

# Temporal Characteristics of Random Processes

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

## 1 Joint Distributions, Independence, and Moments

# First Order Distribution Function

$N$  Gaussian random variables

## Definition

For one particular time  $t_1$ , the distribution function associated with the random variable  $X_1 = X(t_1)$

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$$

the density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

# Second Order Distribution Function

$N$  Gaussian random variables

## Definition

For one particular time  $t_1, t_2$ , the distribution function associated with the random variables  $X_1 = X(t_1)$  and  $X_2 = X(t_2)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

the density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

# $N$ -th Order Distribution Function

$N$  Gaussian random variables

## Definition

For one particular time  $t_1, t_2, \dots, t_N$ ,

the distribution function

associated with the random variables

$$X_1 = X(t_1), X_2 = X(t_2), \dots, X_N = X(t_N)$$

$$F_X(x_1, \dots, x_N; t_1, \dots, t_N) = P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

the density function

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{\partial^N F_X(x_1, \dots, x_N; t_1, \dots, t_N)}{\partial x_1 \cdots \partial x_N}$$

# Statistical Independence

$N$  Gaussian random variables

## Definition

Two processes  $X(t)$ ,  $Y(t)$  are statistically independent if the random variable group  $X(t_1), X(t_2), \dots, X(t_N)$  is independent of the group  $Y(t'_1), Y(t'_2), \dots, Y(t'_M)$  for any choice of time  $t_1, t_2, \dots, t_N, t'_1, t'_2, \dots, t'_M$

Independence requires that the joint density be factorable by group

$$f_{X,Y}(x_1, \dots, x_N, y_1, \dots, y_M; t_1, \dots, t_N, t'_1, \dots, t'_M) \\ = f_X(x_1, \dots, x_N; t_1, \dots, t_N) f_Y(y_1, \dots, y_M; t'_1, \dots, t'_M)$$

# The 1st order moment

$N$  Gaussian random variables

## Definition

The mean of a random process

$$m_X(t) = E[X(t)]$$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx$$

$$m_X[n] = E[X[n]]$$



# The autocorrelation function

$N$  Gaussian random variables

## Definition

The correlation of a random process at two instants of time  $X(t_1)$  and  $X(t_2)$ , in general varies with  $t_1$  and  $t_2$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XX}[n, n + k] = E[X[n]X[n + k]]$$

# The autocovariance function

$N$  Gaussian random variables

## Definition

$$C_{XX}(t, t + \tau) = E[\{X(t) - m_X(t)\}\{X(t + \tau) - m_X(t + \tau)\}]$$

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)$$

$$C_{XX}[n, n + k] = E[\{X[n] - m_X[n]\}\{X[n + k] - m_X[n + k]\}]$$

$$C_{XX}[n, n + k] = R_{XX}[n, n + k] - m_X[n]m_X[n + k]$$

## The variance of a random process

 $N$  Gaussian random variables

## Definition

$$C_{XX}(t, t + \tau) = E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}]$$

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)$$

$$\tau = 0$$

$$C_{XX}(t, t) = R_{XX}(t, t) - (m_X(t))^2 = \sigma_X^2(t)$$

# The cross-correlation function

$N$  Gaussian random variables

## Definition

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)]$$

# The cross-covariance function (1)

$N$  Gaussian random variables

## Definition

$$C_{XX}(t, t + \tau) = E\{\{X(t) - m_X(t)\}\{X(t + \tau) - m_X(t + \tau)\}\}$$

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)$$

$$C_{XY}(t, t + \tau) = E\{\{X(t) - m_X(t)\}\{Y(t + \tau) - m_Y(t + \tau)\}\}$$

$$C_{XY}(t, t + \tau) = R_{XY}(t, t + \tau) - m_X(t)m_Y(t + \tau)$$

## The cross-covariance function (2)

 $N$  Gaussian random variables

## Definition

$$C_{XX}[n, n+k] = E[\{X[n] - m_X[n]\}\{X[n+k] - m_X[n+k]\}]$$

$$C_{XX}[n, n+k] = R_{XX}[n, n+k] - m_X[n]m_X[n+k]$$

$$C_{XY}[n, n+k] = E[\{X[n] - m_X[n]\}\{Y[n+k] - m_Y[n+k]\}]$$

$$C_{XY}[n, n+k] = R_{XY}[n, n+k] - m_X[n]m_Y[n+k]$$

## DT and CT relations

 $N$  Gaussian random variables

## Definition

$$m_X[n] = m_Y(nT_s)$$

$$R_{XX}[n, n+k] = R_{YY}(nT_s(n+k)T_s)$$

$$C_{XX}[n, n+k] = C_{YY}(nT_s(n+k)T_s)$$





