Differentiation of Continuous Functions

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

1. Forward Difference
   - Forward Difference

Outline

Forward Difference Approximation (1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

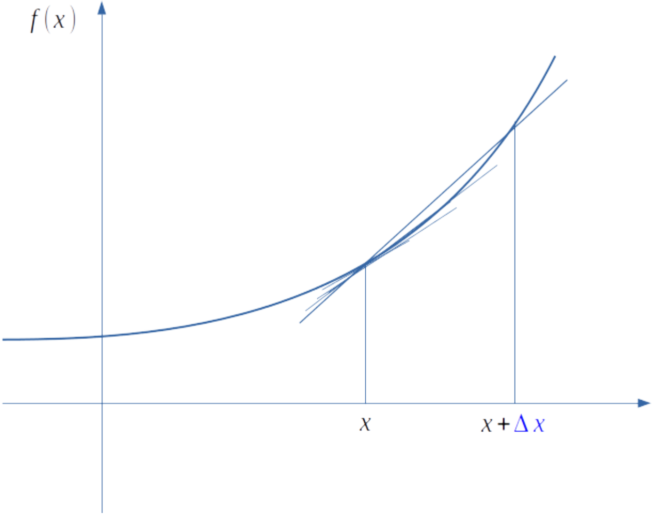


Figure: forward difference approximation

Backward Difference Approximation (1)

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$f'(x) \approx \frac{f(x-\Delta x) - f(x)}{-\Delta x}$$
$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (2)

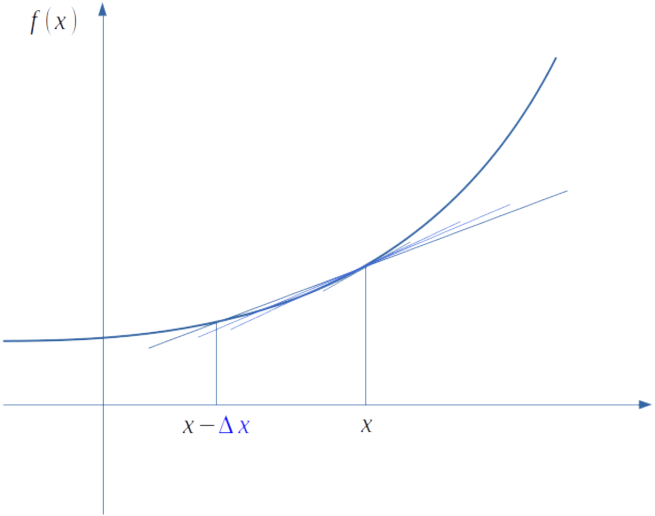


Figure: backward difference approximation

Backward Difference Approximation of the First Derivatives (1)

a backward difference approximation
as you are taking a point backward from x.

To find the value of $f\prime(x)$ at $x = x_i$,
we may choose another point Δx behind as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$f(x_{i+1}) - f(x_i) - (+f'(x_i)(\Delta x)$$

Derive Forward Difference Approximation from Taylor Series (1)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x)^1 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

Derive Forward Difference Approximation from Taylor Series (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \cdots$$

subtracting eq(2) from eq(1)

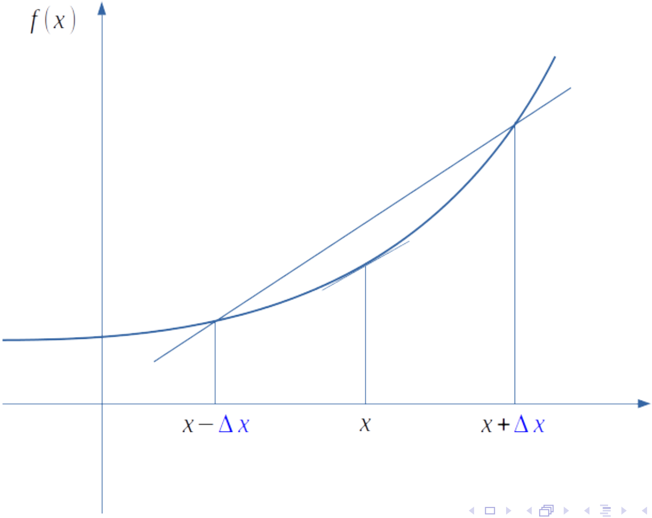
$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O\left((\Delta x)^2\right)$$

Derive Forward Difference Approximation from Taylor Series (3)



Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line $\to$ the tangent line
- the slope of the tangent line

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$
$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Forward Difference