

Differentiation of Continuous Functions

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Based on
Introduction to Matrix Algebra, Autar Kaw
<https://ma.mathforcollege.com>

Outline

- 1 Forward Difference
 - Forward Difference

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Forward Difference Approximation (1)

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\end{aligned}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

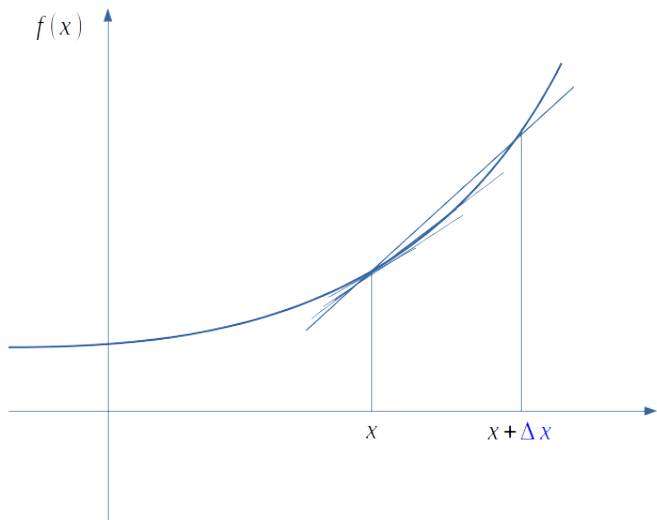


Figure: forward difference approximation

Backward Difference Approximation (1)

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$\begin{aligned} f'(x) &\approx \frac{f(x - \Delta x) - f(x)}{-\Delta x} \\ &= \frac{f(x) - f(x - \Delta x)}{\Delta x} \end{aligned}$$

Backward Difference Approximation (2)

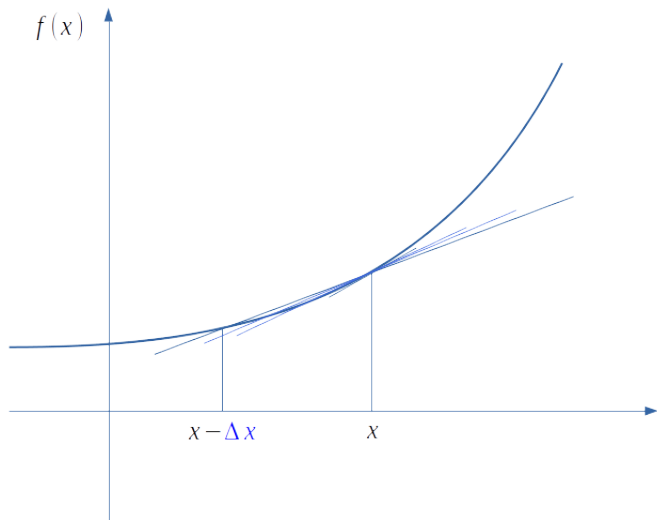


Figure: backward difference approximation

Backward Difference Approximation of the First Derivatives (1)

a backward difference approximation
as you are taking a point backward from x .

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx behind as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\ &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \end{aligned}$$

Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \dots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f(x_{i+1}) - f(x_i) = f'(x_i)(\Delta x) + \dots$$

Derive Forward Difference Approximation from Taylor Series (1)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \dots$$

$$f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x)^1 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

Derive Forward Difference Approximation from Taylor Series (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \dots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \dots$$

subtracting eq(2) from eq(1)

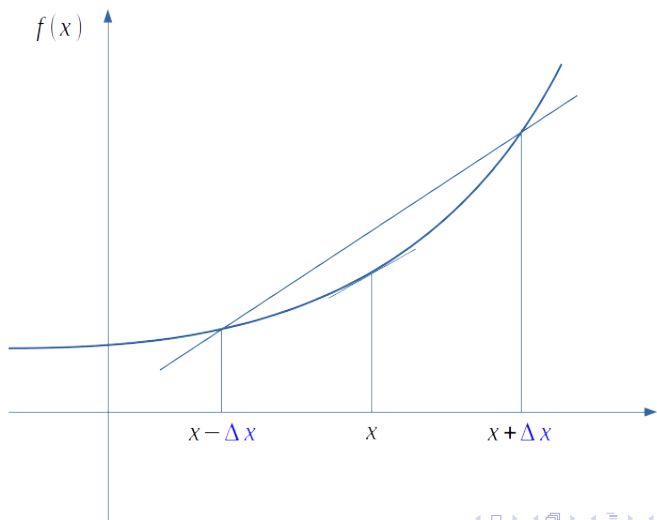
$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \dots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \dots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O((\Delta x)^2)$$

Derive Forward Difference Approximation from Taylor Series (3)



Tangent Lines

- as $h \rightarrow 0$, $Q \rightarrow P$
and the **secant line** \rightarrow the **tangent line**
- the slope of the **tangent line**

$$\begin{aligned}m_{\text{tangent}} &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}\end{aligned}$$

