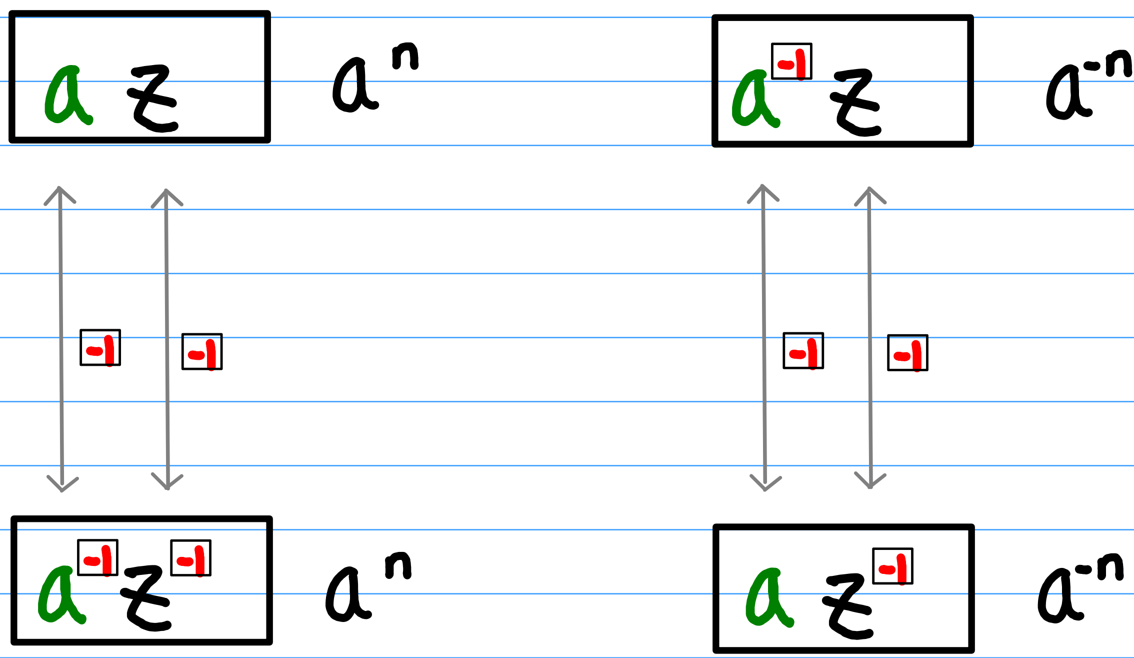
Laurent Series and z-Transform - Geometric Series Combinations

A

20200418 Sat

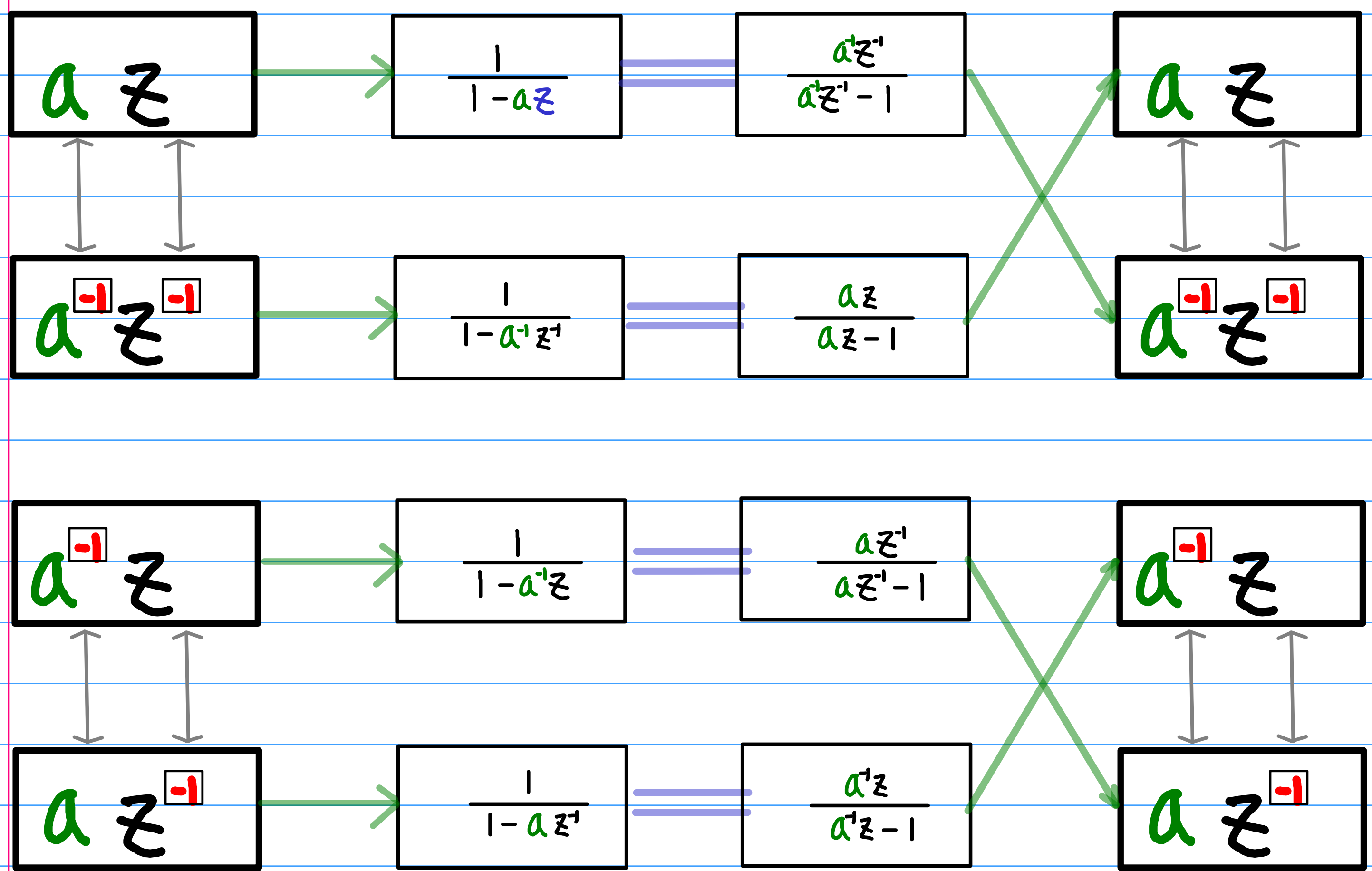
Combinations of a and z
-- common ratio in a geometric series



the same formula,
different representations

Geometric Series

common ratio **common ratio**



two equivalent representations
of geometric series

the same formula with different ROCs

different Geometric Series

complementary ranges

common ratio			inversed common ratio
$a z$	causal u(n): $\frac{1}{1-az}$ $\quad \lvert z\rvert < a^{-1}$	anti-causal u(-n-1): $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$	$a^{-1} z^{-1}$
	$a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots$	$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$	
$a^{-1} z^{-1}$	anti-causal u(-n): $\frac{1}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$	causal u(n-1): $-\frac{az}{1-az}$ $\quad \lvert z\rvert < a^{-1}$	$a z$
	$a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots$	$-(a^1 z^1 + a^2 z^2 + a^3 z^3 + \cdots)$	
$a^{-1} z$	causal u(n): $\frac{1}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$	anti-causal u(-n-1): $-\frac{az^{-1}}{1-az^{-1}}$ $\quad \lvert z\rvert > a$	$a z^{-1}$
	$a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \cdots$	$-(a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots)$	
$a z^{-1}$	anti-causal u(-n): $\frac{1}{1-az^{-1}}$ $\quad \lvert z\rvert > a$	causal u(n-1): $-\frac{a^{-1}z}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$	$a^{-1} z$
	$a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \cdots$	$-(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \cdots)$	

geometric series starting with a unit term	geometric series starting with a non-unit term
non-shifted range u(n), u(-n)	shifted range u(n-1), u(-n-1)

the same formula
with different ROCs

different Geometric Series

common ratio			the same common ratio
$a\,z$	causal u(n) $\frac{1}{1-az}$ $\;|z|<a^{-1}$ $a^0z^0 + a^1z^1 + a^2z^2 + \cdots$	left shifted u(-n-1) $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\;|z|>a^{-1}$ $-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots)$	$a^{-1}z^{-1}$
$a^{-1}z^{-1}$	anti-causal u(-n) $\frac{1}{1-a^{-1}z^{-1}}$ $\;|z|>a^{-1}$ $a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$	right shifted u(n-1) $-\frac{az}{1-az}$ $\;|z|<a^{-1}$ $-(a^1z^1 + a^2z^2 + a^3z^3 + \cdots)$	$a\,z$
$a^{-1}z$	causal u(n) $\frac{1}{1-a^{-1}z}$ $\;|z|<a$ $a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$	left shifted u(-n-1) $-\frac{az^{-1}}{1-az^{-1}}$ $\;|z|>a$ $-(a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots)$	$a\,z^{-1}$
$a\,z^{-1}$	anti-causal u(-n) $\frac{1}{1-az^{-1}}$ $\;|z|>a$ $a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$	right shifted u(n-1) $-\frac{a^{-1}z}{1-a^{-1}z}$ $\;|z|<a$ $-(a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots)$	$a^{-1}z$

geometric series starting with a unit term	geometric series starting with a non-unit term
non-shifted range u(n), u(-n)	shifted range u(n-1), u(-n-1)

Geometric Power Series Property (1)

Each representation has it own ROC
(Region of Convergence)

common ratio $a z$ $\longrightarrow$ $|z| < a^{-1}$ ROC

common ratio $a^{-1} z^{-1}$ $\longrightarrow$ $|z| > a^{-1}$ ROC

common ratio $a^{-1} z$ $\longrightarrow$ $|z| < a$ ROC

common ratio $a z^{-1}$ $\longrightarrow$ $|z| > a$ ROC

Geometric Power Series Property (2)

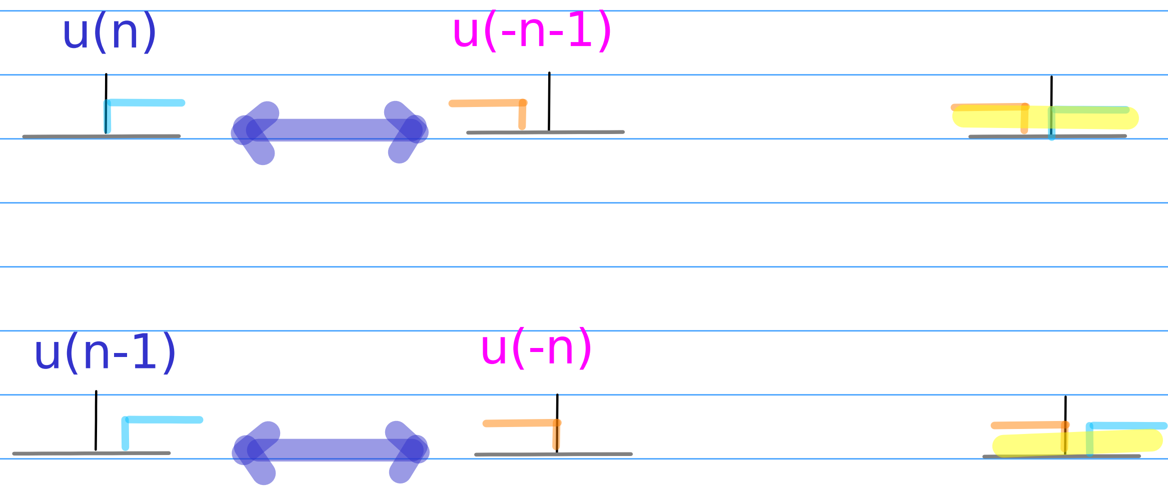
Starting terms

		geometric series starting with **a unit term**	geometric series starting with **a non-unit term (common ratio)**		
z	causal	$\dfrac{1}{1-az}$	$-\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	anti-causal	z^{-1}
z^{-1}	anti-causal	$\dfrac{1}{1-a^{-1}z^{-1}}$	$-\dfrac{az}{1-az}$	causal	z
z	causal	$\dfrac{1}{1-a^{-1}z}$	$-\dfrac{az^{-1}}{1-az^{-1}}$	anti-causal	z^{-1}
z^{-1}	anti-causal	$\dfrac{1}{1-az^{-1}}$	$-\dfrac{a^{-1}z}{1-a^{-1}z}$	causal	z

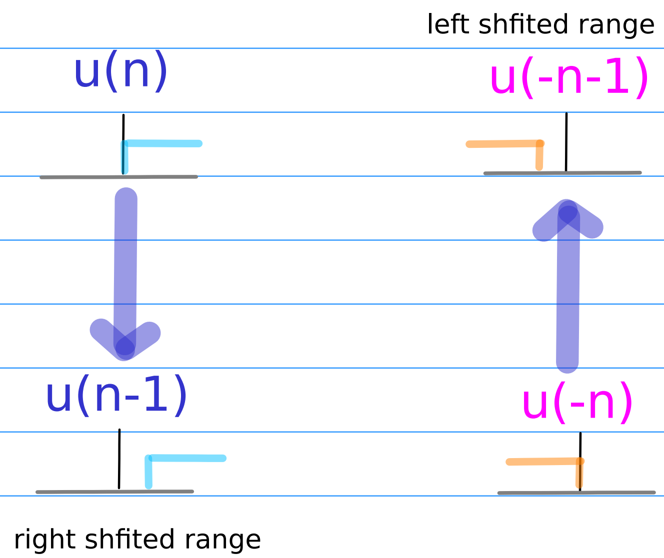
related to shifting

Geometric Power Series Property (3)

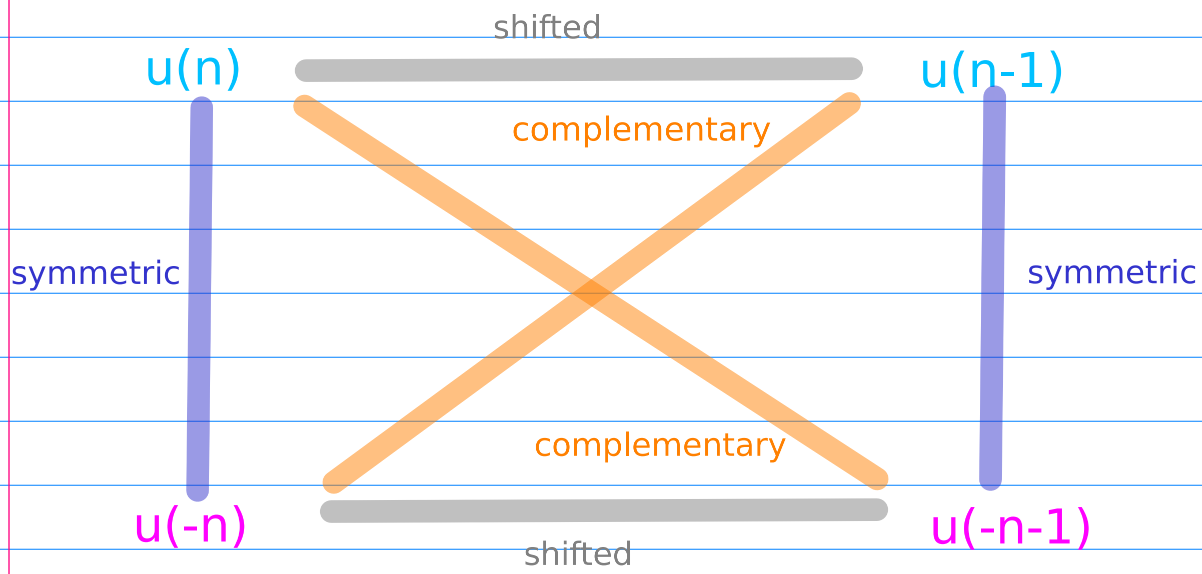
Complementary Ranges



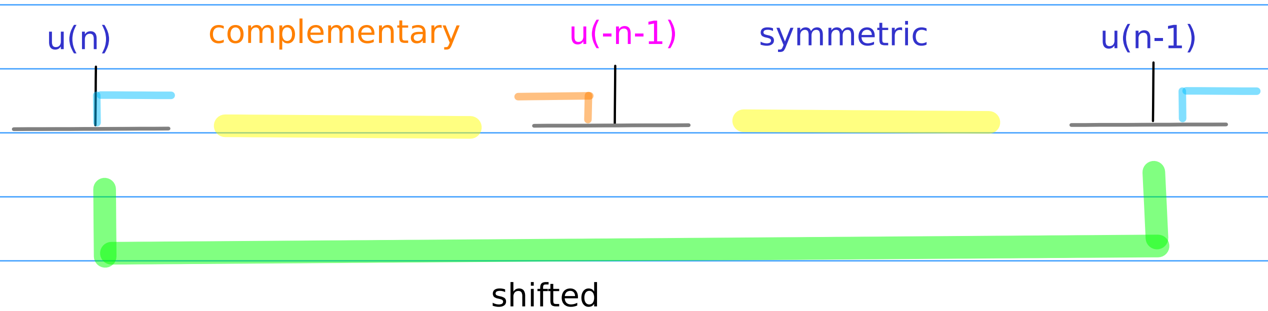
Shifted Ranges

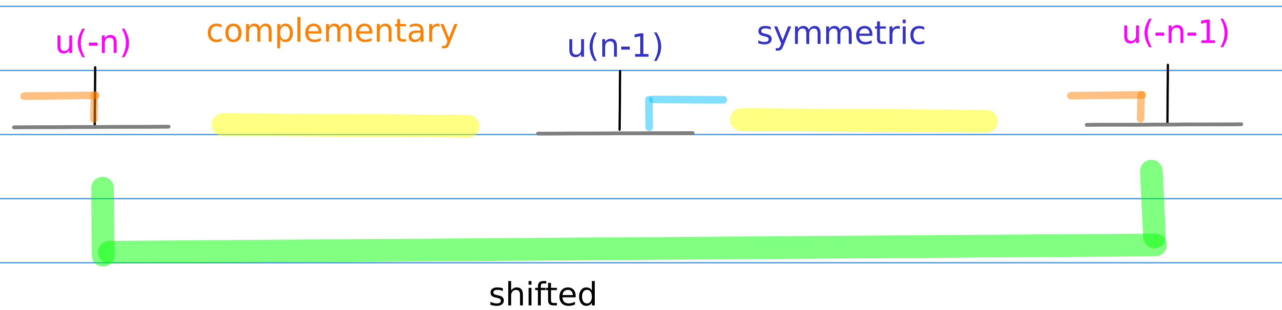


Geometric Power Series Property (4)



u(n) complementary u(-n-1) symmetric u(n-1)
u(-n) complementary u(n-1) symmetric u(-n-1)





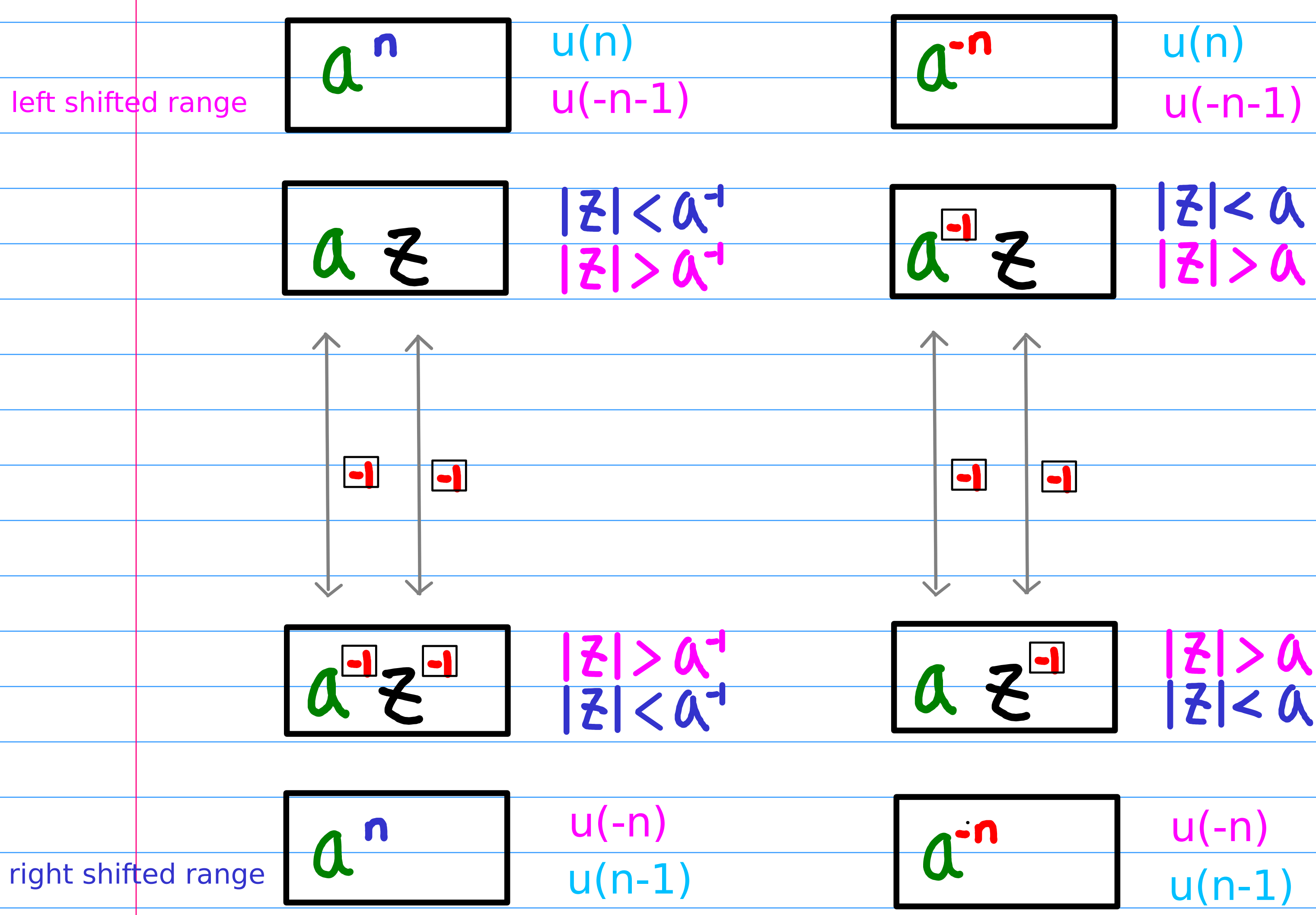
Geometric Power Series Property (5)

non-shifted range	shifted range
u(n), u(-n)	u(n-1), u(-n-1)
geometric series starting with	geometric series starting with
a unit term	a non-unit term (common ratio)

u(n)	$\frac{1}{1-az}$	complementary	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	u(-n-1)
u(-n)	$\frac{1}{1-a^{-1}z^{-1}}$	complementary	$-\frac{az}{1-az}$	u(n-1)
u(n)	$\frac{1}{1-a^{-1}z}$	complementary	$-\frac{az^{-1}}{1-az^{-1}}$	u(-n-1)
u(-n)	$\frac{1}{1-az^{-1}}$	complementary	$-\frac{a^{-1}z}{1-a^{-1}z}$	u(n-1)

u(n)	$\frac{1}{1-az}$	shifted	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	u(-n-1)
u(-n)	$\frac{1}{1-a^{-1}z^{-1}}$	shifted	$-\frac{az}{1-az}$	u(n-1)
u(n)	$\frac{1}{1-a^{-1}z}$	shifted	$-\frac{az^{-1}}{1-az^{-1}}$	u(-n-1)
u(-n)	$\frac{1}{1-az^{-1}}$	shifted	$-\frac{a^{-1}z}{1-a^{-1}z}$	u(n-1)

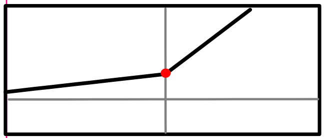
Common Ratio and ROC



Each common ratio has two represenations Sequences

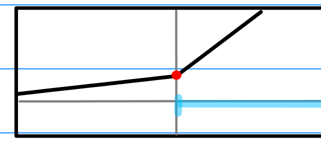
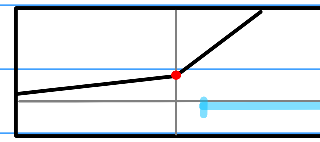
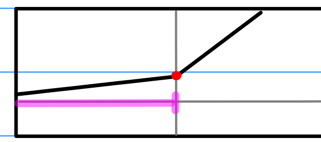
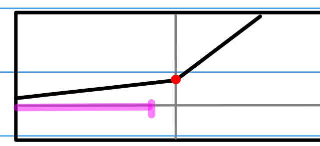
Each representation has it own ROC Ranges

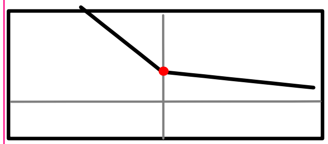
The two representations have complementary ROC's complementary ROC's

a^n 

Geometric Series Combinations (1)

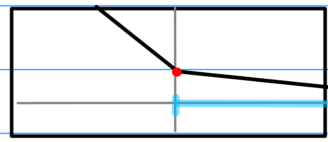
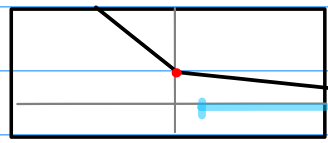
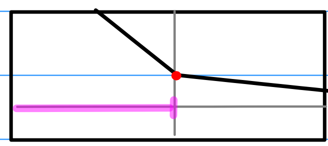
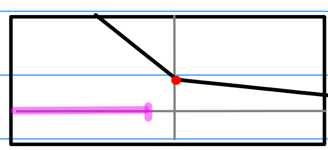
* inverted relation is ignored

Common Ratio		2 Geometric Series	2 Sequences
$\boxed{a\,z}$		$\dfrac{1}{1-az}$ $\lvert z\rvert < a^{-1}$ $a^n\ u(n)$	 a^n
	right shifted	$\dfrac{az}{1-az}$ $\lvert z\rvert < a^{-1}$ $a^n\ u(n-1)$	 a^n
$\boxed{a^{\boxed{-1}} z^{\boxed{-1}}}$		$\dfrac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$ $a^n\ u(-n)$	 a^n
	left shifted	$\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$ $a^n\ u(-n-1)$	 a^n

a^{-n} 

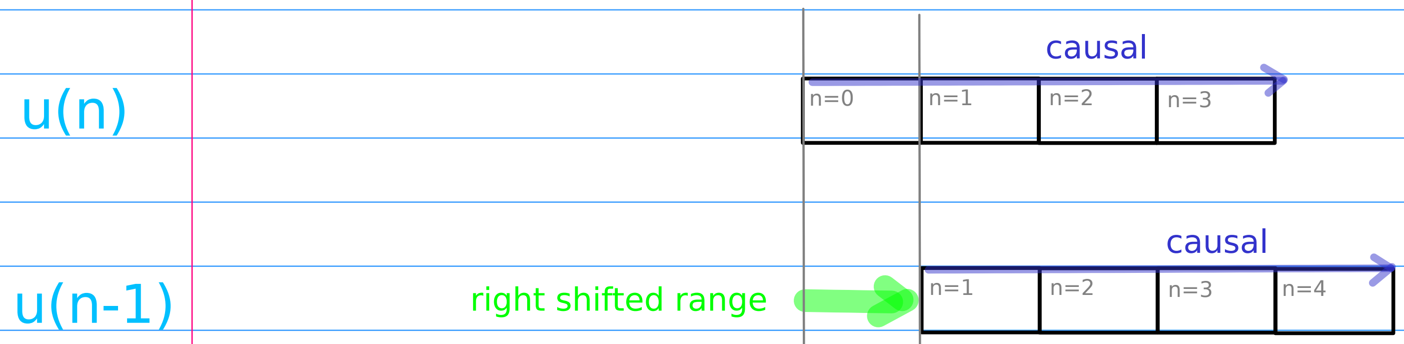
Geometric Series Combinations (2)

* inverted relation is ignored

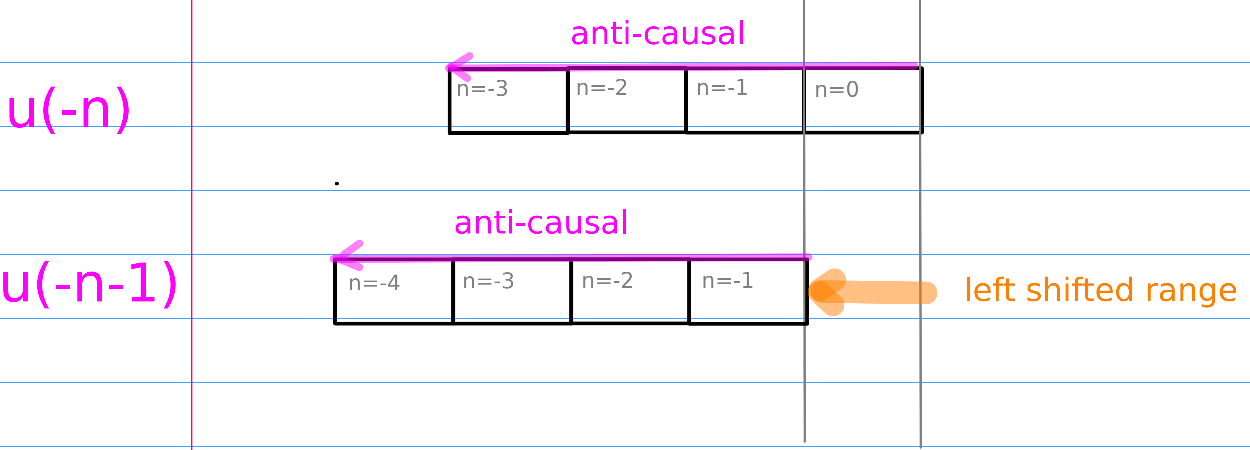
Common Ratio	**2 Geometric Series**	**2 Sequences**
$a^{\boxed{-1}} z$	$\frac{1}{1-a^{-1}z}$ $\lvert z \rvert < a$ $a^{-n}\, u(n)$	 a^{-n}
right shifted	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z \rvert < a$ $a^{-n}\, u(n-1)$	 a^{-n}
$a\, z^{\boxed{-1}}$	$\frac{1}{1-a z^{-1}}$ $\lvert z \rvert > a$ $a^{-n}\, u(-n)$	 a^{-n}
left shifted	$\frac{a z^{-1}}{1-a z^{-1}}$ $\lvert z \rvert > a$ $a^{-n}\, u(-n-1)$	 a^{-n}

Shift Relations of Ranges

Right Shifted Range Relation

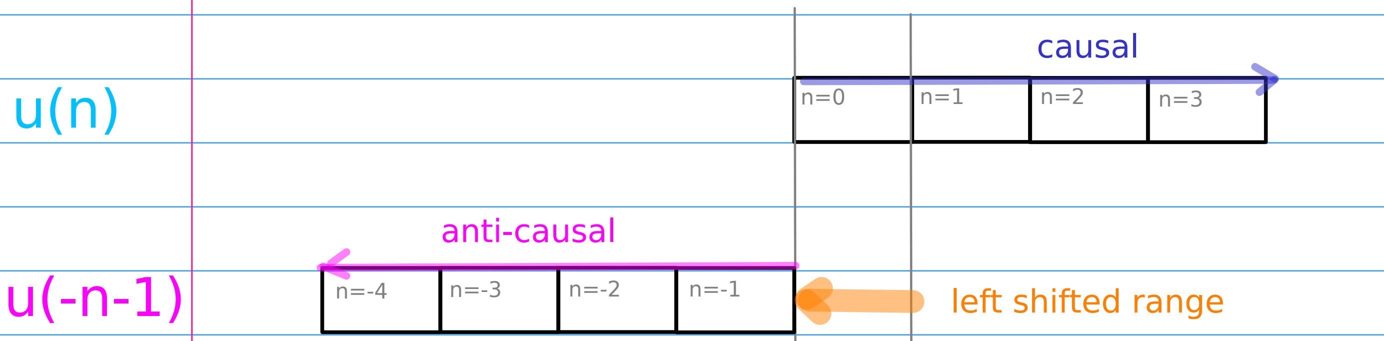


Left Shifted Range Relation

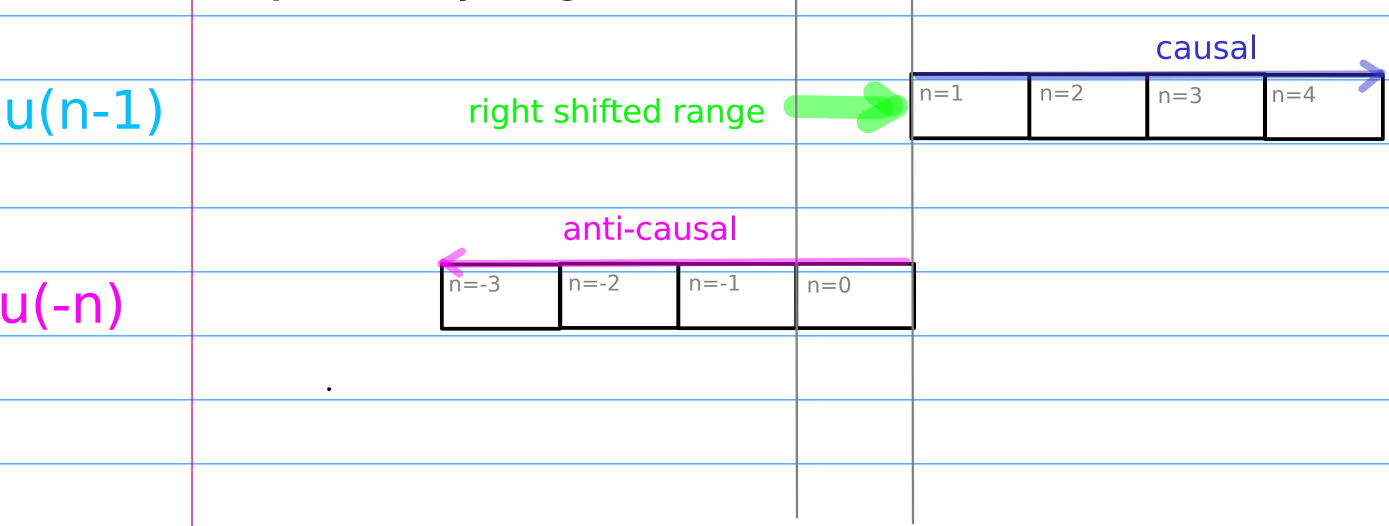


Complementary Relations of Ranges

Complementary Range Relation



Complementary Range Relation



[Complementary Range & Inverted Relation]

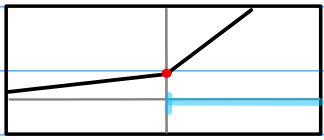
* inverted relation is ignored

$a \ z$

$\frac{1}{1-az}$ $|z|<a^{-1}$

$a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots$

a^n u(n)



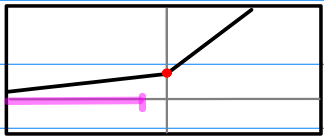
$\Big\Vert$ -1

$a^{\boxed{-1}} z^{\boxed{-1}}$

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$

$a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots$

a^n u(-n-1)

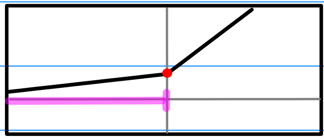


$a^{\boxed{-1}} z^{\boxed{-1}}$

$\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$

$a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \dots$

a^n u(-n)



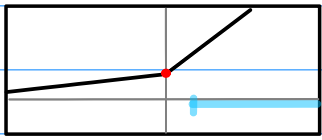
$\Big\Vert$ -1

$a \ z$

$\frac{az}{1-az}$ $|z|<a^{-1}$

$a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots$

a^n u(n-1)

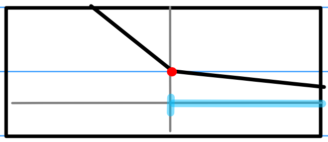


$a^{\boxed{-1}} z$

$\frac{1}{1-a^{-1}z}$ $|z|<a$

$a^0 z^0 + a^{-1} z^1 + a^{-2} z^2 + \dots$

a^{-n} u(n)



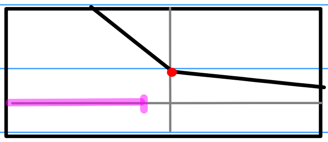
$\Big\Vert$ -1

$a \ z^{\boxed{-1}}$

$\frac{az^{-1}}{1-az^{-1}}$ $|z|>a$

$a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots$

a^{-n} u(-n-1)

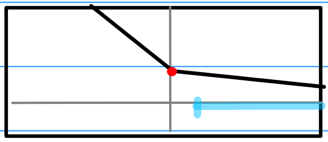


$a \ z^{\boxed{-1}}$

$\frac{1}{1-az^{-1}}$ $|z|>a$

$a^0 z^0 + a^1 z^{-1} + a^2 z^{-2} + \dots$

a^{-n} u(n-1)



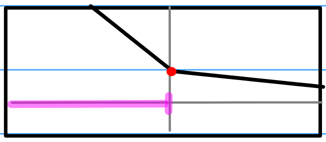
$\Big\Vert$ -1

$a^{\boxed{-1}} z$

$\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a$

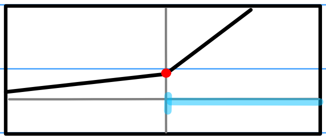
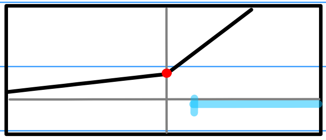
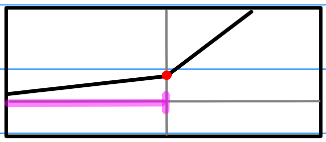
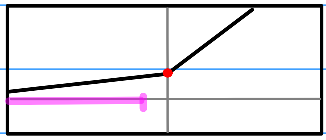
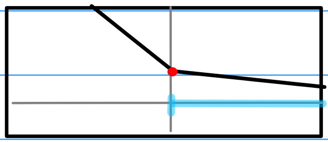
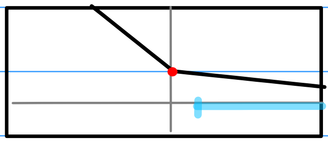
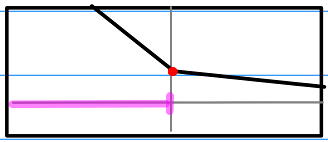
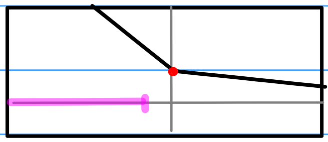
$a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots$

a^{-n} u(-n)



[Shifted Range Relation]

* inverted relation is ignored

az	$\frac{1}{1-az}$ $\|z\|<a^{-1}$	$a^0z^0 + a^1z^1 + a^2z^2 + \cdots$ a^n u(n)	
az	$\frac{az}{1-az}$ $\|z\|<a^{-1}$	$a^1z^1 + a^2z^2 + a^3z^3 + \cdots$ a^n u(n-1)	
$a^{\boxed{-1}}z^{\boxed{-1}}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	$a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots$ a^n u(-n)	
$a^{\boxed{-1}}z^{\boxed{-1}}$	$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\|z\|>a^{-1}$	$a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \cdots$ a^n u(-n-1)	
$a^{\boxed{-1}}z$	$\frac{1}{1-a^{-1}z}$ $\|z\|<a$	$a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots$ a^{-n} u(n)	
$a^{\boxed{-1}}z$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\|z\|<a$	$a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \cdots$ a^{-n} u(n-1)	
$az^{\boxed{-1}}$	$\frac{1}{1-az^{-1}}$ $\|z\|>a$	$a^0z^0 + a^1z^{-1} + a^2z^{-2} + \cdots$ a^{-n} u(-n)	
$az^{\boxed{-1}}$	$\frac{az^{-1}}{1-az^{-1}}$ $\|z\|>a$	$a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$ a^{-n} u(-n-1)	

each formula has two geometric series

- two common ratios with inversed relation

$a\ z$	$\frac{1}{1-az}$ $\quad \|z\|<a^{-1}$	$a^{n}u(n)$
		Complementary Ranges
$a^{-1}z^{-1}$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad \|z\|>a^{-1}$	$-a^{n}u(-n-1)$
$a^{-1}z^{-1}$	$\frac{1}{1-a^{-1}z^{-1}}$ $\quad \|z\|>a^{-1}$	$a^{n}u(-n)$
		Complementary Ranges
$a\ z$	$-\frac{az}{1-az}$ $\quad \|z\|<a^{-1}$	$-a^{n}u(n-1)$
$a^{-1}z$	$\frac{1}{1-a^{-1}z}$ $\quad \|z\|<a$	$a^{-n}u(n)$
		Complementary Ranges
$a\ z^{-1}$	$-\frac{az^{-1}}{1-az^{-1}}$ $\quad \|z\|>a$	$-a^{-n}u(-n-1)$
$a\ z^{-1}$	$\frac{1}{1-az^{-1}}$ $\quad \|z\|>a$	$a^{-n}u(-n)$
		Complementary Ranges
$a^{-1}z$	$-\frac{a^{-1}z}{1-a^{-1}z}$ $\quad \|z\|<a$	$-a^{-n}u(n-1)$

each common ratio is associated with
2 different sequences (represenations)

$a\,z$	$\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n}\,u(n)$
			Shifted Ranges
$a\,z$	$-\dfrac{az}{1-az}$	$\lvert z\rvert < a^{-1}$	$-a^{n}\,u(n-1)$
$a^{\boxed{-1}}z^{\boxed{-1}}$	$\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$a^{n}\,u(-n)$
			Shifted Ranges
$a^{\boxed{-1}}z^{\boxed{-1}}$	$-\dfrac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n}\,u(-n-1)$
$a^{\boxed{-1}}\,z$	$\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$	$a^{-n}\,u(n)$
			Shifted Ranges
$a^{\boxed{-1}}\,z$	$-\dfrac{a^{-1}z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$-a^{-n}\,u(n-1)$
$a\,z^{\boxed{-1}}$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$	$a^{-n}\,u(-n)$
			Shifted Ranges
$a\,z^{\boxed{-1}}$	$-\dfrac{az^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-a^{-n}\,u(-n-1)$

Making Shifted Sequences

Shifting Geometric Power Series Property (1)

$*z$ **Right Shifted** $n \rightarrow n-1$

SHR.Rng

$u(n) \rightarrow u(n-1)$

$u(-n-1) \rightarrow u(-n)$

SHR.Exp

$a^{n} \rightarrow a^{n-1}$

$a^{-n} \rightarrow a^{-n+1}$

$/z$ **Left Shifted** $n \rightarrow n+1$

SHL.Rng

$u(n-1) \rightarrow u(n)$

$u(-n) \rightarrow u(-n-1)$

SHL.Exp

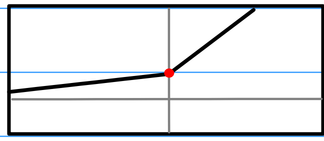
$a^{n} \rightarrow a^{n+1}$

$a^{-n} \rightarrow a^{-n-1}$

$*a$	**Left Shifted**	**SHL.Exp**	$a^{n} \longrightarrow a^{n+1}$
	Right Shifted	**SHR.Exp**	$a^{-n} \longrightarrow a^{-n+1}$

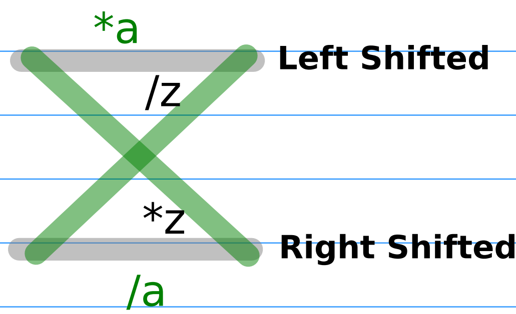
$/a$	**Right Shifted**	**SHR.Exp**	$a^{n} \longrightarrow a^{n-1}$
	Left Shifted	**SHL.Exp**	$a^{-n} \longrightarrow a^{-n-1}$

Shifting Geometric Power Series Property (2)

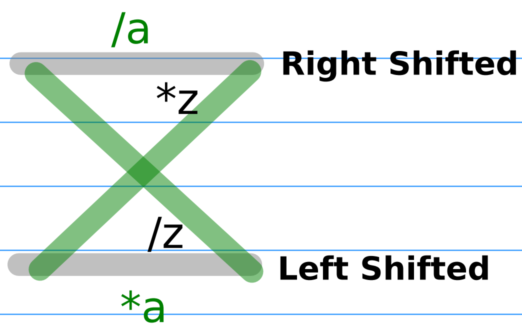


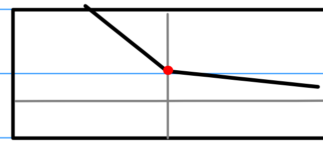
a^n

Causal Sequences



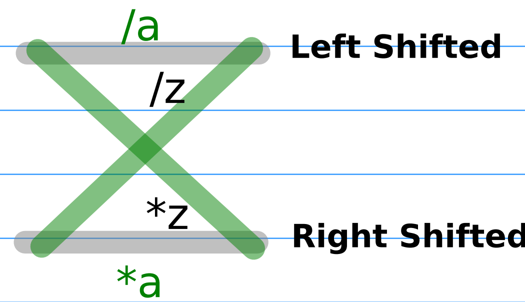
Anti-Causal Sequences



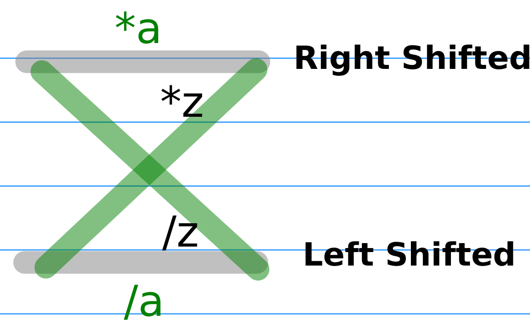


a^{-n}

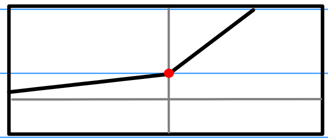
Causal Sequences

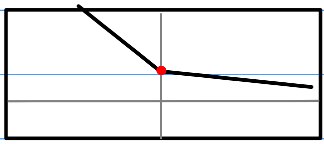


Anti-Causal Sequences



Shifting Geometric Power Series Property (3)

 a^n

 a^{-n}

Causal Sequences

a^n u(n)	*a	a^{n+1} u(n)	
	/z		
	*z		
a^n u(n-1)	/a	a^{n-1} u(n-1)	

Causal Sequences

a^{-n} u(n)	/a	a^{-n-1} u(n)
	/z	
	*z	
a^{-n} u(n-1)	*a	a^{-n+1} u(n-1)

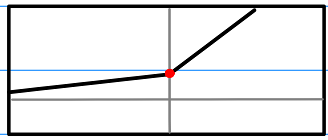
Anti-Causal Sequences

a^n u(-n)	/a	a^{n-1} u(-n)
	*z	
	/z	
a^n u(-n-1)	*a	a^{n+1} u(-n-1)

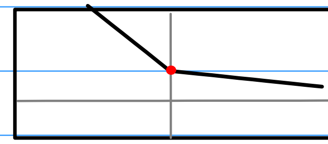
Anti-Causal Sequences

a^{-n} u(-n)	*a	a^{-n+1} u(-n)
	*z	
	/z	
a^{-n} u(-n-1)	/a	a^{-n-1} u(-n-1)

Shifting Geometric Power Series Property (4)



a^n



a^{-n}

Causal Sequences

$\frac{1}{1-az}$	*a	$\frac{a}{1-az}$
	/z	
	*z	
$\frac{az}{1-az}$	/a	$\frac{z}{1-az}$

Causal Sequences

$\frac{1}{1-a^{-1}z}$	/a	$\frac{a^{-1}}{1-a^{-1}z}$
	/z	
	*z	
$\frac{a^{-1}z}{1-a^{-1}z}$	*a	$\frac{z}{1-a^{-1}z}$

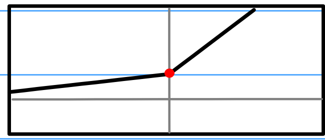
Anti-Causal Sequences

$\frac{1}{1-a^{-1}z^{-1}}$	/a	$\frac{a^{-1}}{1-a^{-1}z^{-1}}$
	*z	
	/z	
$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$	*a	$\frac{z^{-1}}{1-a^{-1}z^{-1}}$

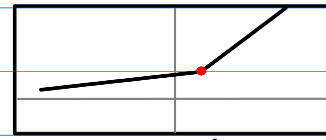
Anti-Causal Sequences

$\frac{1}{1-az^{-1}}$	*a	$\frac{a}{1-az^{-1}}$
	*z	
	/z	
$\frac{az^{-1}}{1-az^{-1}}$	/a	$\frac{z^{-1}}{1-az^{-1}}$

Shifting exponential functions

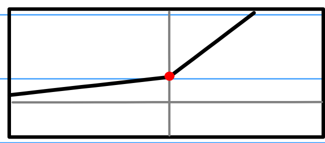


a^n

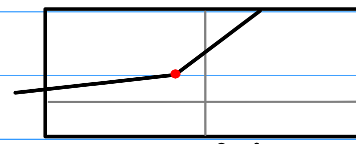


right shift

a^{n-1}

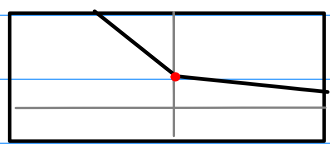


a^n

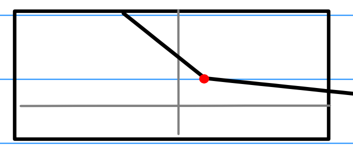


left shift

a^{n+1}

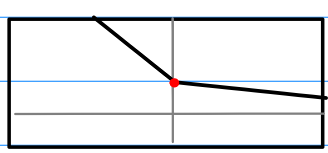


a^{-n}

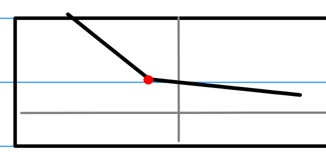


right shift

a^{-n+1}



a^{-n}



left shift

a^{-n-1}

$\boxed{*a}$

Left Shifted $a^n \longrightarrow a^{n+1}$

Right Shifted $a^{-n} \longrightarrow a^{-n+1}$

$\boxed{/a}$

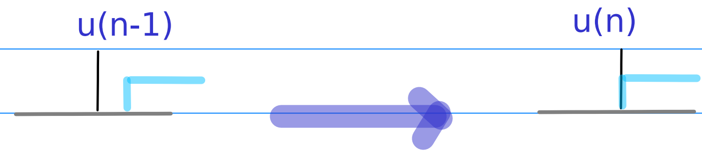
Right Shifted $a^n \longrightarrow a^{n-1}$

Left Shifted $a^{-n} \longrightarrow a^{-n-1}$

Shifting of a Range

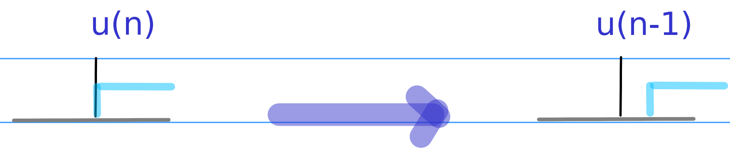
SHL.Rng

$n \to n+1$



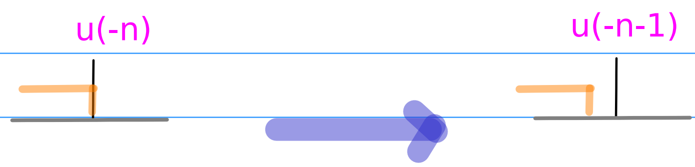
SHR.Rng

$n \to n-1$



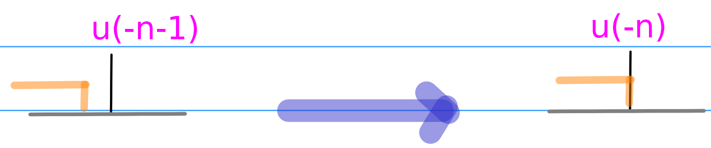
SHL.Rng

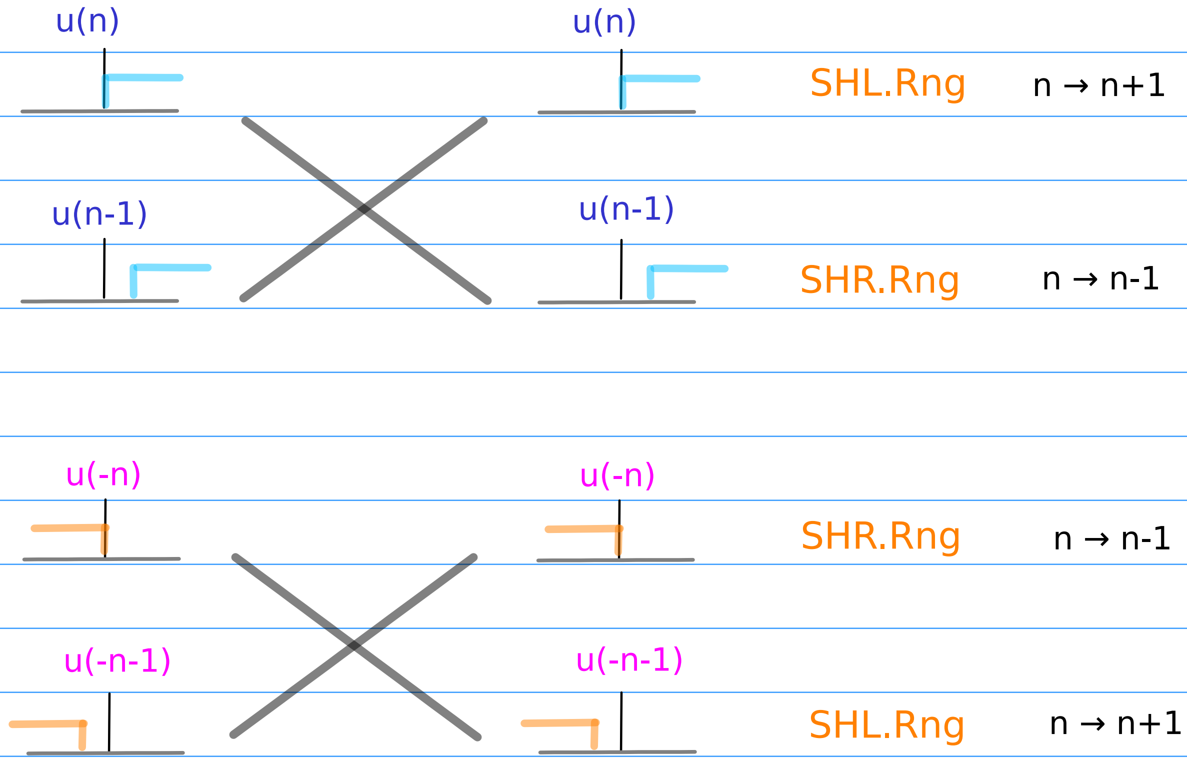
$n \to n+1$



SHR.Rng

$n \to n-1$





u(n)
u(n)
SHL.Rng
n → n+1
u(n-1)
u(n-1)
SHR.Rng
n → n-1
u(-n)
u(-n)
SHR.Rng
n → n-1
u(-n-1)
u(-n-1)
SHL.Rng
n → n+1

Left Shifting Sequences

0 1 2

$(a^0, a^1, a^2, \dots)$ $a^n\ u(n)$

the same range

$(a^1, a^2, a^3, \dots)$ $a^{n+1}\ u(n)$

0 1 2

one shifted-out

---

0 1 2

$(0, a^1, a^2, \dots)$ $a^n\ u(n-1)$

left shifted range

$(a^1, a^2, a^3, \dots)$ $a^{n+1}\ u(n)$

0 1 2

zero shifted-out

-3 -2 -1

$(\dots, a^{-3}, a^{-2}, a^{-1})$ $a^n\ u(-n-1)$

the same range

$(\dots, a^{-2}, a^{-1}, a^0)$ $a^{n+1}\ u(-n-1)$

-3 -2 -1

one shifted-in

-2 -1 0

$(\dots, a^{-2}, a^{-1}, a^0)$ $a^n\ u(-n)$

left shifted range

$(\dots, a^{-1}, a^0, 0)$ $a^{n+1}\ u(-n-1)$

-2 -1 0

zero shifted-in

$a^n \rightarrow a^{n+1}$

SHL.Rng, $a^n \rightarrow a^{n+1}$

$a^{-n} \rightarrow a^{-n-1}$

SHL.Rng, $a^{-n} \rightarrow a^{-n-1}$

---

0 1 2

$(a^0, a^{-1}, a^{-2}, \dots)$ $a^{-n}\ u(n)$

the same range

$(a^{-1}, a^{-2}, a^{-3}, \dots)$ $a^{-n-1}\ u(n)$

0 1 2

one shifted-out

---

0 1 2

$(0, a^{-1}, a^{-2}, \dots)$ $a^{-n}\ u(n-1)$

left shifted range

$(a^{-1}, a^{-2}, a^{-3}, \dots)$ $a^{-n-1}\ u(n)$

0 1 2

zero shifted-out

-3 -2 -1

$(\dots, a^3, a^2, a^1)$ $a^{-n}\ u(-n-1)$

the same range

$(\dots, a^2, a^1, a^0)$ $a^{-n-1}\ u(-n-1)$

-3 -2 -1

one shifted-in

-2 -1 0

$(\dots, a^2, a^1, a^0)$ $a^{-n}\ u(-n)$

left shifted range

$(\dots, a^1, a^0, 0)$ $a^{-n-1}\ u(-n-1)$

-2 -1 0

zero shifted-in

Right Shifting Sequences

$\begin{array}{cccc} 1 & 2 & 3 & \\ (a^1, & a^2, & a^3, & \cdots) \end{array}$	$a^n\, u(n-1)$
	the same range
$\begin{array}{cccc} (a^0, & a^1, & a^2, & \cdots) \\ & 1 & 2 & 3 \end{array}$	$a^{n-1}\, u(n-1)$
	one shifted-in

$\begin{array}{cccc} 0 & 1 & 2 & \\ (a^0, & a^1, & a^2, & \cdots) \end{array}$	$a^n\, u(n)$
	right shifted range
$\begin{array}{cccc} (0, & a^0, & a^1, & \cdots) \\ 0 & 1 & 2 & \end{array}$	$a^{n-1}\, u(n-1)$
	zero shifted-in

$\begin{array}{cccc} & -2 & -1 & 0 \\ (\cdots, & a^{-2}, & a^{-1}, & a^0) \end{array}$	$a^n\, u(-n)$
	the same range
$\begin{array}{cccc} (\cdots, & a^{-3}, & a^{-2}, & a^{-1}) \\ & -2 & -1 & 0 \end{array}$	$a^{n-1}\, u(-n)$
	one shifted-out

$\begin{array}{cccc} & -3 & -2 & -1 \\ (\cdots, & a^{-2}, & a^{-1}, & 0) \end{array}$	$a^n\, u(-n-1)$
	right shifted range
$\begin{array}{cccc} (\cdots, & a^{-3}, & a^{-2}, & a^{-1}) \\ & -3 & -2 & -1 \end{array}$	$a^{n-1}\, u(-n)$
	zero shifted-out

$a^n \rightarrow a^{n-1}$ SHR.Rng, $a^n \rightarrow a^{n-1}$

$a^{-n} \rightarrow a^{-n+1}$ SHR.Rng, $a^{-n} \rightarrow a^{-n+1}$

$\begin{array}{cccc} 1 & 2 & 3 & \\ (a^{-1}, & a^{-2}, & a^{-3}, & \cdots) \end{array}$	$a^{-n}\, u(n-1)$
	the same range
$\begin{array}{cccc} (a^0, & a^{-1}, & a^{-2}, & \cdots) \\ & 1 & 2 & 3 \end{array}$	$a^{-n+1}\, u(n-1)$
	one shifted-in

$\begin{array}{cccc} 0 & 1 & 2 & \\ (a^0, & a^{-1}, & a^{-2}, & \cdots) \end{array}$	$a^{-n}\, u(n)$
	right shifted range
$\begin{array}{cccc} (0, & a^0, & a^{-1}, & \cdots) \\ 0 & 1 & 2 & \end{array}$	$a^{-n+1}\, u(n-1)$
	zero shifted-in

$\begin{array}{cccc} & -2 & -1 & 0 \\ (\cdots, & a^2, & a^1, & a^0) \end{array}$	$a^{-n}\, u(-n)$
	the same range
$\begin{array}{cccc} (\cdots, & a^3, & a^2, & a^1) \\ & -2 & -1 & 0 \end{array}$	$a^{-n+1}\, u(-n)$
	one shifted-out

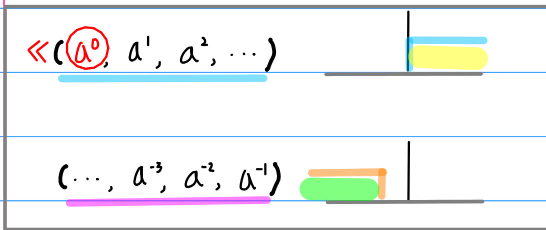
$\begin{array}{cccc} & -2 & -1 & 0 \\ (\cdots, & a^2, & a^1, & 0) \end{array}$	$a^{-n}\, u(-n-1)$
	right shifted range
$\begin{array}{cccc} (\cdots, & a^3, & a^2, & a^1) \\ & -2 & -1 & 0 \end{array}$	$a^{-n+1}\, u(-n)$
	zero shifted-out

Original Sequence $\longrightarrow$ Shifted Sequence

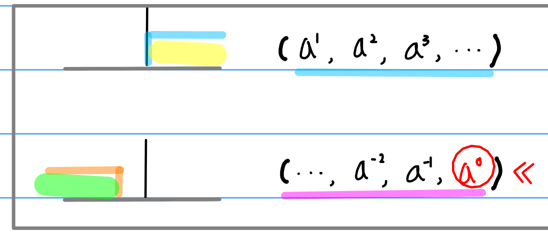
Original Sequence		Shifted Sequence
$\ll(a^0, a^1, a^2, \cdots)$	ID.Rng	$(a^1, a^2, a^3, \cdots)$
$(\cdots, a^{-3}, a^{-2}, a^{-1})$	ID.Rng	$(\cdots, a^{-2}, a^{-1}, a^0)\ll$
$\ll(0, a^1, a^2, \cdots)$	SHL.Rng	$(a^1, a^2, a^3, \cdots)$
$(\cdots, a^{-2}, a^{-1}, a^0)$	SHL.Rng	$(\cdots, a^{-1}, a^0, 0)\ll$
$(a^1, a^2, a^3, \cdots)$	ID.Rng	$\gg(a^0, a^1, a^2, \cdots)$
$(\cdots, a^{-2}, a^{-1}, a^0)\gg$	ID.Rng	$(\cdots, a^{-3}, a^{-2}, a^{-1})$
$(a^0, a^1, a^2, \cdots)$	SHR.Rng	$\gg(0, a^0, a^1, \cdots)$
$(\cdots, a^{-2}, a^{-1}, 0)\gg$	SHR.Rng	$(\cdots, a^{-3}, a^{-2}, a^{-1})$

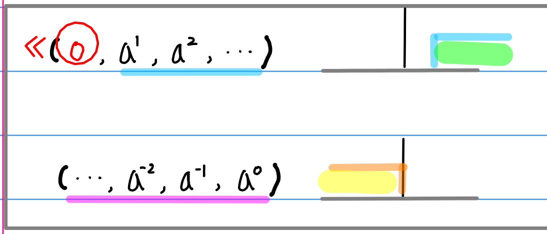
Original Sequence

Shifted Sequence

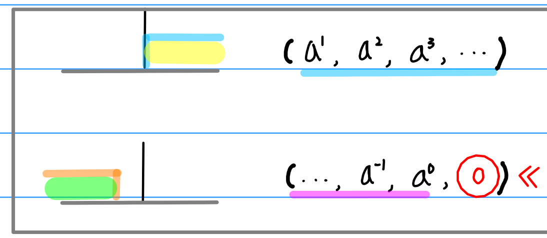


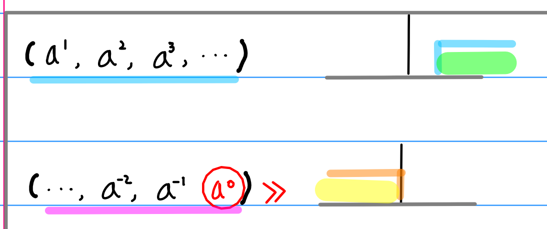
* no shift
* non-zero shift in
* a new value introduced



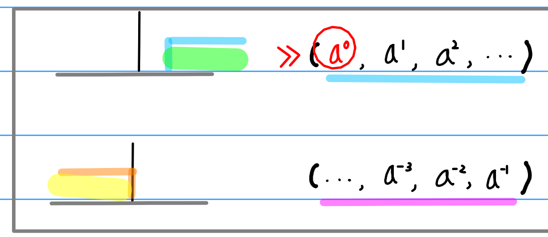


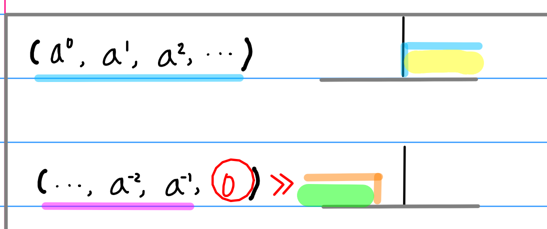
* left shift
* zero shift in
* the same set of values



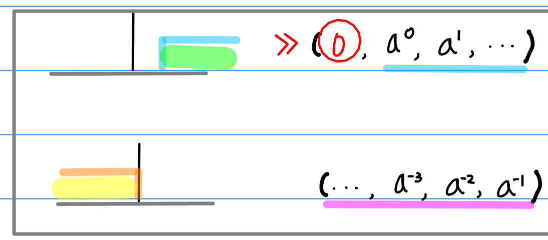


* no shift
* non-zero shift in
* a new value introduced





* right shift
* zero shift in
* the same set of values



Making Shifted Sequences

making left shifted sequences

causal	anti-causal
the same set of slots	the same set of slots
left shifted set of samples	left shifted set of samples

causal	anti-causal
left shifted set of slots	left shifted set of slots
the same set of samples	the same set of samples

making right shifted sequences

causal	anti-causal
the same set of slots	the same set of slots
right shifted set of samples	right shifted set of samples

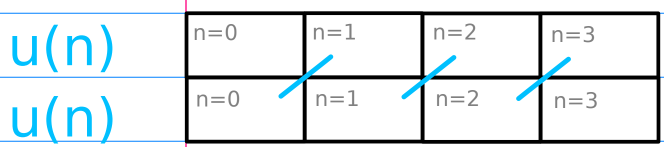
causal	anti-causal
right shifted set of slots	right shifted set of slots
the same set of samples	the same set of samples

Making Shifted Sequences

making left shifted sequences

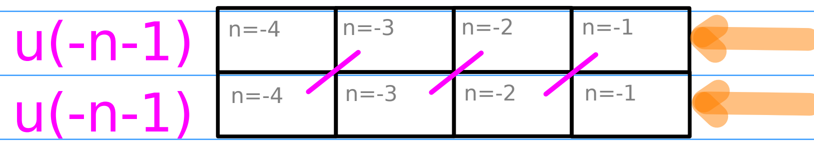
the same set of slots

left shifted set of samples



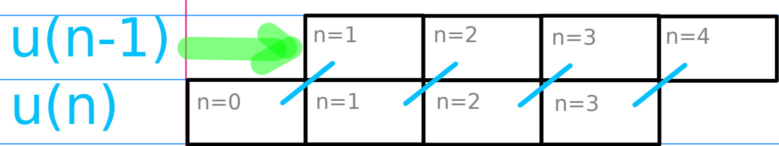
the same set of slots

left shifted set of samples



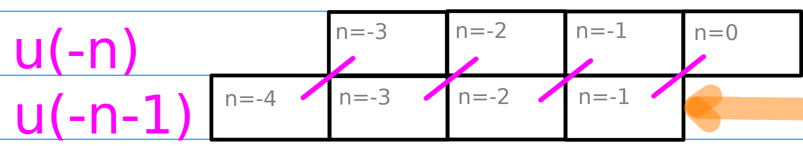
left shifted set of slots

the same set of samples



left shifted set of slots

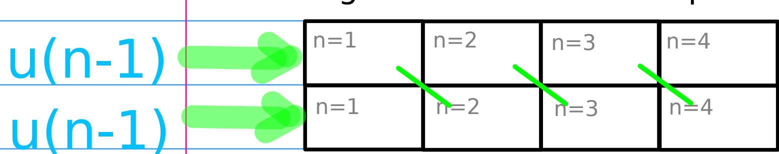
the same set of samples



making right shifted sequences

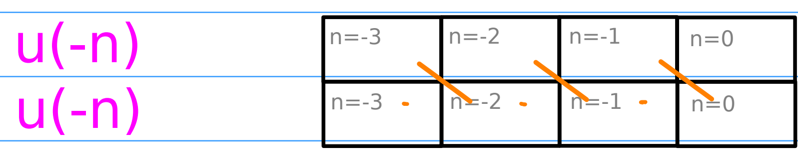
the same set of slots

right shifted set of samples



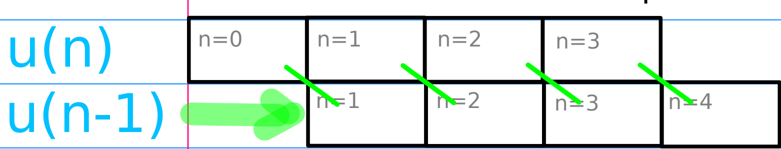
the same set of slots

right shifted set of samples



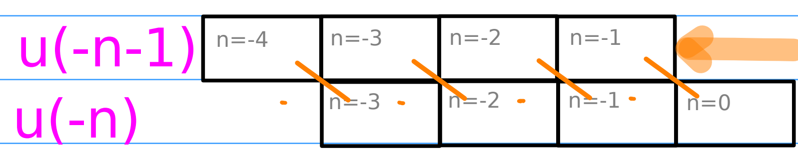
right shifted set of slots

the same set of samples



right shifted set of slots

the same set of samples



Two Types of Left-Shifted Causal Sequences

the same fixed slots

$u(n)$	n=0	n=1	n=2	n=3
$u(n)$	n=0	n=1	n=2	n=3

left-shift samples

$a^0 \quad a^1 \quad a^2 \quad a^3$

$a^1 \quad a^2 \quad a^3 \quad a^4$

a^n	a^0	a^1	a^2	a^3
a^{n+1}	a^1	a^2	a^3	a^4

left-shifted sequence (I)

the same set of slots

left shifted set of samples

right-shift pre-slot

$u(n-1)$		n=1	n=2	n=3	n=4
$u(n)$	n=0	n=1	n=2	n=3	

fixed samples

$a^1 \quad a^2 \quad a^3 \quad a^4$

$a^1 \quad a^2 \quad a^3 \quad a^4$

a^n		a^1	a^2	a^3	a^4
a^{n+1}	a^1	a^2	a^3	a^4	

left-shifted sequence (II)

left shifted set of slots

the same set of samples

Two Types of Left-Shifted Anti-Causal Sequences

left shift both slots

u(-n-1)	n=-4	n=-3	n=-2	n=-1
u(-n-1)	n=-4	n=-3	n=-2	n=-1

left-shift samples

a^{-4}	a^{-3}	a^{-2}	a^{-1}
a^{-3}	a^{-2}	a^{-1}	a^{0}

a^n	a^{-4}	a^{-3}	a^{-2}	a^{-1}
a^{n+1}	a^{-3}	a^{-2}	a^{-1}	a^{0}

left-shifted sequence (I)

the same set of slots

left shifted set of samples

left-shift post-slot

u(-n)		n=-3	n=-2	n=-1	n=0
u(-n-1)	n=-4	n=-3	n=-2	n=-1	

fixed samples

a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{-3}	a^{-2}	a^{-1}	a^{0}

a^n		a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{n+1}	a^{-3}	a^{-2}	a^{-1}	a^{0}	

left-shifted sequence (II)

left shifted set of slots

the same set of samples

Two Types of Right-Shifted Causal Sequences

right shift both slots

$u(n-1)$	n=1	n=2	n=3	n=4
$u(n-1)$	n=1	n=2	n=3	n=4

right shift post-samples

a^1	a^2	a^3	a^4
a^0	a^1	a^2	a^3

a^n	a^1	a^2	a^3	a^4
a^{n-1}	a^0	a^1	a^2	a^3

right-shifted sequence (I)

the same set of slots

right shifted set of samples

right shift post-slot

$u(n)$	n=0	n=1	n=2	n=3	
$u(n-1)$		n=1	n=2	n=3	n=4

fixed samples

a^0	a^1	a^2	a^3
a^0	a^1	a^2	a^3

a^n	a^0	a^1	a^2	a^3	
a^{n-1}		a^0	a^1	a^2	a^3

right-shifted sequence (II)

right shifted set of slots

the same set of samples

Two Types of Right-Shifted Anti-Causal Sequence

the same fixed slots

u(-n)	n=-3	n=-2	n=-1	n=0
u(-n)	n=-3	n=-2	n=-1	n=0

right shift post-samples

a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{-4}	a^{-3}	a^{-2}	a^{-1}

a^{n}	a^{-3}	a^{-2}	a^{-1}	a^{0}
a^{n-1}	a^{-4}	a^{-3}	a^{-2}	a^{-1}

right-shifted sequence (I)

the same set of slots

right shifted set of samples

left shift pre-slot

u(-n-1)	n=-4	n=-3	n=-2	n=-1	
u(-n)		n=-3	n=-2	n=-1	n=0

fixed samples

a^{-4}	a^{-3}	a^{-2}	a^{-1}
a^{-4}	a^{-3}	a^{-2}	a^{-1}

a^{n}	a^{-4}	a^{-3}	a^{-2}	a^{-1}	
a^{n-1}		a^{-4}	a^{-3}	a^{-2}	a^{-1}

right-shifted sequence (II)

right shifted set of slots

the same set of samples