

Temporal Characteristics of Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 Random Variables
- 2 Random Processes
- 3 Stochastic Process

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a particular value : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, \dots, s_n\}$
- an **outcome** (an element of S) : s

$$s \rightarrow X(s)$$

$$x = X(s)$$

$$s \rightarrow x$$

Understanding Random Variables (1)

- **random variables** are used to quantify **outcomes** of a random occurrence, and therefore, can take on many **values**.
- **random variables** are required to be **measurable** and are typically real numbers.

for example, the letter **X** may be designated to represent the *sum* of the resulting numbers after *three dice* are rolled.

therefore, X could be 3 ($1 + 1 + 1$), 18 ($6 + 6 + 6$), or somewhere between 3 and 18

<https://www.investopedia.com/terms/r/random-variable.asp>

Understanding Random Variables (2)

- A **random variable** is different from an algebraic variable.
 - The variable in an algebraic equation is an unknown value that can be calculated.
 - The equation $10 + x = 13$ shows that we can calculate the specific value for x which is 3.
 - a **random variable** has a set of values, and any of those values *could be* the resulting **outcome** as seen in the example of the dice above.

<https://www.investopedia.com/terms/r/random-variable.asp>

Understanding Random Variables (3)

- A typical example of a random variable is the **outcome** of a **coin toss**.
 - Consider a **probability distribution** in which the outcomes of a random event are not **equally likely** to happen.
 - If the **random variable** Y is the **number** of **heads** we get from **tossing two coins**, then Y could be 0, 1, or 2.
(no heads, one head, or both heads)

<https://www.investopedia.com/terms/r/random-variable.asp>

Understanding Random Variables (4)

- However, the two coins land in four different ways: TT, HT, TH, and HH. Therefore, the $P(Y=0) = 1/4$ since we have one chance of getting no heads (i.e., two tails [TT] when the coins are tossed). Similarly, the probability of getting two heads (HH) is also $1/4$. Notice that getting one head has a likelihood of occurring twice: in HT and TH. In this case, $P(Y=1) = 2/4 = 1/2$.

<https://www.investopedia.com/terms/r/random-variable.asp>

Formal definition of a random variable

A random variable X is a measurable function $X: \Omega \rightarrow E$ from a set of possible outcomes Ω to a measurable space E .

The technical axiomatic definition requires Ω to be a sample space of a probability triple (Ω, \mathcal{F}, P)

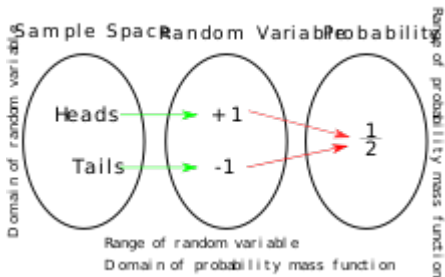
A random variable is often denoted by capital roman letters such as X, Y, Z, T .

The probability that X takes on a value in a measurable set $S \subseteq E$ is written as

$$P(X \in S) = P(\{\omega \in \Omega \mid X(\omega) \in S\})$$

https://en.wikipedia.org/wiki/Random_variable

Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

https://en.wikipedia.org/wiki/Random_variable

Probability Space (1)

In probability theory, a probability space or a probability triple (Ω, \mathcal{F}, P) is a mathematical construct that provides a formal model of a random process or "experiment".

For example, one can define a probability space which models the throwing of a die

https://en.wikipedia.org/wiki/Probability_space

Probability Space (2)

A probability space consists of three elements

A sample space, Ω , which is the set of all possible outcomes.

An event space, which is a set of events \mathcal{F} ,

an event being a set of outcomes in the sample space.

A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

https://en.wikipedia.org/wiki/Probability_space

Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be $\{1, 2, 3, 4, 5, 6\}$. For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as $\{5\}$ ("the die lands on 5"), as well as complex events such as $\{2, 4, 6\}$ ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example, $\{5\}$ would be mapped to $1 / 6$ $1/6$, and $\{2, 4, 6\}$ would be mapped to $3/6 = 1/2$.

https://en.wikipedia.org/wiki/Probability_space

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t, s)$$

assigning a **time function** to every **outcome** s_i

$$s_i \rightarrow x(t, s_i)$$

Random Process (2)

A random process

the family of such **time functions** is called a **random process**

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a random variable X is a rule which assigns a number to every outcome e of an experiment.

The random variable is a function $X(e)$ that maps the set of experiment outcomes to the set of numbers.

A random process is a rule that maps every outcome e of an experiment to a function $X(t, e)$.

A random process is usually conceived of as a function of time, but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

The function $X(u, v, e)$ would be a function whose value depended on the location (u, v) and the outcome e .

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents a family or ensemble of **time functions**

$X(t, s)$ represents

- a **single time function** $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a **sample function**,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a **random process** $X(t)$
for a given **outcome** s_j

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t)$$

$$s_1 \rightarrow x_1(t)$$

$$X(t, s_2) = x_2(t)$$

$$s_2 \rightarrow x_2(t)$$

...

...

$$X(t, s_n) = x_n(t)$$

$$s_n \rightarrow x_n(t)$$

$S = \{s_1, s_2, s_3, \dots, s_n\}$ a sample space

$X(t) = \{x_1(t), x_2(t), x_3(t), \dots, x_n(t)\}$ a random process

Random variables with time

a **random process** $X(t,s)$ represents a **single time function** when t is a variable and s is fixed at an outcome

a random process $X(t,s)$ represents a **single random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \quad \text{random variable}$$

$$X(t, s) = X(t) \quad \text{random process}$$

An alphabet

the **alphabet** of $X(t)$

the set of its possible values

- the values of **time** t for which a **random process** is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a **random process** is defined
 - continuous time
 - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
 - continuous alphabet
 - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
 - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
 - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
 - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
 - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic** if **future values** of any sample function cannot be predicted exactly from **observed past values**
- A process is **deterministic** if **future values** of any sample function can be predicted from **observed past values**

Deterministic Random Process Example (1)

$$X(t) = A \cos(\omega_0 t + \Theta)$$

A , Θ , or ω_0 (or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i}t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \quad \implies \quad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a **family of functions**. Imagine a giant strip chart recording in which each pen is identified with a different θ . This family of functions is traditionally called an **ensemble**.
- A **single function** $X(t, \theta_k)$ is selected by the **outcome** θ_k . This is just a **time function** that we could call $X_k(t)$. Different **outcomes** give us different **time functions**.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a **random variable**. Its value depends on the **outcome** θ .
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a **number**.

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>
<https://en.wiktionary.org/wiki/stochastic>

Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is n -dimensional **Euclidean space** \mathbb{R}^n or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted, by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$, although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to the **random variable** with the **index** t , and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write, for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space** (Ω, \mathcal{F}, P) ,

- Ω is a **sample space**,
- \mathcal{F} is a σ -**algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T ,
- all take values in the same **mathematical space** S , which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** (Ω, \mathcal{F}, P) and a **measurable space** (S, Σ) , a **stochastic process** is a **collection** of S -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point $t \in T$ had the meaning of time, so $X(t)$ is a **random variable** representing a value observed at time t .

A **stochastic process** can also be written as $\{X(t, \omega) : t \in T\}$ to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a S^T -valued **random variable**, where S^T is the space of all the possible functions from the set T into the space S .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n -dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t, \omega) : t \in T\}$ is a **stochastic process**, then for any point $\omega \in \Omega$, the mapping $X(\cdot, \omega) : T \rightarrow S$, is called a **sample function**, a **realization**, or, particularly when T is interpreted as time, a **sample path** of the **stochastic process** $\{X(t, \omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S .

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process

