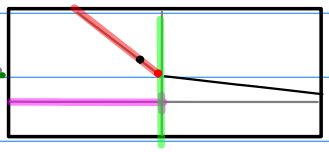
Symmetric Range

(8)



$a^{-n}\ u(n-1)$

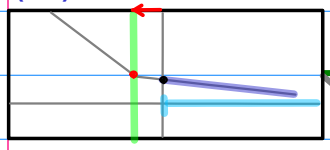
(4)



$a^{-n}\ u(-n)$

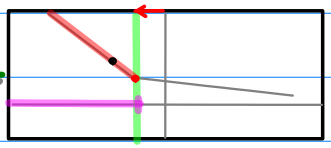
Complementary Range

(2')



$a^{-n-1}\ u(n)$

(6')

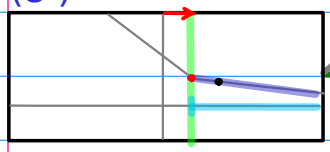


$a^{-n-1}\ u(-n-1)$

Complementary Range

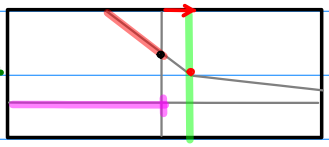
Symmetric Range

(8')



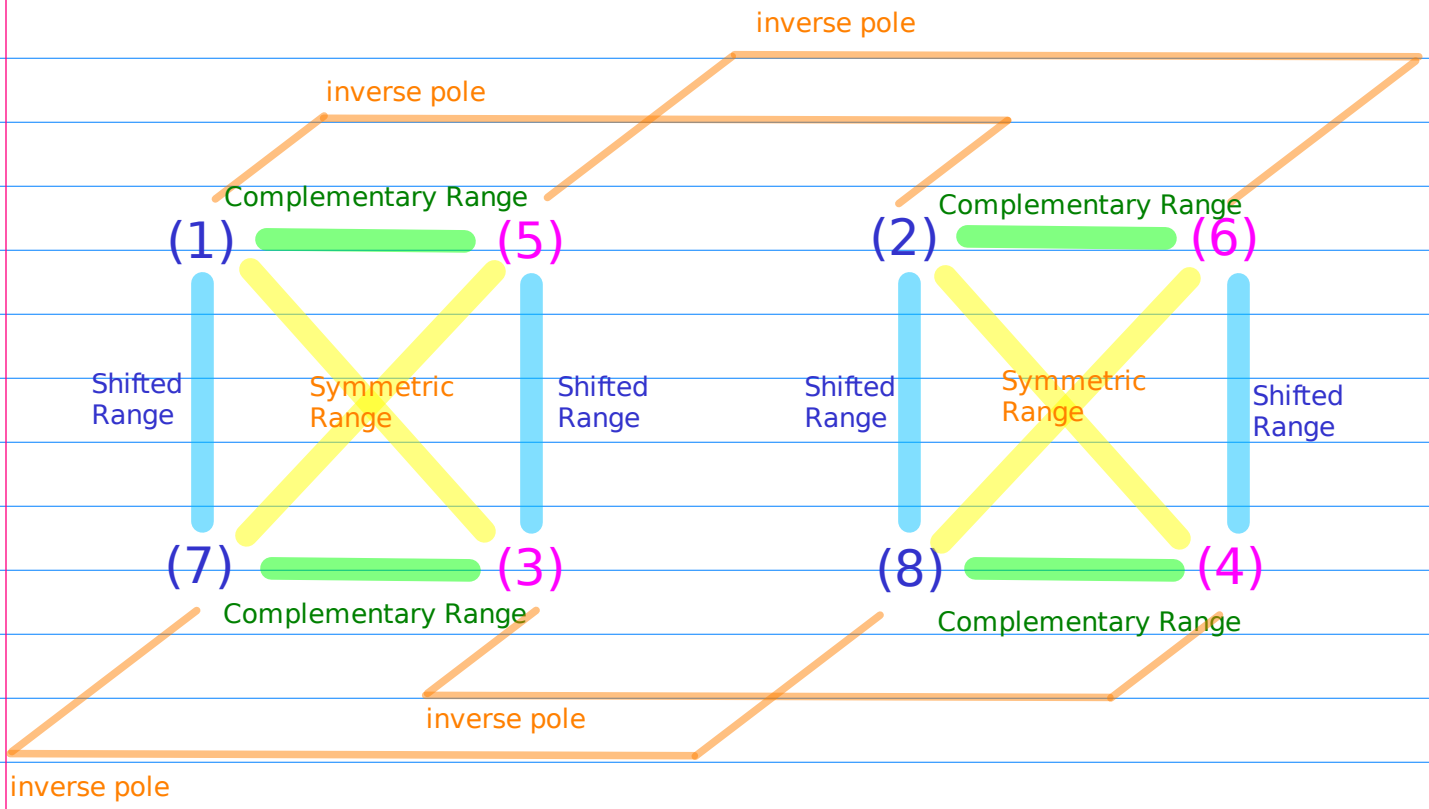
$a^{-n+1}\ u(n-1)$

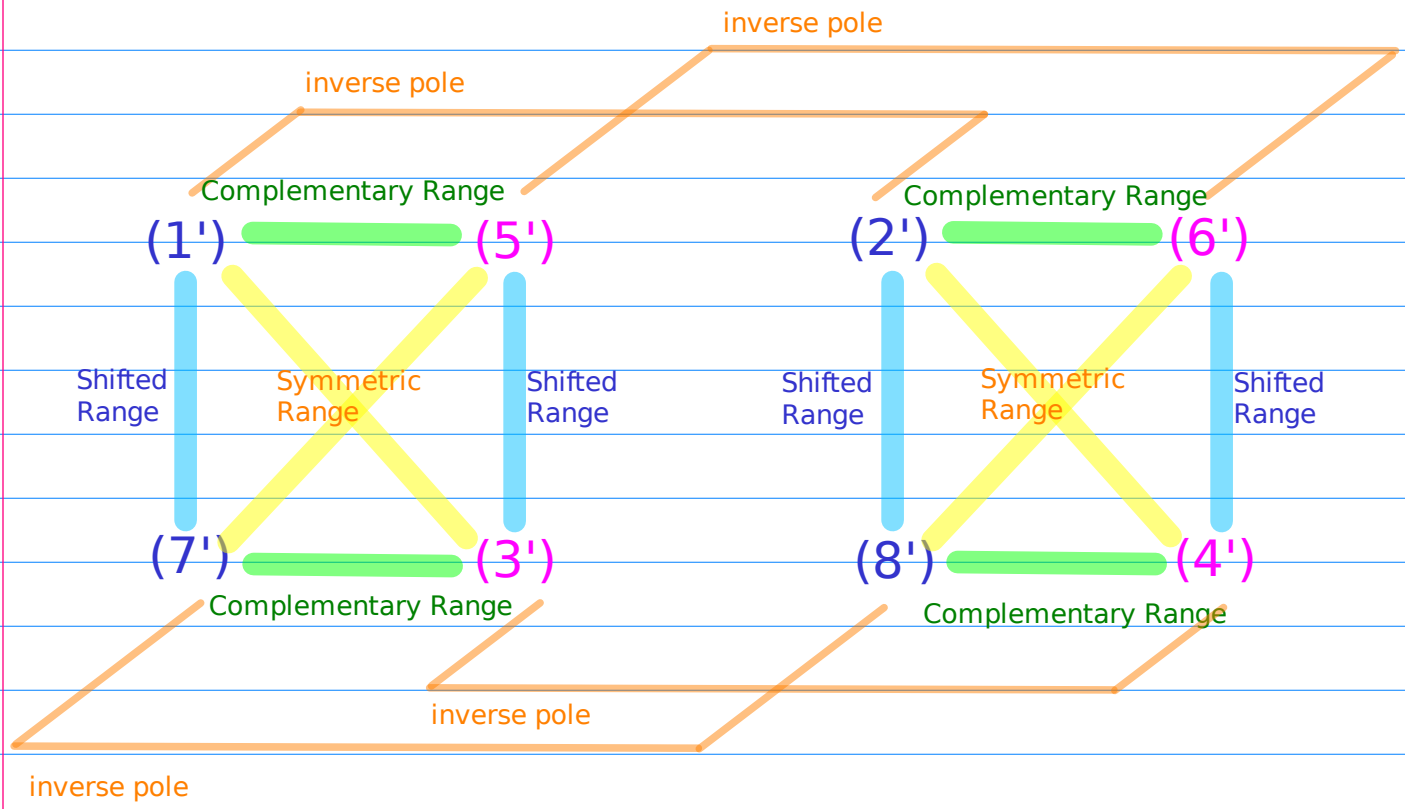
(4')



$a^{-n+1}\ u(-n)$

Complementary Range


inverse pole
inverse pole
Complementary Range
(1)
(5)
Shifted Range
Symmetric Range
Shifted Range
(7)
(3)
Complementary Range
inverse pole
inverse pole
Complementary Range
(2)
(6)
Shifted Range
Symmetric Range
Shifted Range
(8)
(4)
Complementary Range


inverse pole
inverse pole
Complementary Range
(1')
(5')
Shifted Range
Symmetric Range
Shifted Range
(7')
(3')
Complementary Range
inverse pole
inverse pole
Complementary Range
(2')
(6')
Shifted Range
Symmetric Range
Shifted Range
(8')
(4')
Complementary Range

a^n ← a^-n

u(n) ↔ u(n-1)

u(-n) ↔ u(-n-1)

reciprocal power

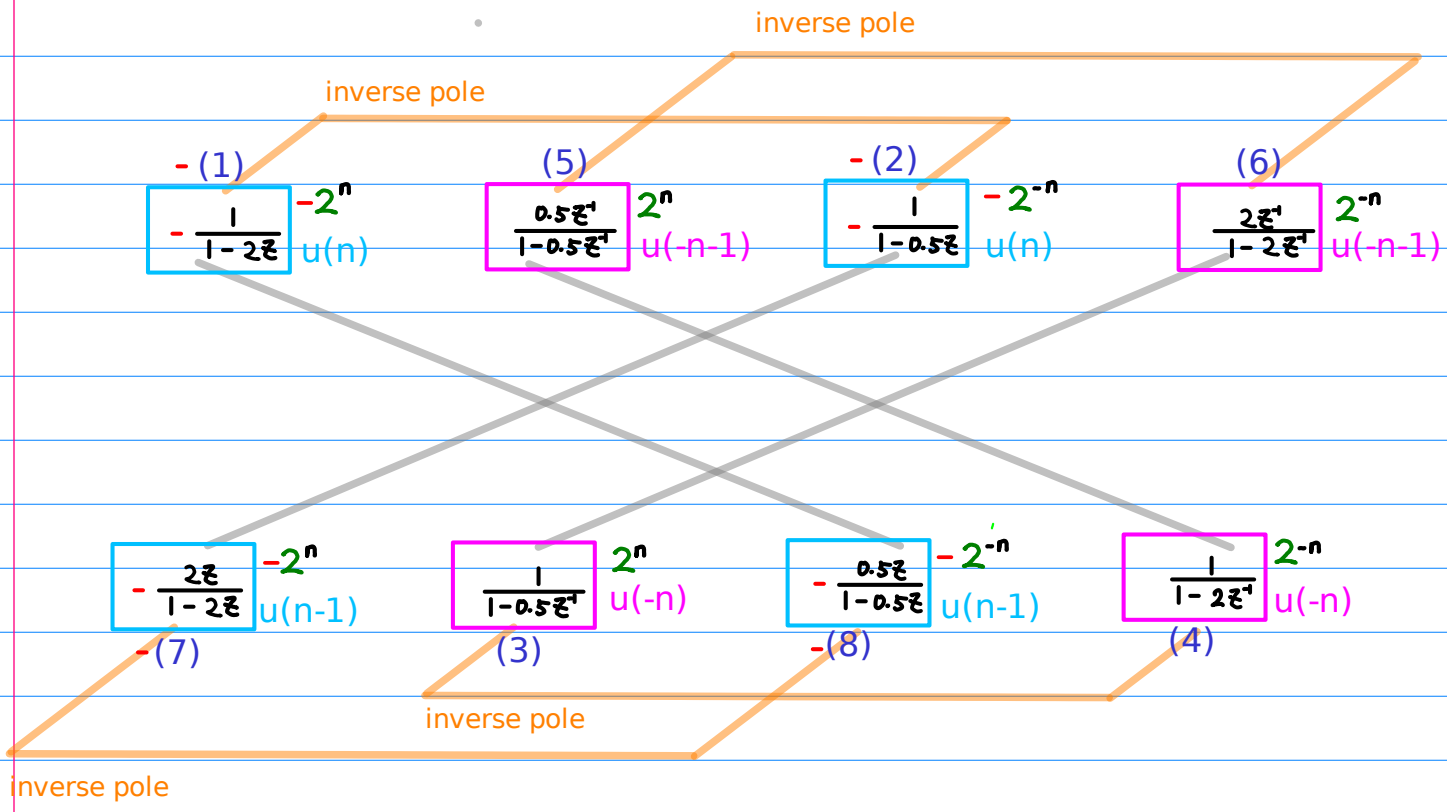
shift of a range

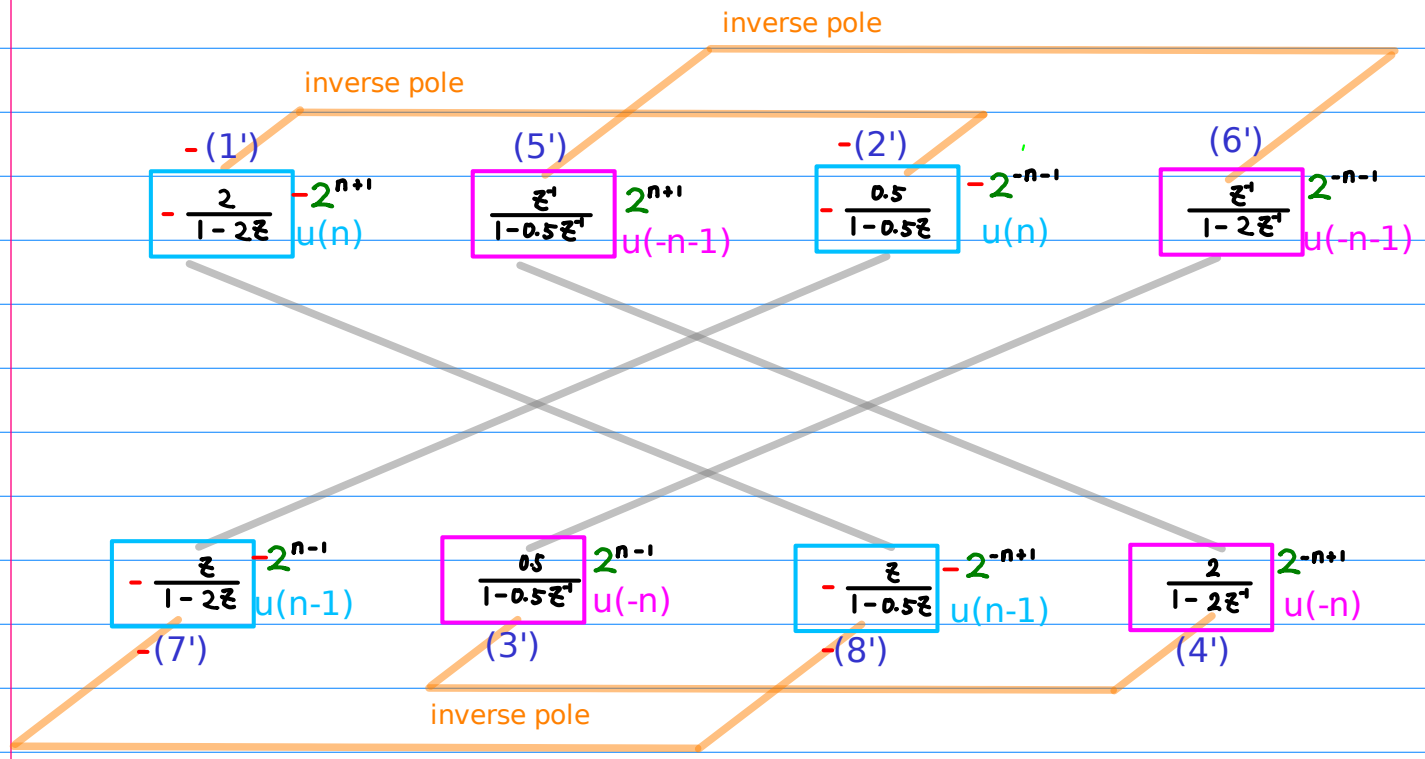
(1) (5)

(7) (3)

(2) (6)

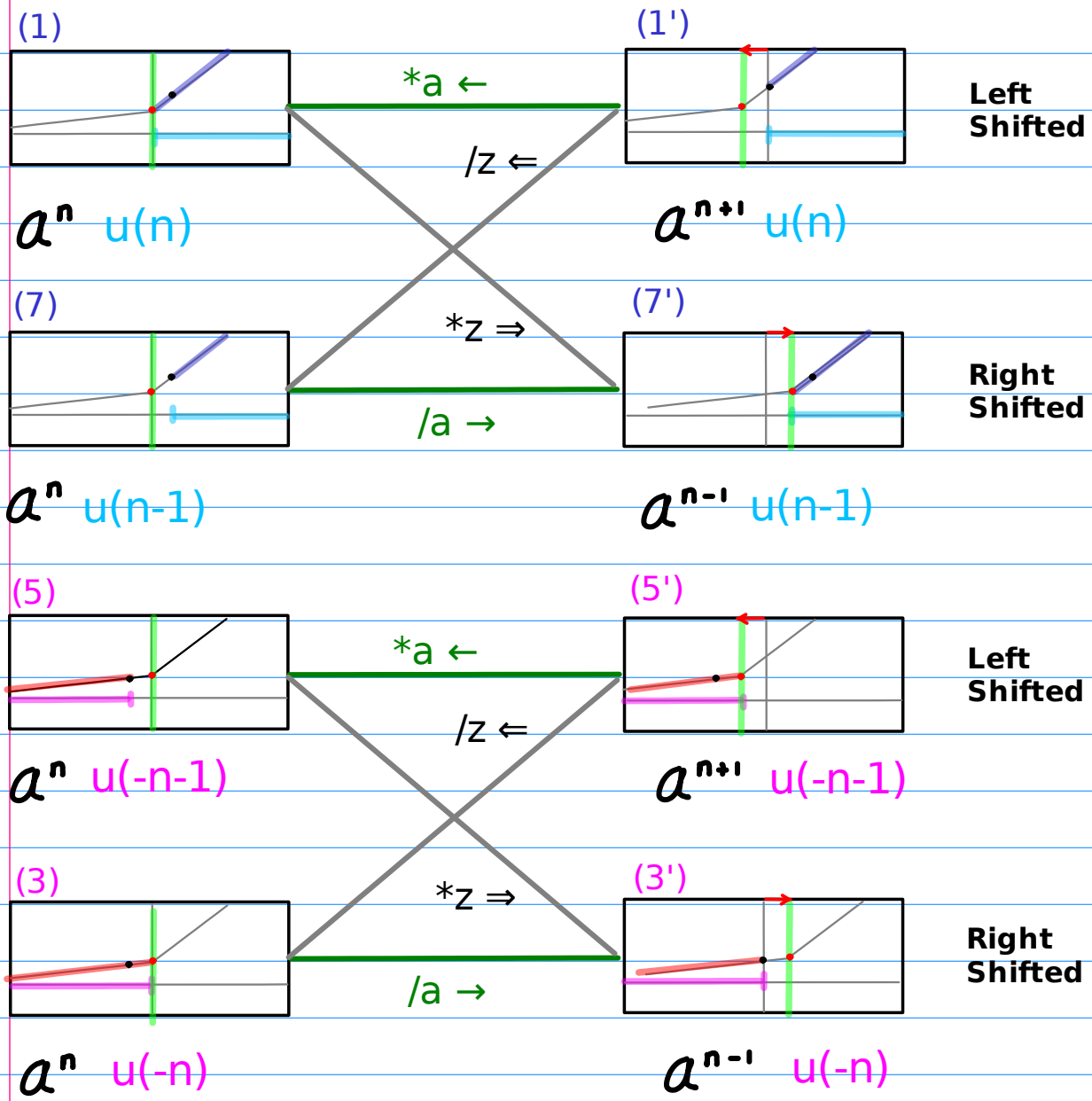
(8) (4)





(1) (1')
(7) (7')
(5) (5')
(3) (3')

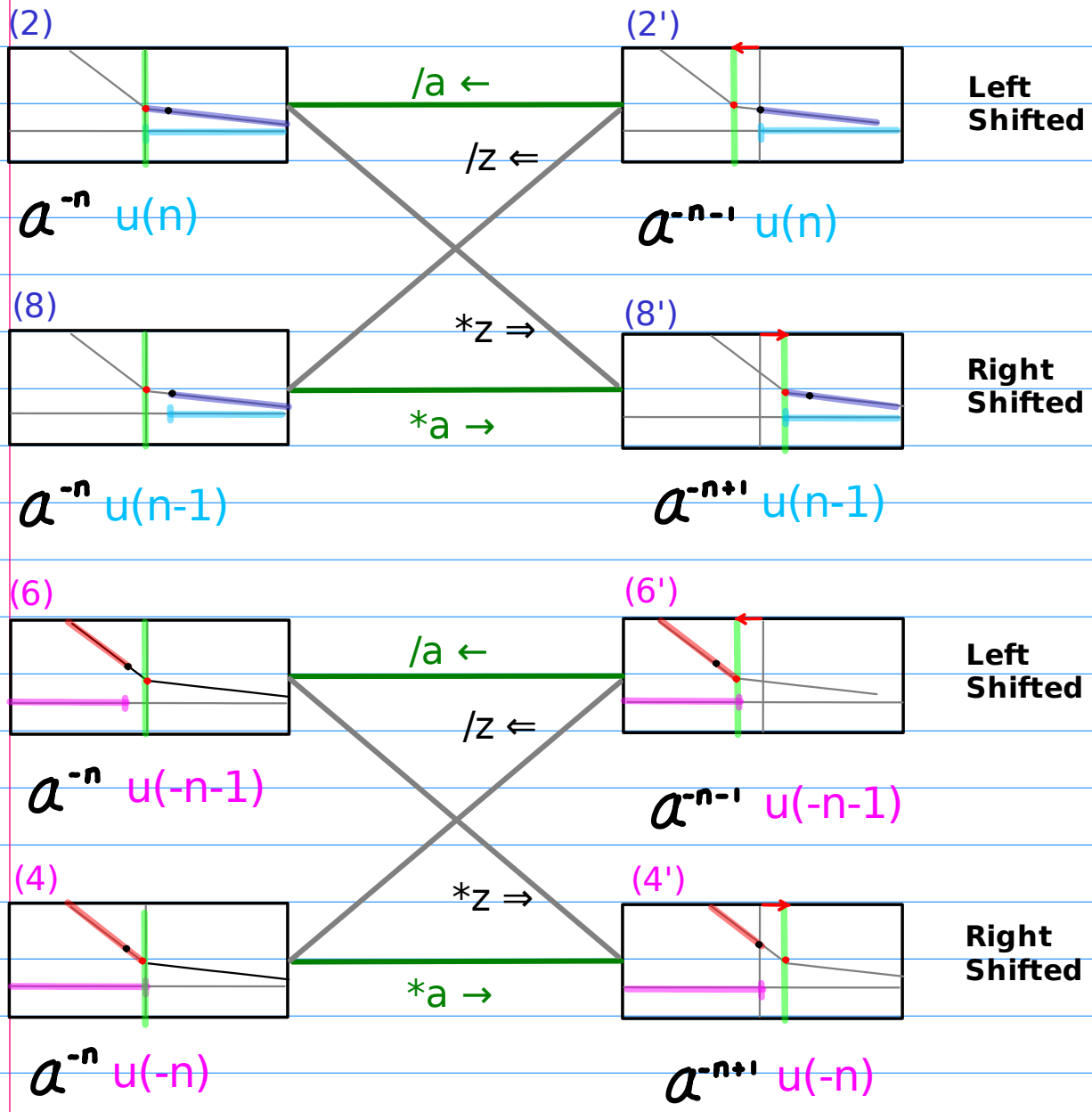
Left / Right Shift



(2) (2')
(8) (8')

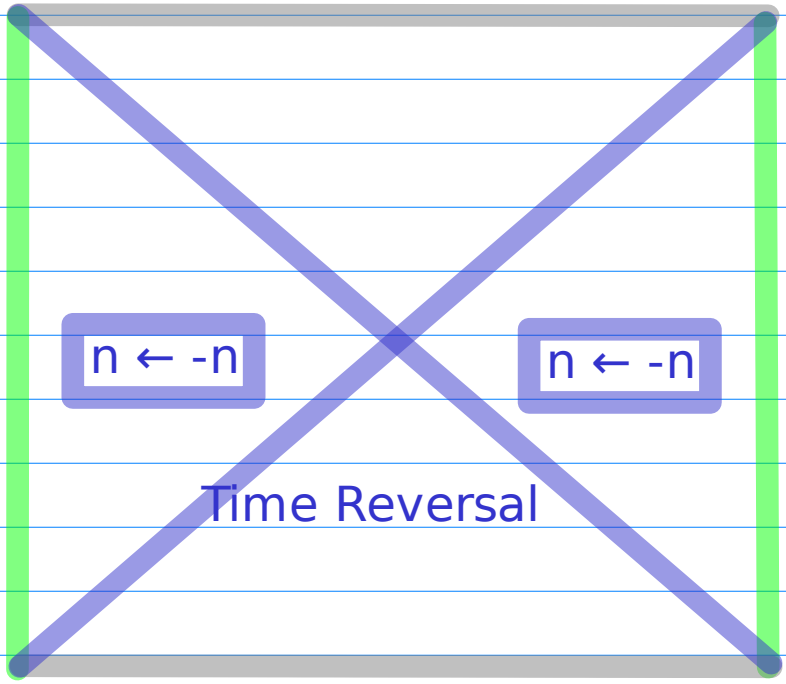
(6) (6')
(4) (4')

Left / Right Shift



reciprocal power,
range shifting

a^n ← a^-n
u(n) ↔ u(n-1)
u(-n) ↔ u(-n-1)



Complementary
Region

Complementary
Region

u(n) ↔ u(-n-1)
u(-n) ↔ u(n-1)

Complementary
Region

a^n ← a^-n
u(n) ↔ u(n-1)
u(-n) ↔ u(-n-1)

reciprocal power,
range shifting

u(n) ↔ u(-n-1)
u(-n) ↔ u(n-1)

Complementary
Region

u(n) → u(-n-1) → u(-n)
u(-n) → u(n-1) → u(n)
u(-n-1) → u(n) → u(n-1)
u(n-1) → u(-n) → u(-n-1)

Time Reversal

a^n ← a^-n
u(n) ↔ u(-n)
u(n-1) ↔ u(-n-1)

n ← -n

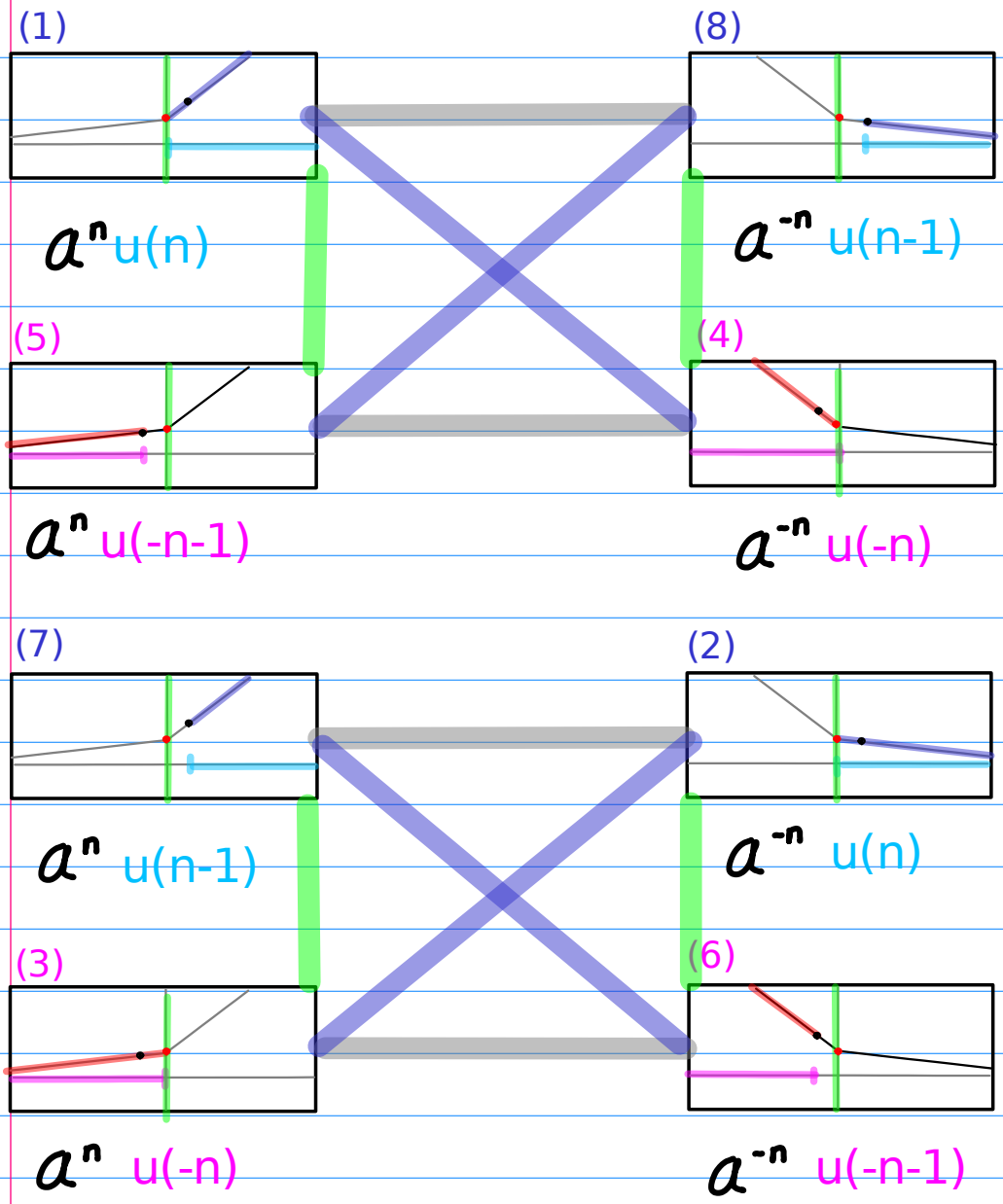
Time Reversal

a^n ← a^-n

u(n) ↔ u(n-1)

u(-n) ↔ u(-n-1)

reciprocal power shift of a range

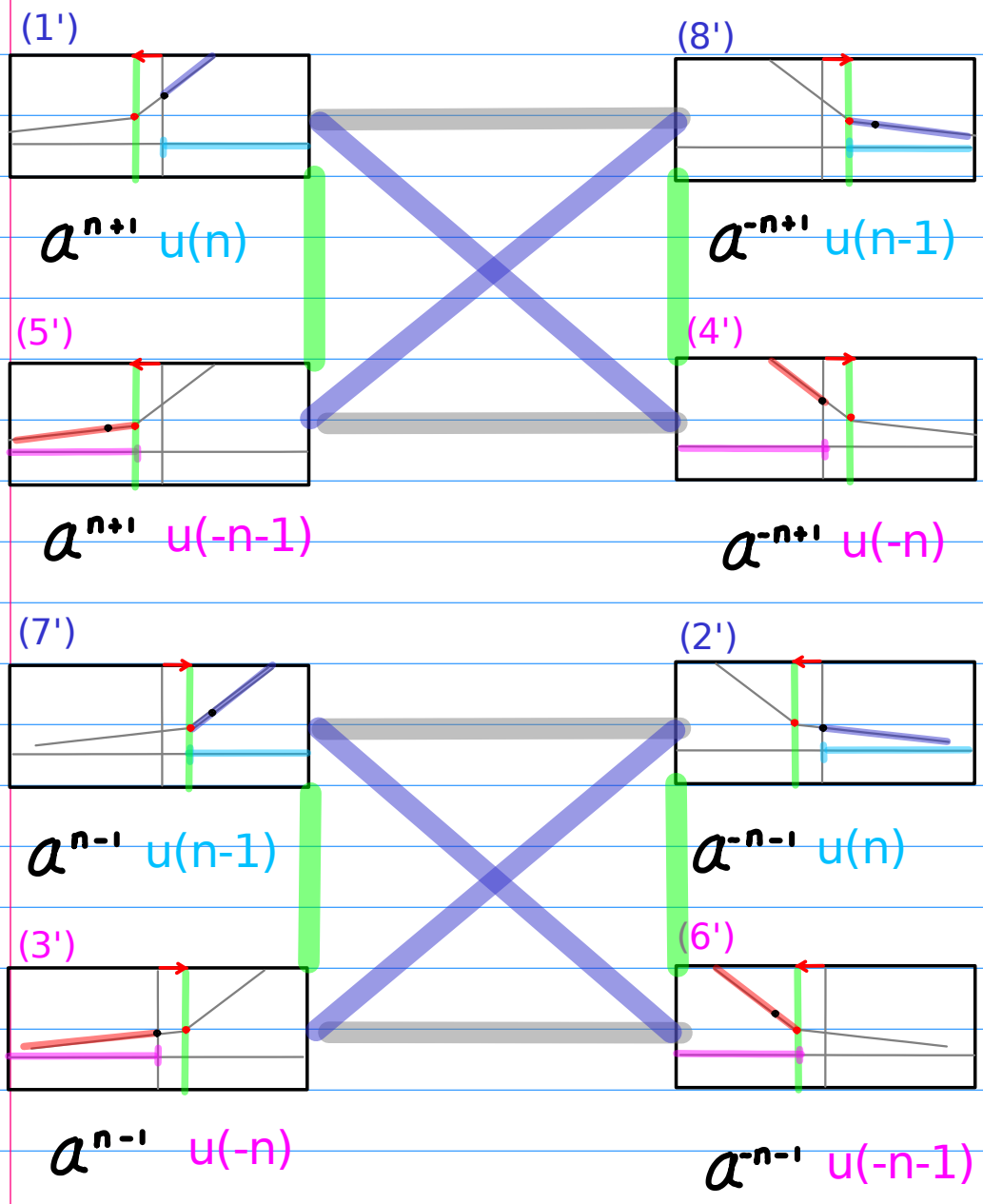


n ← -n

Time Reversal

a^n ← a^-n
u(n) ↔ u(n-1)
u(-n) ↔ u(-n-1)

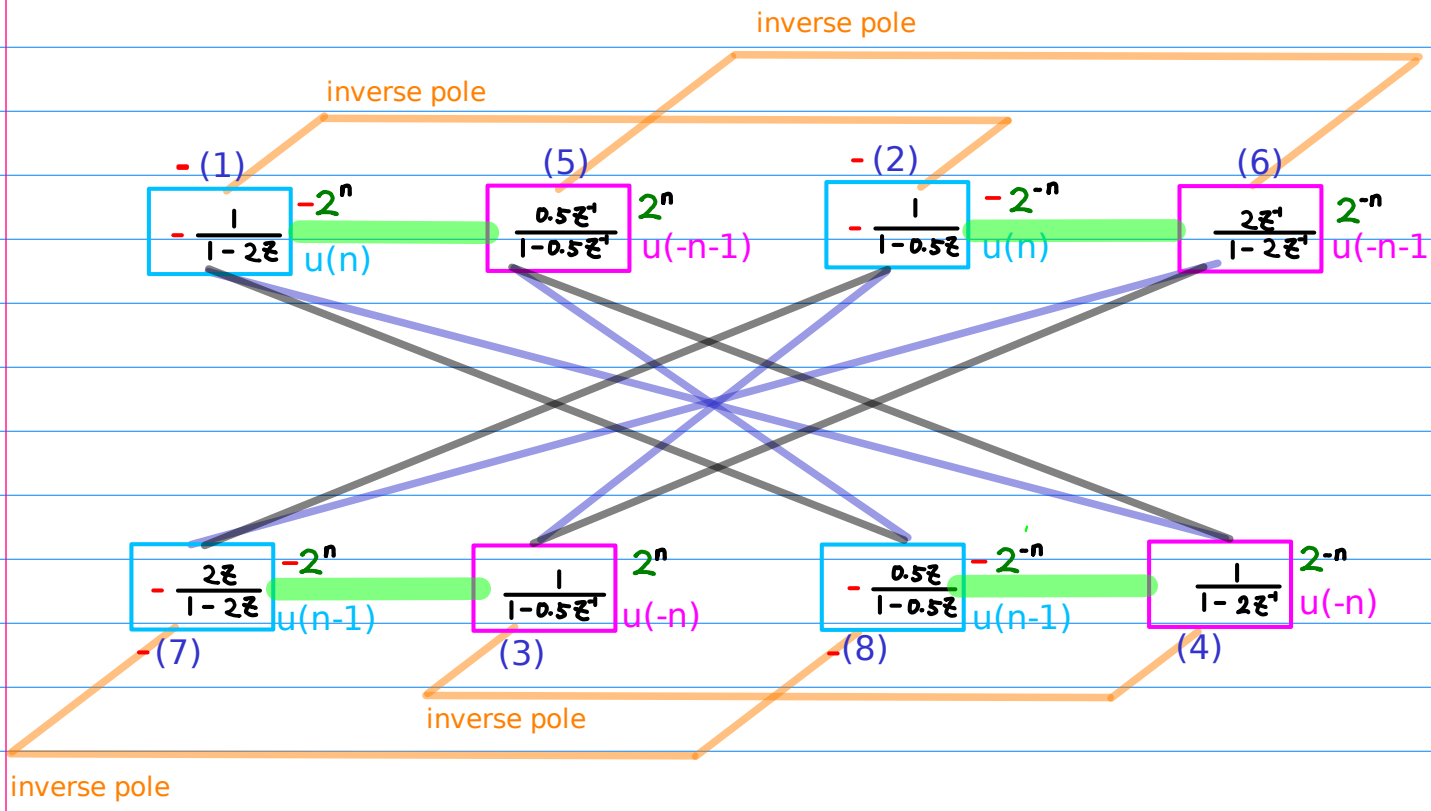
reciprocal power
shift of a range

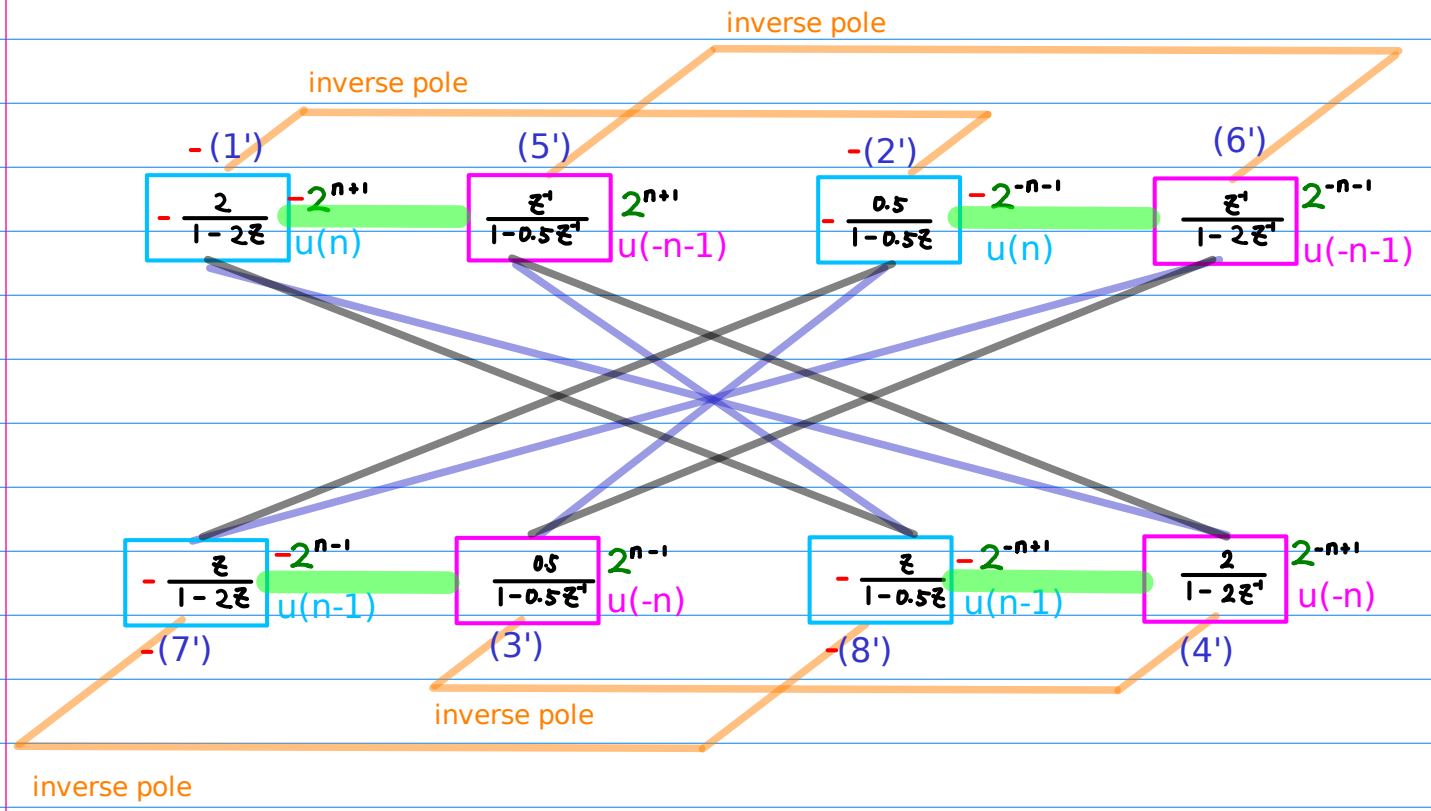


Time Reversal $-1, z^{-1}$

(1) (5)
(7) (3)

(2) (6)
(8) (4)





(1) (5)
(8) (4)

(2) (6)
(7) (3)

(1') (5')
(8') (4')

(2')(6')
(7')(3')

Partial fractions and geometric power series

$P^{-1} = 0.5$

$P = 2$

-(1) $-\frac{1}{1-2z}$ $-2^n\ u(n)$

(5) $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^n\ u(-n-1)$

-(2) $-\frac{1}{1-0.5z}$ $-2^{-n}\ u(n)$

(6) $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n}\ u(-n-1)$

-(8) $-\frac{0.5z}{1-0.5z}$ $-2^n\ u(n-1)$

(4) $\frac{1}{1-2z^{-1}}$ $2^n\ u(-n)$

-(7) $-\frac{2z}{1-2z}$ $-2^{-n}\ u(n-1)$

(3) $\frac{1}{1-0.5z^{-1}}$ $2^{-n}\ u(-n)$

-(1') $-\frac{2}{1-2z}$ $-2^{n+1}\ u(n)$

(5') $\frac{z^{-1}}{1-0.5z^{-1}}$ $2^{n+1}\ u(-n-1)$

-(2') $-\frac{0.5}{1-0.5z}$ $-2^{-n-1}\ u(n)$

(6') $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1}\ u(-n-1)$

-(8') $-\frac{z}{1-0.5z}$ $-2^{-n+1}u(n-1)$

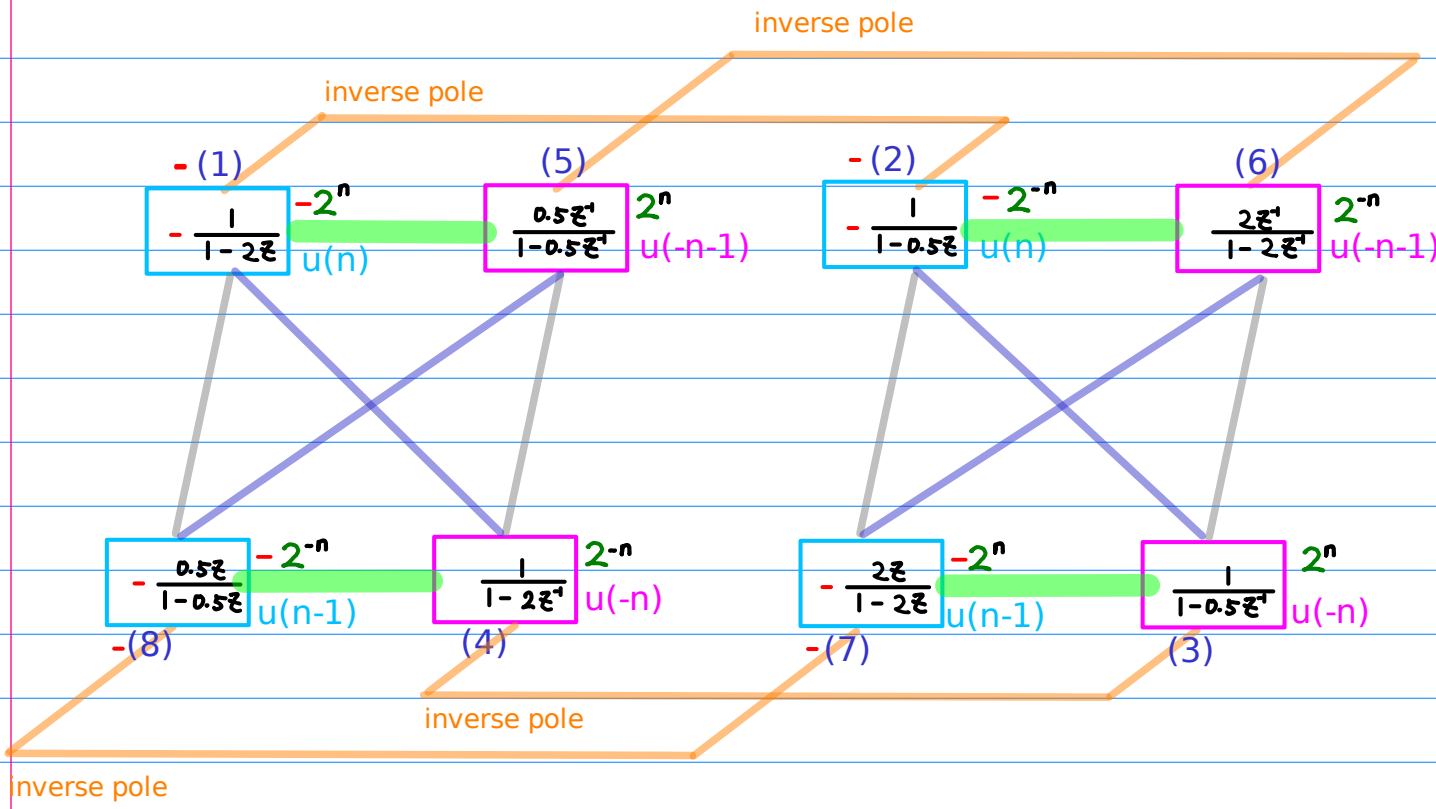
(4') $\frac{2}{1-2z^{-1}}$ $2^{-n+1}\ u(-n)$

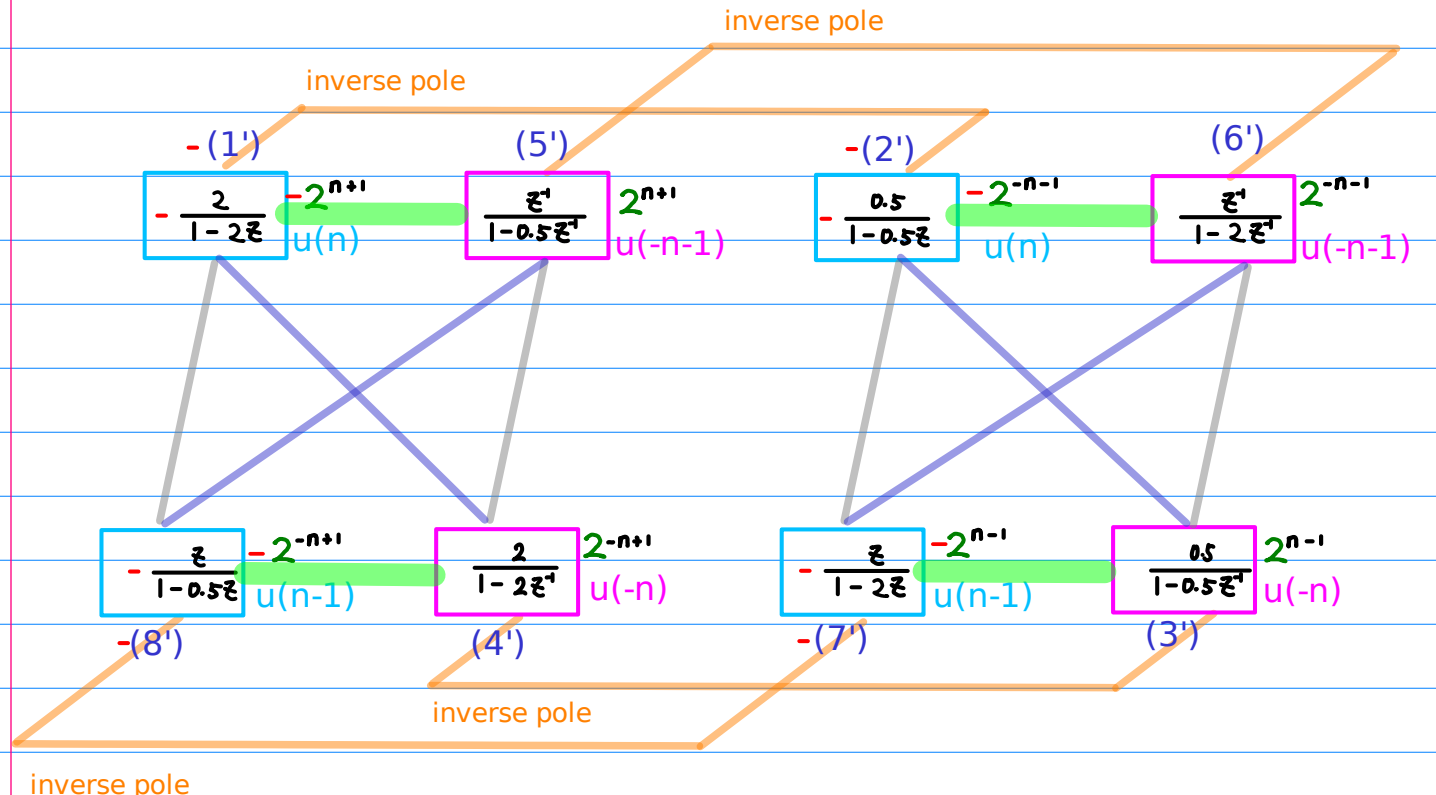
-(7') $-\frac{z}{1-2z}$ $-2^{n-1}\ u(n-1)$

(3') $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1}\ u(-n)$

Time Reversal $-1, z^{-1}$

(1) (5) (2) (6)
(8) (4) (7) (3)





Time Reversal in z-Transform

Time reversal x[-n] X(z^-1)

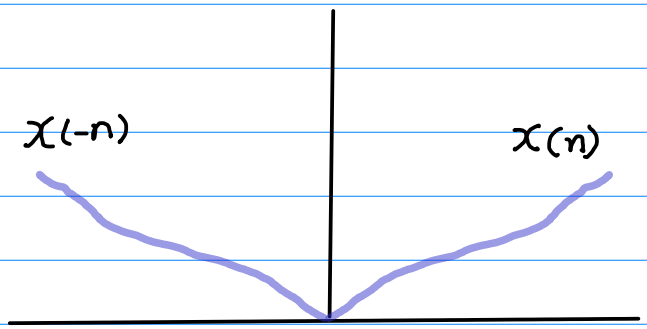
.

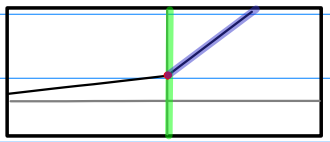
$$\mathcal{Z}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{m}$$

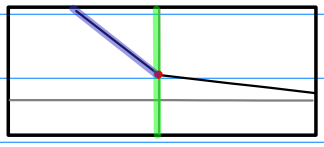
$$= \sum_{m=-\infty}^{\infty} x(m)\left(z^{-1}\right)^{-m}$$

$$= X(z^{-1})$$





2^n
u(n)



2^{-n}
u(-n)

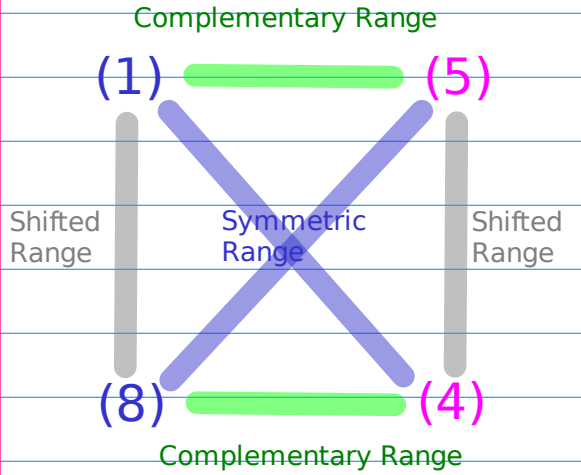
(1) (5)
(7) (3)

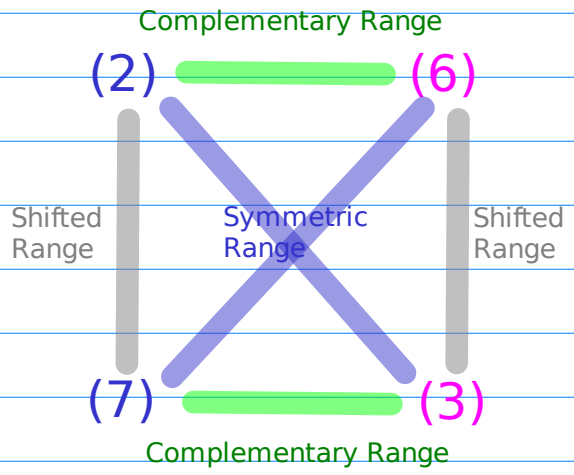
(2) (6)
(8) (4)

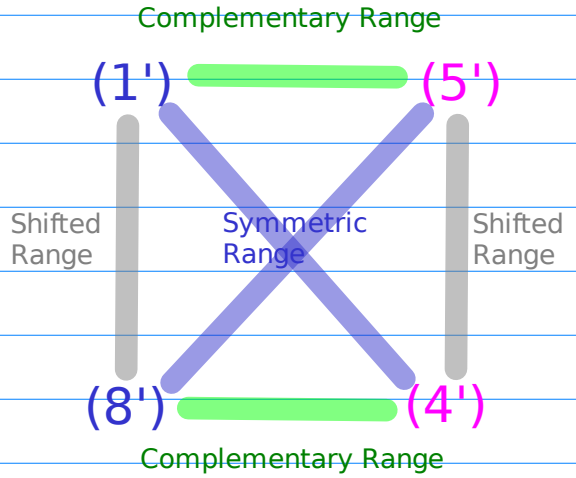
(1') (5')
(7') (3')

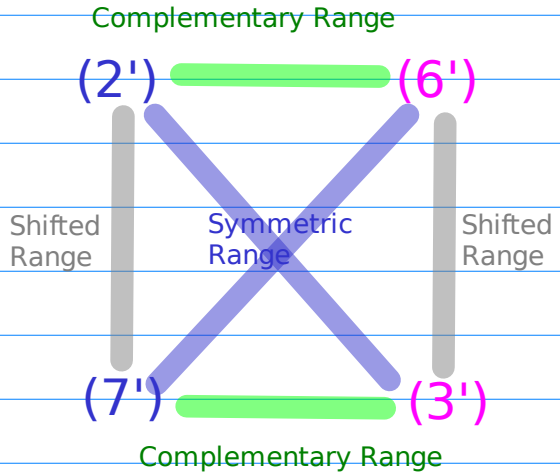
(2') (6')
(8') (4')

Complementary, Shifted, Symmetric Ranges

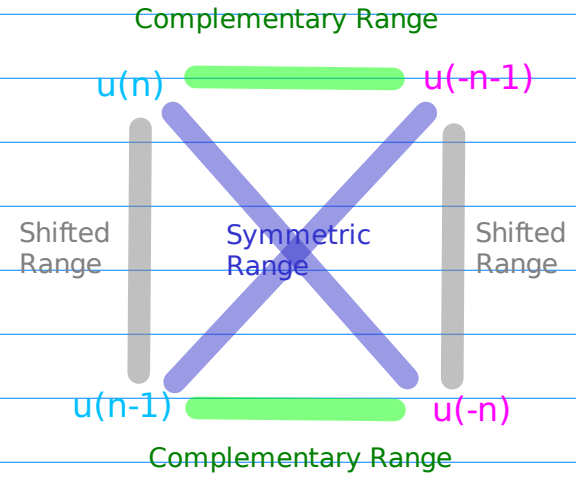


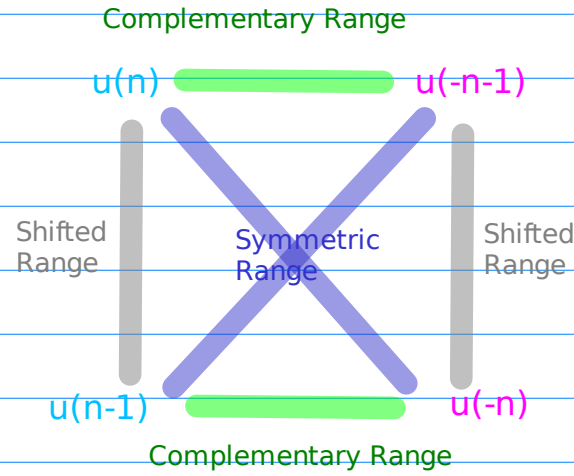


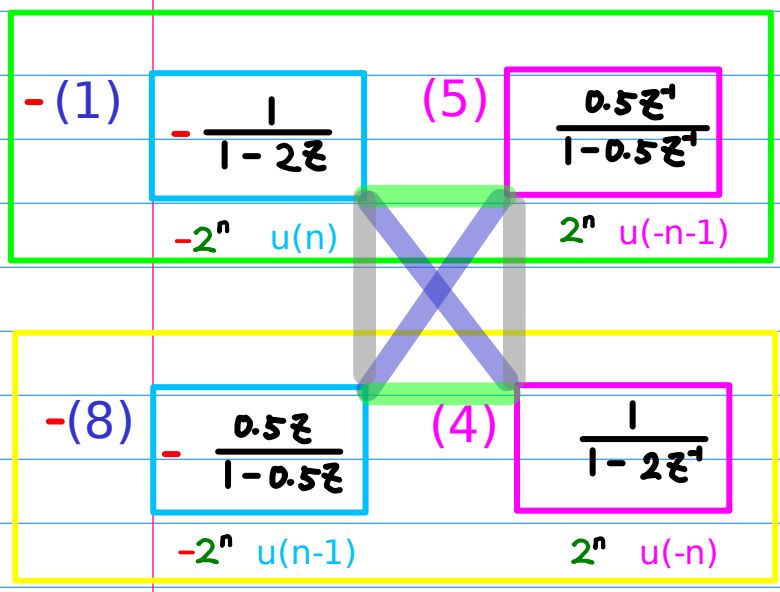


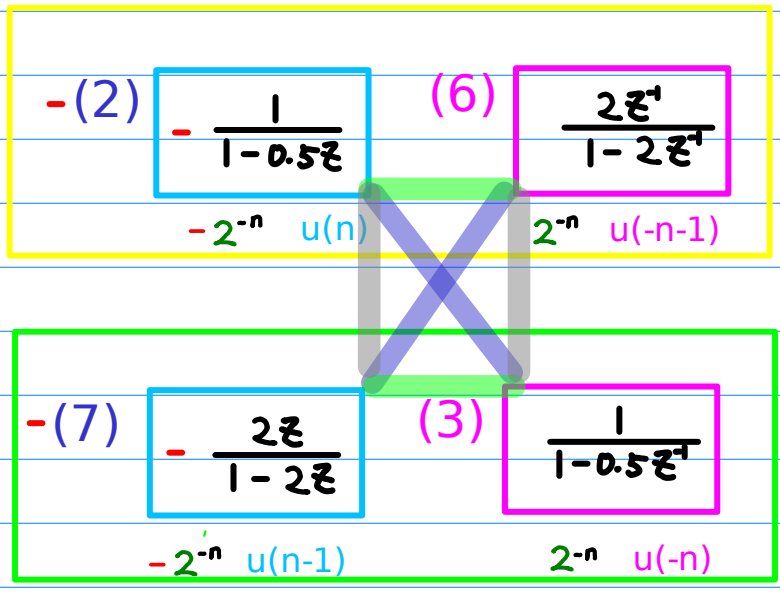


$a^n \quad a^{-n}$



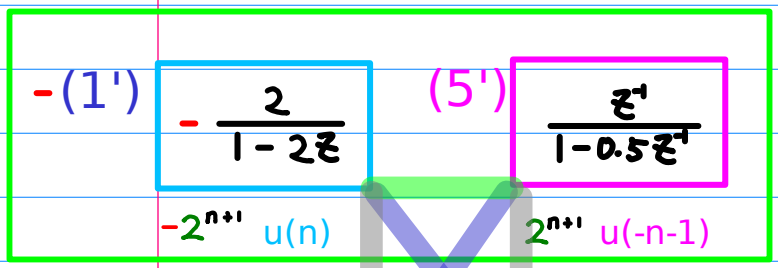


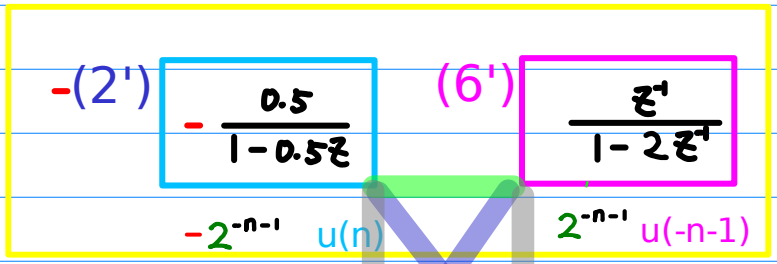

-(1)
$-\frac{1}{1-2z}$
$-2^n\ u(n)$
(5)
$\frac{0.5z^{-1}}{1-0.5z^{-1}}$
$2^n\ u(-n-1)$

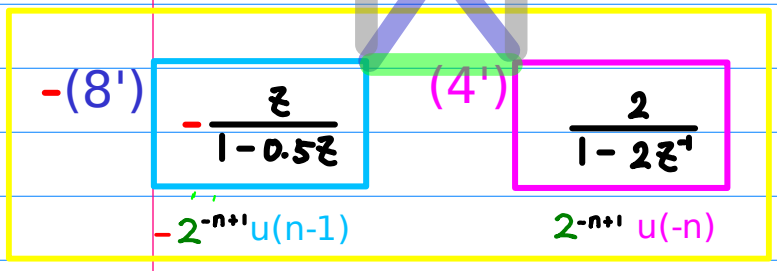

-(2)
$-\frac{1}{1-0.5z}$
$-2^{-n}\ u(n)$
(6)
$\frac{2z^{-1}}{1-2z^{-1}}$
$2^{-n}\ u(-n-1)$

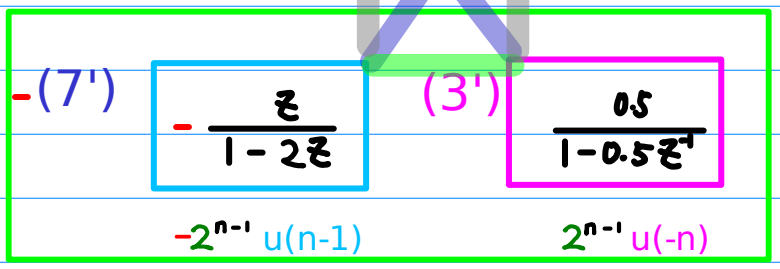
-(8)
$-\frac{0.5z}{1-0.5z}$
$-2^n\ u(n-1)$
(4)
$\frac{1}{1-2z^{-1}}$
$2^n\ u(-n)$

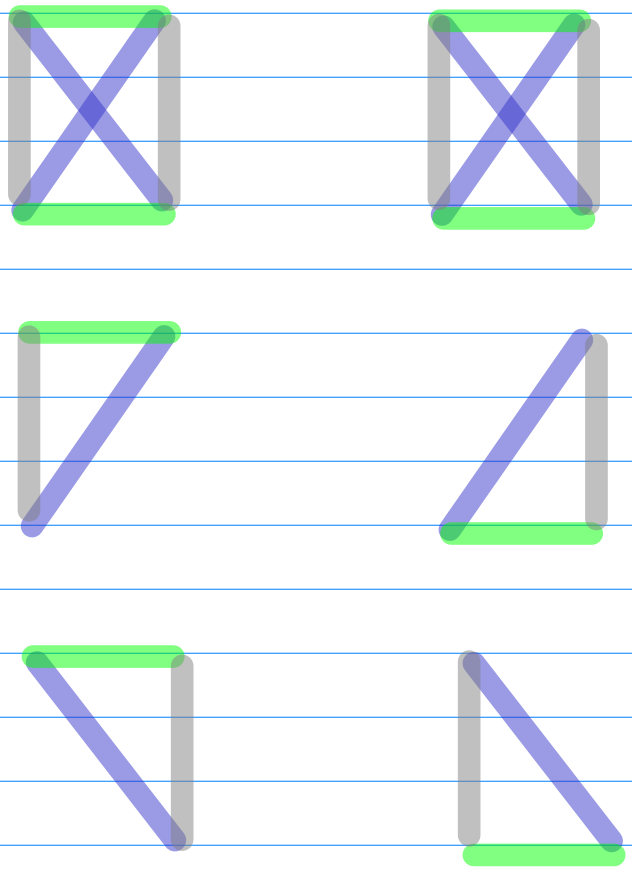
-(7)
$-\frac{2z}{1-2z}$
$-2^{-n}\ u(n-1)$
(3)
$\frac{1}{1-0.5z^{-1}}$
$2^{-n}\ u(-n)$


-(1')
$-\frac{2}{1-2z}$
$-2^{n+1}\ u(n)$
(5')
$\frac{z^{-1}}{1-0.5z^{-1}}$
$2^{n+1}\ u(-n-1)$


-(2')
$-\frac{0.5}{1-0.5z}$
$-2^{-n-1}\ u(n)$
(6')
$\frac{z^{-1}}{1-2z^{-1}}$
$2^{-n-1}\ u(-n-1)$

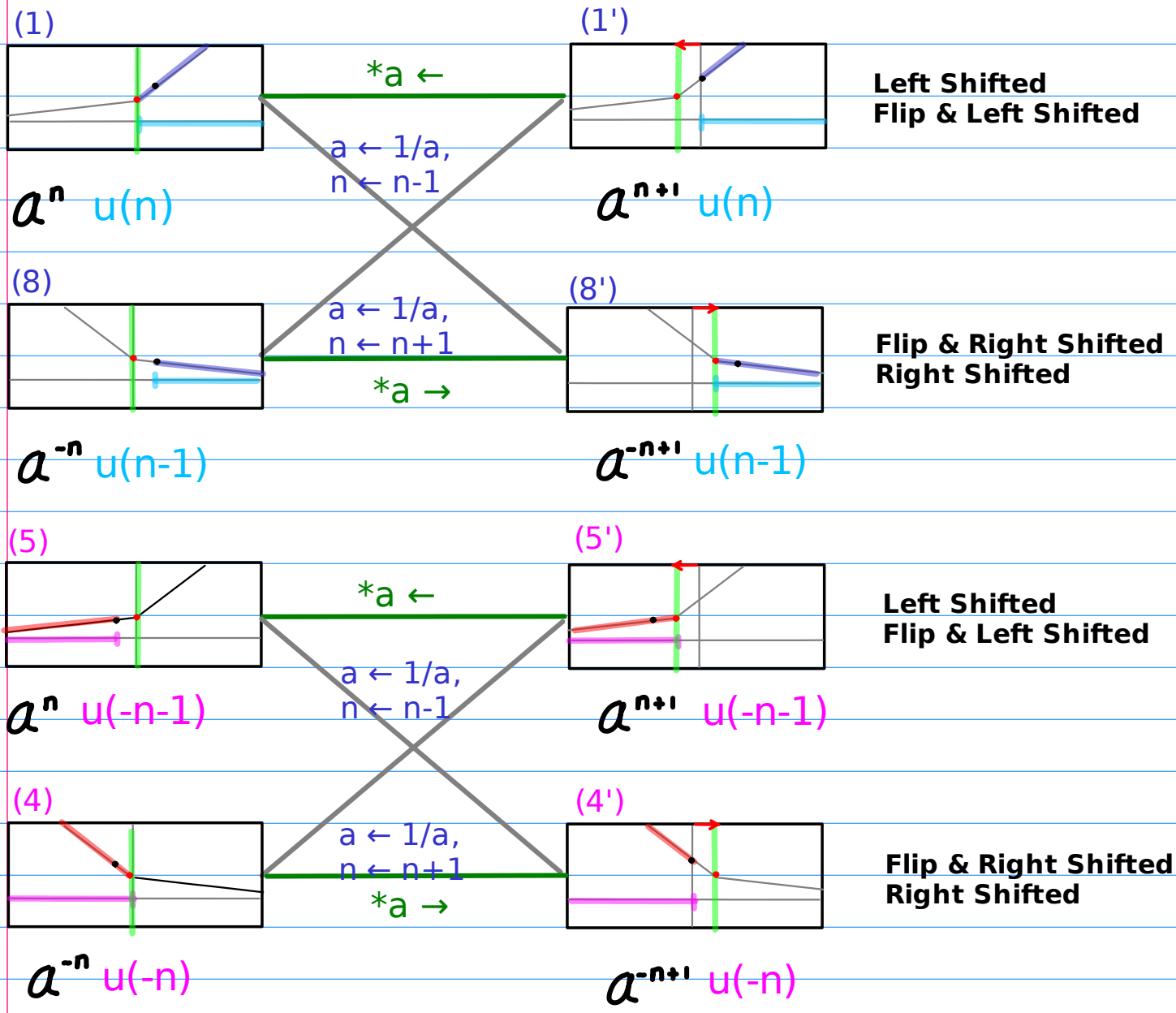

-(8')
$-\frac{z}{1-0.5z}$
$-2^{-n+1}u(n-1)$
(4')
$\frac{2}{1-2z^{-1}}$
$2^{-n+1}\ u(-n)$


-(7')
$-\frac{z}{1-2z}$
$-2^{n-1}\ u(n-1)$
(3')
$\frac{0.5}{1-0.5z^{-1}}$
$2^{n-1}\ u(-n)$



(1) (1')
(8) (8')
(5) (5')
(4) (4')

Flip & Left / Right Shift



(2) (2')
(7) (7')
(6) (6')
(3) (3')

Flip & Left / Right Shift