

Ergodic Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 Averages and Ergodicity
- 2 Mean Ergodic Processes
- 3 Correlation Ergodic Processes

Average

N Gaussian random variables

Definition

$$\bar{m}_x = \frac{1}{N} \sum_{i=1}^N X_i(t)$$

—

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

—

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

Expectation of Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

$$E[\bar{X}_T] = E[A_T[x(t)]] = \bar{X}$$

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

$$E[R_T(\tau)] = E[A_T[x(t)x(t+\tau)]] = R_{XX}(\tau)$$

Ergodicity Theorem

N Gaussian random variables

Definition

$$\lim_{n \rightarrow \infty} E [(X_n - X)^2] = 0$$

$$A[\bullet] = \lim_{n \rightarrow \infty} A_T[\bullet]$$

Conditions

N Gaussian random variables

- ① $X(t)$ has a finite constant mean \bar{X} for all t
- ② $X(t)$ is bounded $x(t) < \infty$ for all t and all $x(t)$
- ③ Bounded time average of $E[|X(t)|]$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[|X(t)|] dt$$

- ④ $X(t)$ is a regular process

$$E \left[|X(t)|^2 \right] = R_{XX}(t, t) < \infty$$

Mean Ergodic

N Gaussian random variables

Definition

A wide sense stationary process $X(t)$ with a constant mean value \bar{X} is called mean-ergodic if $\bar{x}_T = A_T[x(t)]$ converges to \bar{X} as $T \rightarrow \infty$

$$\lim_{T \rightarrow \infty} E [(\bar{x}_T - \bar{X})^2] = 0$$

$$\lim_{T \rightarrow \infty} \sigma_{\bar{x}_T} = 0$$

Variance of \bar{X}_T (1) N Gaussian random variables

Definition

$$\begin{aligned}\sigma_{\bar{X}_T}^2 &= E \left[\left\{ \frac{1}{2T} \int_{-T}^T (X(t) - \bar{X}) dt \right\}^2 \right] \\ &= E \left[\left(\frac{1}{2T} \right)^2 \left\{ \int_{-T}^T (X(t) - \bar{X}) dt \right\} \left\{ \int_{-T}^T (X(t_1) - \bar{X}) dt_1 \right\} \right] \\ &= E \left[\left(\frac{1}{2T} \right)^2 \int_{-T}^T (X(t) - \bar{X}) (X(t_1) - \bar{X}) dt dt_1 \right] \\ &= \left(\frac{1}{2T} \right)^2 \int_{-T}^T \int_{-T}^T E [(X(t) - \bar{X}) (X(t_1) - \bar{X})] dt dt_1 \\ &= \left(\frac{1}{2T} \right)^2 \int_{-T}^T C_{XX}(t, t_1) dt dt_1\end{aligned}$$

Variance of \bar{x}_T (2)

N Gaussian random variables

Definition

$$\sigma_{\bar{x}_T}^2 = \left(\frac{1}{2T}\right)^2 \int_{-T}^T C_{XX}(t, t_1) dt dt_1$$

$$C_{XX}(t, t_1) = C_{XX}(T), \quad \tau = t_1 - t, \quad dt_1 = dT$$

$$\sigma_{\bar{x}_T}^2 = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^T \int_{\tau=-T-t}^T C_{XX}(\tau) dt d\tau$$

Variance of \bar{x}_T (3) N Gaussian random variables

Definition

using the symmetry $C_{XX}(-\tau) = C_{XX}(\tau)$

$$\sigma_{\bar{x}_T}^2 = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau$$

a necessary and sufficient condition for a WSS process $X(t)$ to be mean ergodic

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau \right\} = 0$$

Mean Ergodic Process - continuous time

N Gaussian random variables

Definition

$X(t)$ is a mean ergodic if

- 1 $C_{XX}(0) < \infty$ and $C_{XX}(\tau) \rightarrow 0$ as $|\tau| \rightarrow \infty$
- 2 $\int_{-\infty}^{\infty} C_{XX}(\tau) d\tau < \infty$

Mean Ergodic Process - discrete time

N Gaussian random variables

Definition

$X[n]$ is a mean ergodic if

$$\lim_{N \rightarrow \infty} \left\{ \frac{1}{2N+1} \sum_{n=-N}^{+N} X[n] \right\} = \bar{X}$$

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2N+1} \sum_{n=-2N}^{+2N} \left(1 - \frac{|n|}{2N+1} \right) C_{XX}[n] \right\} = 0$$

