Ergodic Random Processes

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Average

N Gaussian random variables

Definition

$$\overline{m}_x = \frac{1}{N} \sum_{i=1}^{N} X_i(t)$$

—

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^{T} [\bullet] dt$$

Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\overline{X}_T = A_T[x(t)] = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

—

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Expectation of Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\overline{X}_T = A_T\left[x(t)\right] = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$$E\left[\overline{X}_T\right] = E\left[A_T\left[x(t)\right]\right] = \overline{X}$$

$$R_T(\tau) = A_T\left[x(t)x(t+\tau)\right] = \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$E\left[R_T(\tau)\right] = E\left[A_T\left[x(t)x(t+\tau)\right]\right] = R_{XX}(\tau)$$

Ergodicity Theorem

N Gaussian random variables

Definition

$$\lim_{n\to\infty} E\left[(X_n - X)^2\right] = 0$$

$$A[\bullet] = \lim_{n\to\infty} A_T[\bullet]$$

Conditions

N Gaussian random variables

1. $X(t)$ has a finite constant mean$\overline{X}$ for all t
2. $X(t)$ is bounded $x(t) < \infty$ for all t and all $x(t)$
3. Bounded time average of E[|X(t)|]

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[|X(t)|]dt$$

4. $X(t)$ is a regular process

$$E\left[|X(t)|^2\right] = R_{XX}(t,t) < \infty$$

Mean Erogodic

N Gaussian random variables

Definition

A wide snese stationary process $X(t)$with a constant mean value $\overline{X}$ is called mean-ergodic if $\overline{x}_T = A_T[x(t)]$ converges to $\overline{X}$ as $T \to \infty$

$$\lim_{T\to\infty} E\left[(\overline{x}_T - \overline{X})^2\right] = 0$$

$$\lim_{T\to\infty} \sigma_{\overline{x}_T} = 0$$

Variance of $\overline{x}_T$ (1)

N Gaussian random variables

Definition

$$\sigma^2_{\overline{x}_T} = E\left[\left\{\frac{1}{2T}\int_{-T}^{T}\left(X(t)-\overline{X}\right)dt\right\}^2\right]$$
$$= E\left[\left(\frac{1}{2T}\right)^2\left\{\int_{-T}^{T}\left(X(t)-\overline{X}\right)dt\right\}\left\{\int_{-T}^{T}\left(X(t_1)-\overline{X}\right)dt_1\right\}\right]$$
$$= E\left[\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\left(X(t)-\overline{X}\right)\left(X(t_1)-\overline{X}\right)dtdt_1\right]$$
$$= \left(\frac{1}{2T}\right)^2\int_{-T}^{T}E\left[\left(X(t)-\overline{X}\right)\left(X(t_1)-\overline{X}\right)\right]dtdt_1$$
$$= \left(\frac{1}{2T}\right)^2\int_{-T}^{T}C_{XX}(t,t_1)dtdt_1$$

Variance of $\overline{x}_T$ (2)

N Gaussian random variables

Definition

$$\sigma^2_{\overline{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{-T}^{T} C_{XX}(t,t_1)dtdt_1$$
$$C_{XX}(t,t_1) = C_{XX}(T), \qquad \tau = t_1 - t, \qquad dt_1 = dT$$
$$\sigma_{\overline{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^{T} \int_{\tau=-T-t}^{T} C_{XX}(\tau)dtd\tau$$

Variance of $\overline{x}_T$ (3)

N Gaussian random variables

Definition

using the symmetry $C_{XX}(-\tau) = C_{XX}(-\tau)$

$$\sigma^2_{\overline{x}_T} = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau$$

a necessary and sufficient condition for a WSS process X(t) to be mean ergodic

$$\lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau \right\} = 0$$

Mean Ergodic Process - continuous time

N Gaussian random variables

Definition

$X(t)$ is a mean ergodic if

1. $C_{XX}(0) < \infty$ and $C_{XX}(\tau) \to 0$ as $|\tau| \to \infty$
2. $\int_{-\infty}^{\infty} C_{XX}(\tau) d\tau < \infty$

Mean Ergodic Process - discrete time

N Gaussian random variables

Definition

$X[n]$ is a mean ergodic if

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{+N}X[n]\right\}=\overline{X}$$

$$\lim_{T\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-2N}^{+2N}\left(1-\frac{|n|}{2N+1}\right)C_{XX}[n]\right\}=0$$