Example Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Gaussian Random Process

$f_X(x_1, \cdots, x_n; t_1, , t_N)$

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) =$$

$$\frac{exp\left\{-(1/2)\left[x - \overline{X}\right]^t [C_X]^{-1} \left[x - \overline{X}\right]\right\}}{\sqrt{(2\pi)^N |[C_X]|}}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \overline{X} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_N \end{bmatrix} \qquad [x - \overline{X}] = \begin{bmatrix} x_1 - \overline{X}_1 \\ x_2 - \overline{X}_2 \\ \vdots \\ x_N - \overline{X}_N \end{bmatrix}$$

The Covariance Matrix (1)

$$\overline{X}_i = E[X_i] = E[X(t_i)]$$

$$\overline{\boldsymbol{X}} = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_N \end{bmatrix} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_N] \end{bmatrix} = \begin{bmatrix} E[X(t_1)] \\ E[X(t_2)] \\ \vdots \\ E[X(t_N)] \end{bmatrix}$$

The Covariance Matrix (2)

$C_{ik} = C_{X_i X_k}$

$$C_{ik} = C_{X_i X_k} = E\left[\left(X_i - \overline{X}_i\right)\left(X_k - \overline{X}_k\right)\right]$$
$$= E\left[\left(X(t_i) - E[X(t_i)]\right)\left(X(t_k) - E[X(t_k)]\right)\right]$$

$$C_{ik} = C_{X_i X_k} = C_{XX}(t_i, t_k)$$
$$= R_{XX}(t_i, t_k) - E[X(t_i)]\, E[X(t_k)]$$

Stationary Gaussian Process

$C_{XX}(\tau), R_{XX}(\tau)$

$$\overline{X}_i = E[X_i] = E[X(t_i)] = \overline{X} = const$$

$$C_{XX}(t_i, t_k) = C_{XX}(t_k - t_i)$$

$$R_{XX}(t_i, t_k) = R_{XX}(t_k - t_i)$$

Jointly Gaussian Process

$X(t_1), \cdots, X(t_N), Y(t'_1),, Y(t'_N)$

the two random processes $X(t)$ and $Y(t)$

are **jointly Gaussian** if the random variables

$X(t_1), ..., X(t_N)$ at times $t_1, ..., t_N$ for $X(t)$ and

$Y(t'_1), ..., Y(t'_M)$ at times $t'_1, ..., t'_M$ for $Y(t)$

are **jointly gaussian** for any N, $t_1, ..., t_N$, and M, $t'_1, ..., t'_M$

Stationary Gaussian Markov Process

$C_{XX}(\tau), C_{XX}[k]$

$$C_{XX}(\tau) = \sigma^2 e^{-\beta|\tau|}$$

$$C_{XX}[k] = \sigma^2 a^{-|k|}$$

$$a = e^{\beta T_S}$$

Poisson Random Process

$f_X(x)$

$$p[X(t)=k]=\frac{(\lambda t)^k e^{-\lambda t}}{k!}, \qquad k=0,1,2,\cdots$$

$$f_X(x)=\sum_{k=0}^{\infty}\frac{(\lambda t)^k e^{-\lambda t}}{k!}\delta(x-k)$$

Poisson Random Process - mean and 2nd moment

$E[X(t)], E[X^2(t)]$

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x) dx = \int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t$$

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = \int_{-\infty}^{\infty} x^2 \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} \delta(x-k) dx$$

$$= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k e^{-\lambda t}}{k!} = \lambda t(1+\lambda t)$$

Poisson Random Process - joint probability density (1)

$P(k_1, k_2)$

$$P[X(t_1) = k_1] = \frac{(\lambda t_1)^{k_1} e^{-\lambda t_1}}{k_1!} \qquad k_1 = 0, 1, 2, \cdots$$

$$P[X(t_2) = k_2 | X(t_1) = k_1] = \frac{[\lambda(t_2 - t_1)]^{k_2 - k_1} e^{-\lambda(t_2 - t_1)}}{(k_2 - k_1)!}$$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

Poisson Random Process - joint probability density (2)

$f_X(x_1, x_2)$

$$P(k_1, k_2) = P[X(t_2) = k_2 | X(t_1) = k_1] \cdot P[X(t_1) = k_1]$$

$$= \frac{(\lambda t_1)^{k_1} [\lambda (t_2 - t_1)]^{k_2 - k_1} e^{-\lambda t_2}}{k_1!(k_2 - k_1)!}$$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=k_1}^{\infty} P(k_1, k_2) \delta(x_1 - k_1) \delta(x_2 - k_2)$$

Counting Process (1)

$i \to i+1 \to i+2 \to \cdots$

A **random process** is called a **counting process** if

1. the possible states are the non-negative integers.
2. for each **state** $i$the only possible transitions are $i \to i$, $i \to i+1$, $i \to i+2$, ...

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Counting Process (2)

$p, (1-p)$

A **counting process** is said to be a **Bernoulli counting process** if

1. the **number of successes** that can occur in each frame is either 0 or 1
2. the **probability** p that a success occurs during any frame is the same for all frames
3. **successes** in non-overlapping frames are independent of one another

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Counting Process (3)

$P(x[n] = k)$

Let $X[n]$ denote the total number of successes
in a **Bernoulli counting process**
at the end of the n-th frame, $n = 1, 2, 3....$
Let the initial state be $X[0] = 0$.
The **probability distribution** of $X[n]$ is the **binomial**

$$P(X[n] = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

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Counting Process (4)

$p = l/n$

If l denotes the expected number of successes in a unit of time, and n is the number of frames in this unit of time, then

$$p = l/n$$

The constant l is called the rate of success, and is estimated by

$$\hat{l} = Number\ of\ successes\ in\ t\ units\ of\ time/t$$

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Counting Process (5)

$D = 1/n$

If there are n frames in the unit of time and we let the **frame length**, denoted by D, be expressed in the same unit as l (seconds, minutes, hours,etc.), then

$$D = 1/n$$

Moreover,

$$p = lD$$

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Bernoulli Random Process (1)

$I[0], I[1], I[2], \cdots$

A **Bernoulli process** is a finite or infinite sequence of independent random variables $I[1], I[2], I[3], \ldots$, such that for each n, the value of $I[n]$ is either 0 or 1;

for all values of n, the probability p that $I[n] = 1$ is the same.

In other words, a **Bernoulli process** is a sequence of independent identically distributed **Bernoulli trials**.

Independence of the trials implies that the process is memoryless.

Bernoulli Random Process (2)

$X[n] = \Sigma_{m=1}^{n} I[m]$

the **Bernoulli random process** at sample index n is $I[n]$
the number of events that have occurred
after sample index 0 and up to n

$$X[n] = \sum_{m=1}^{n} I[m]$$

the **binomial counting process** is an example of what is called a **sum process**, since it can be obtained by summing the values of another random process

Bernoulli Random Process (3)

$f_X(x)$

the density function for $X[n]$is represented by
a binomial density function

$$f_X(x) = \sum_{k=0}^{n} P(k)\delta(x-k)$$

$$P(k) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

the mean and the variance of the binomial counting process

$$E\left[X\left[n\right]\right] = np$$
$$Var\left[X\left[n\right]\right] = np(1-p)$$

Binomial Counting Process

$f_X(x_1, x_2)$

$$f_X(x_1, x_2) = \sum_{k_1=0}^{n_1} \sum_{k_2=k_1}^{n_2} P(k_1, k_2)\delta(x_1 - k_1)\delta(x_2 - k_2)$$

$$P(k_1, k_2) = P\left[X[n_1] = k_1, X[n_2] = k_2\right]$$

$$= \begin{pmatrix} n_2 - n_1 \\ k_2 - k_1 \end{pmatrix} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix} p^{k_2} (1-p)^{n_2 - k_2}$$

$$P(k) = \frac{(np)^k e^{-np}}{k!} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$