Laurent Series and z-Transform
- Geometric Series
Reciprocity Properties B

20191031 Tue

2 formulas Simple Pole Form

$\boxed{\dfrac{1}{z - p}}$ $\boxed{\dfrac{1}{z^{-1} - p}}$

2 representations each Geometric Series Form

causal anti- causal

$$\boxed{\frac{1}{z-p}} \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z}} \triangleq f(z) = X(z^{-1})$$

$$\| \qquad \|$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}} \triangleq Y(z) = g(z^{-1})$$

causal anti- causal

causal anti- causal

$$\boxed{\frac{1}{z^{-1}-p}} \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq X(z) = f(z^{-1})$$

$$\| \qquad \|$$

$$\boxed{\frac{z}{1-pz}} \triangleq g(z) = Y(z^{-1})$$

causal anti- causal

Simple Pole Form | Geometric Series Form

$$f(z) = g(z^{-1}) \qquad f(z^{-1}) = g(z)$$

$\bar{f}(z)$ $\bar{f}(z^{-1})$

$\bar{g}(z^{-1})$ $\bar{g}(z)$

$$\begin{array}{l} f(z) = f(a, z) \\ \Vert \\ g(z^{-1}) = g(a, z^{-1}) \end{array} \qquad \begin{array}{l} f(z^{-1}) = f(a, z^{-1}) \\ \Vert \\ g(z) = g(a, z) \end{array}$$

$$\begin{array}{l} \bar{f}(z) = f(a^{-1}, z) \\ \Vert \\ \bar{g}(z^{-1}) = g(a^{-1}, z^{-1}) \end{array} \qquad \begin{array}{l} \bar{f}(z^{-1}) = f(a^{-1}, z^{-1}) \\ \Vert \\ \bar{g}(z) = g(a^{-1}, z) \end{array}$$

Geometric Series : $f(z)$, $g(z^{-1})$, $\bar{f}(z)$, $\bar{g}(z^{-1})$

①

$f(z) = -\dfrac{a}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = -a^{n+1}$	$(n \geq 0)$

②

$f(z^{-1}) = -\dfrac{a}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = -\left(\dfrac{1}{a}\right)^{n-1}$	$(n < 1)$

③

$g(z^{-1}) = \dfrac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = a^{n+1}$	$(n < 0)$

④

$g(z) = \dfrac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = \left(\dfrac{1}{a}\right)^{n-1}$	$(n \geq 1)$

⑤

$\bar{f}(z) = -\dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = -\left(\dfrac{1}{a}\right)^{n+1}$	$(n \geq 0)$

⑥

$\bar{f}(z^{-1}) = -\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = -a^{n-1}$	$(n < 1)$

⑦

$\bar{g}(z^{-1}) = \dfrac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = \left(\dfrac{1}{a}\right)^{n+1}$	$(n < 0)$

⑧

$\bar{g}(z) = \dfrac{z}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = a^{n-1}$	$(n \geq 1)$

simple pole models with a unit nominator

$a^{-1} f(z)$, $z\, g(z^{-1})$, $a\, \bar{f}(z)$, $z\, \bar{g}(z^{-1})$

①

$a^{-1} f(z) = -\frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = -a^n$	$(n \geq 0)$

②

$a^{-1} f(z^{-1}) = -\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = -(\frac{1}{a})^n$	$(n < 1)$

③

$z\, g(z^{-1}) = \frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = a^n$	$(n < 1)$

④

$z^{-1} g(z) = \frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = (\frac{1}{a})^n$	$(n \geq 0)$

⑤

$a\bar{f}(z) = -\frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = -(\frac{1}{a})^n$	$(n \geq 0)$

⑥

$a\bar{f}(z^{-1}) = -\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = -a^n$	$(n < 1)$

⑦

$z\,\bar{g}(z^{-1}) = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = (\frac{1}{a})^n$	$(n < 1)$

⑧

$z^{-1}\bar{g}(z) = \frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = a^n$	$(n \geq 0)$

	Inv(z) / Inv(z)		Inv(z) / Inv(z) simple model with a unit nominator	z expression / ROC
(1)	$f(z) = -\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{a^{-1} f(z)} = -\frac{1}{1-az} = -\frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$
	Inv(z)	Inv(z)	Inv(z)	Inv(z)
(2)	$f(z^{-1}) = -\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{a^{-1} f(z^{-1})} = -\frac{1}{1-az^{-1}} = -\frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
(3)	$g(z^{-1}) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{z\, g(z^{-1})} = \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
	Inv(z)	Inv(z)	Inv(z)	Inv(z)
(4)	$g(z) = \frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{z^{-1} g(z)} = \frac{1}{1-a^{-1}z} = \frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
(5)	$\bar{f}(z) = -\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{a\, \bar{f}(z)} = -\frac{1}{1-a^{-1}z} = -\frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
	Inv(z)	Inv(z)	Inv(z)	Inv(z)
(6)	$\bar{f}(z^{-1}) = -\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{a\, \bar{f}(z^{-1})} = -\frac{1}{1-a^{-1}z^{-1}} = -\frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
(7)	$\bar{g}(z^{-1}) = \frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{z\, \bar{g}(z^{-1})} = \frac{1}{1-az^{-1}} = \frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
	Inv(z)	Inv(z)	Inv(z)	Inv(z)
(8)	$\bar{g}(z) = \frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{z^{-1} \bar{g}(z)} = \frac{1}{1-az} = \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$

	Neg(n) Sym(n)	Neg(n) Sym(n)	a sequence Range
(1)	$a_n = -a^{n+1} \quad (n \geq 0)$	$a_n = -a^{n} \quad (n \geq 0)$	
	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	
(2)	$a_n = -\left(\frac{1}{a}\right)^{n-1} \quad (n < 1)$	$a_n = -\left(\frac{1}{a}\right)^{n} \quad (n < 1)$	
(3)	$a_n = a^{n+1} \quad (n < 0)$	$a_n = a^{n} \quad (n < 1)$	
	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	
(4)	$a_n = \left(\frac{1}{a}\right)^{n-1} \quad (n \geq 1)$	$a_n = \left(\frac{1}{a}\right)^{n} \quad (n \geq 0)$	
(5)	$a_n = -\left(\frac{1}{a}\right)^{n+1} \quad (n \geq 0)$	$a_n = -\left(\frac{1}{a}\right)^{n} \quad (n \geq 0)$	
	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	
(6)	$a_n = -a^{n-1} \quad (n < 1)$	$a_n = -a^{n} \quad (n < 1)$	
(7)	$a_n = \left(\frac{1}{a}\right)^{n+1} \quad (n < 0)$	$a_n = \left(\frac{1}{a}\right)^{n} \quad (n < 1)$	
	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	Neg(n) $\updownarrow$ Sym(n) $\updownarrow$	
(8)	$a_n = a^{n-1} \quad (n \geq 1)$	$a_n = a^{n} \quad (n \geq 0)$	

	Inv(a) Inv(a)		Inv(a) Inv(a) simple model with a unit nominator	z expression ROC
(1)	$f(z) = -\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{a^{-1} f(z)} = -\frac{1}{1-az} = -\frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$
	Inv(a)	Inv(a)	Inv(a)	Inv(a)
(5)	$\bar{f}(z) = -\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{a \bar{f}(z)} = -\frac{1}{1-a^{-1}z} = -\frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
(2)	$f(z^{-1}) = -\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{a^{-1} f(z^{-1})} = -\frac{1}{1-az^{-1}} = -\frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
	Inv(a)	Inv(a)	Inv(a)	Inv(a)
(6)	$\bar{f}(z^{-1}) = -\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{a \bar{f}(z^{-1})} = -\frac{1}{1-a^{-1}z^{-1}} = -\frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
(3)	$g(z^{-1}) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{z\, g(z^{-1})} = \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
	Inv(a)	Inv(a)	Inv(a)	Inv(a)
(7)	$\bar{g}(z^{-1}) = \frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{z\, \bar{g}(z^{-1})} = \frac{1}{1-az^{-1}} = \frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
(4)	$g(z) = \frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{z^{-1} g(z)} = \frac{1}{1-a^{-1}z} = \frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
	Inv(a)	Inv(a)	Inv(a)	Inv(a)
(8)	$\bar{g}(z) = \frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{z^{-1} \bar{g}(z)} = \frac{1}{1-az} = \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$

	Inv(a)	Id	Inv(a)	Id
			a sequence	range
(1)	$a_n = -a^{n+1}$	$(n \geq 0)$	$a_n = -a^{n}$	$(n \geq 0)$
	Inv(a) $\Updownarrow$	$\|$	Inv(a) $\Updownarrow$	$\|$
(5)	$a_n = -\left(\frac{1}{a}\right)^{n+1}$	$(n \geq 0)$	$a_n = -\left(\frac{1}{a}\right)^{n}$	$(n \geq 0)$
(2)	$a_n = -\left(\frac{1}{a}\right)^{n-1}$	$(n < 1)$	$a_n = -\left(\frac{1}{a}\right)^{n}$	$(n < 1)$
	Inv(a) $\Updownarrow$	$\|$	Inv(a) $\Updownarrow$	$\|$
(6)	$a_n = -a^{n-1}$	$(n < 1)$	$a_n = -a^{n}$	$(n < 1)$
(3)	$a_n = a^{n+1}$	$(n < 0)$	$a_n = a^{n}$	$(n < 1)$
	Inv(a) $\Updownarrow$	$\|$	Inv(a) $\Updownarrow$	$\|$
(7)	$a_n = \left(\frac{1}{a}\right)^{n+1}$	$(n < 0)$	$a_n = \left(\frac{1}{a}\right)^{n}$	$(n < 1)$
(4)	$a_n = \left(\frac{1}{a}\right)^{n-1}$	$(n \geq 1)$	$a_n = \left(\frac{1}{a}\right)^{n}$	$(n \geq 0)$
	Inv(a) $\Updownarrow$	$\|$	Inv(a) $\Updownarrow$	$\|$
(8)	$a_n = a^{n-1}$	$(n \geq 1)$	$a_n = a^{n}$	$(n \geq 0)$

	Id Comp(z)		Sign, Inv(a,z) Comp(z)		z expression ROC
			simple model with a unit nominator		
(1)	$f(z) = -\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{a^{-1} f(z)} = -\frac{1}{1-az}$	$= -\frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$
	$\Vert$	$\frac{a\,z}{a\,z} \rightleftarrows \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}}$ comp(z)	$\not\Vert$	sign, inv(a,z)	comp(z)
(3)	$g(z^{-1}) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{z\, g(z^{-1})} = \frac{1}{1-a^{-1}z^{-1}}$	$= \frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
(2)	$f(z^{-1}) = -\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{a^{-1} f(z^{-1})} = -\frac{1}{1-az^{-1}}$	$= -\frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
	$\Vert$	$\frac{a\,z^{-1}}{a\,z^{-1}} \rightleftarrows \frac{a^{-1}z}{a^{-1}z}$ comp(z)	$\not\Vert$	sign, inv(a,z)	comp(z)
(4)	$g(z) = \frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{z^{-1} g(z)} = \frac{1}{1-a^{-1}z}$	$= \frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
(5)	$\bar f(z) = -\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{a\, \bar f(z)} = -\frac{1}{1-a^{-1}z}$	$= -\frac{az^{-1}}{az^{-1}-1}$	$\lvert z\rvert < a$
	$\Vert$	$\frac{a^{-1}z}{a^{-1}z} \rightleftarrows \frac{a\,z^{-1}}{a\,z^{-1}}$ comp(z)	$\not\Vert$	sign, inv(a,z)	comp(z)
(7)	$\bar g(z^{-1}) = \frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{z\, \bar g(z^{-1})} = \frac{1}{1-az^{-1}}$	$= \frac{a^{-1}z}{a^{-1}z-1}$	$\lvert z\rvert > a$
(6)	$\bar f(z^{-1}) = -\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$(n < 1)$	$\boxed{a\, \bar f(z^{-1})} = -\frac{1}{1-a^{-1}z^{-1}}$	$= -\frac{az}{az-1}$	$\lvert z\rvert > a^{-1}$
	$\Vert$	$\frac{a^{-1}z^{-1}}{a^{-1}z^{-1}} \rightleftarrows \frac{a\,z}{a\,z}$ comp(z)	$\not\Vert$	sign, inv(a,z)	comp(z)
(8)	$\bar g(z) = \frac{z}{1-az}$	$(n \geq 1)$	$\boxed{z^{-1} \bar g(z)} = \frac{1}{1-az}$	$= \frac{a^{-1}z^{-1}}{a^{-1}z^{-1}-1}$	$\lvert z\rvert < a^{-1}$

	Sign Comp(z)	Sign Comp(z)	a sequence range
(1)	$a_n = -a^{n+1}$ $(n \geq 0)$	$a_n = -a^{n}$ $(n \geq 0)$	
	sign comp(n)	sign comp(n)	
(3)	$a_n = a^{n+1}$ $(n < 0)$	$a_n = a^{n}$ $(n < 1)$	
(2)	$a_n = -\left(\frac{1}{a}\right)^{n-1}$ $(n < 1)$	$a_n = -\left(\frac{1}{a}\right)^{n}$ $(n < 1)$	
	sign comp(n)	sign comp(n)	
(4)	$a_n = \left(\frac{1}{a}\right)^{n-1}$ $(n \geq 1)$	$a_n = \left(\frac{1}{a}\right)^{n}$ $(n \geq 0)$	
(5)	$a_n = -\left(\frac{1}{a}\right)^{n+1}$ $(n \geq 0)$	$a_n = -\left(\frac{1}{a}\right)^{n}$ $(n \geq 0)$	
	sign comp(n)	sign comp(n)	
(7)	$a_n = \left(\frac{1}{a}\right)^{n+1}$ $(n < 0)$	$a_n = \left(\frac{1}{a}\right)^{n}$ $(n < 1)$	
(6)	$a_n = -a^{n-1}$ $(n < 1)$	$a_n = -a^{n}$ $(n < 1)$	
	sign comp(n)	sign comp(n)	
(8)	$a_n = a^{n-1}$ $(n \geq 1)$	$a_n = a^{n}$ $(n \geq 0)$	

	Inv(z) / Inv(z)		Inv(z) / Inv(z) simple model with a unit nominator	z expression / ROC
(1)	$f(z) = -\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{a^{-1} f(z)} = -\frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
	$\cdot a^{-1}$	id	$\cdot a$	id
(2)	$f(z^{-1}) = -\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{a^{-1} f(z^{-1})} = -\frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
	$\cdot a^{-1}$	id	$\cdot a$	id
(3)	$g(z^{-1}) = \frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{z\, g(z^{-1})} = \frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
	$\cdot z$	id	$\cdot z^{-1}$	id
(4)	$g(z) = \frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{z^{-1} g(z)} = \frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
	$\cdot z^{-1}$	id	$\cdot z$	id
(5)	$\bar{f}(z) = -\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\boxed{a\, \bar{f}(z)} = -\frac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
	$\cdot a$	id	$\cdot a^{-1}$	id
(6)	$\bar{f}(z^{-1}) = -\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\boxed{a\, \bar{f}(z^{-1})} = -\frac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
	$\cdot a$	id	$\cdot a^{-1}$	id
(7)	$\bar{g}(z^{-1}) = \frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\boxed{z\, \bar{g}(z^{-1})} = \frac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
	$\cdot z$	id	$\cdot z^{-1}$	id
(8)	$\bar{g}(z) = \frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$\boxed{z^{-1} \bar{g}(z)} = \frac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
	$\cdot z^{-1}$	id	$\cdot z$	id

	Neg(n) Sym(n)	Neg(n) Sym(n)				a sequence Range
(1)	$a_n = -a^{n+1}$ $\cdot a^{-1}$	$(n \geq 0)$ id	$\rightarrow$	$\leftarrow$	$a_n = -a^{n}$ $\cdot a$	$(n \geq 0)$ id
(2)	$a_n = -\left(\frac{1}{a}\right)^{n-1}$ $\cdot a^{-1}$	$(n < 1)$ id	$\rightarrow$	$\leftarrow$	$a_n = -\left(\frac{1}{a}\right)^{n}$ $\cdot a$	$(n < 1)$ id
(3)	$a_n = a^{n+1}$ $\cdot a^{-1}$	$(n < 0)$ sym(comp(n))	$\rightarrow$	$\leftarrow$	$a_n = a^{n}$ $\cdot a$	$(n < 1)$ sym(comp(n))
(4)	$a_n = \left(\frac{1}{a}\right)^{n-1}$ $\cdot a^{-1}$	$(n \geq 1)$ sym(comp(n))	$\rightarrow$	$\leftarrow$	$a_n = \left(\frac{1}{a}\right)^{n}$ $\cdot a$	$(n \geq 0)$ sym(comp(n))
(5)	$a_n = -\left(\frac{1}{a}\right)^{n+1}$ $\cdot a$	$(n \geq 0)$ id	$\rightarrow$	$\leftarrow$	$a_n = -\left(\frac{1}{a}\right)^{n}$ $\cdot a^{-1}$	$(n \geq 0)$ id
(6)	$a_n = -a^{n-1}$ $\cdot a$	$(n < 1)$ id	$\rightarrow$	$\leftarrow$	$a_n = -a^{n}$ $\cdot a^{-1}$	$(n < 1)$ id
(7)	$a_n = \left(\frac{1}{a}\right)^{n+1}$ $\cdot a$	$(n < 0)$ sym(comp(n))	$\rightarrow$	$\leftarrow$	$a_n = \left(\frac{1}{a}\right)^{n}$ $\cdot a^{-1}$	$(n < 1)$ sym(comp(n))
(8)	$a_n = a^{n-1}$ $\cdot a$	$(n \geq 1)$ sym(comp(n))	$\rightarrow$	$\leftarrow$	$a_n = a^{n}$ $\cdot a^{-1}$	$(n \geq 0)$ sym(comp(n))

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$
$\bar{f}(z)$	$\bar{f}(z^{-1})$
$\bar{g}(z^{-1})$	$\bar{g}(z)$

(1)

$a_n = -a^{n+1}$	$(n \geq 0)$
$f(z) = -\dfrac{a}{1-az}$	$\lvert z \rvert < a^{-1}$

(1) ↔ (2): neg(n) sym(n), inv(z) inv(z)

(2)

$a_n = -\left(\frac{1}{a}\right)^{n-1}$	$(n < 1)$
$f(z^{-1}) = -\dfrac{a}{1-az^{-1}}$	$\lvert z \rvert > a$

(1) ↔ (3): sign comp(z), sign comp(n)

(2) ↔ (4): sign comp(z), sign comp(n)

(3)

$g(z^{-1}) = \dfrac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z \rvert > a^{-1}$
$a_n = a^{n+1}$	$(n < 0)$

(3) ↔ (4): inv(z) inv(z), neg(n) sym(n)

(4)

$g(z) = \dfrac{z}{1-a^{-1}z}$	$\lvert z \rvert < a$
$a_n = \left(\frac{1}{a}\right)^{n-1}$	$(n \geq 1)$

(5)

$a_n = -\left(\frac{1}{a}\right)^{n+1}$	$(n \geq 0)$
$\bar{f}(z) = -\dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z \rvert < a$

(5) ↔ (6): neg(n) sym(n), inv(z) inv(z)

(6)

$a_n = -a^{n-1}$	$(n < 1)$
$\bar{f}(z^{-1}) = -\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z \rvert > a^{-1}$

(5) ↔ (7): sign comp(z), sign comp(n)

(6) ↔ (8): sign comp(z), sign comp(n)

(7)

$\bar{g}(z^{-1}) = \dfrac{z^{-1}}{1-az^{-1}}$	$\lvert z \rvert > a$
$a_n = \left(\frac{1}{a}\right)^{n+1}$	$(n < 0)$

(7) ↔ (8): inv(z) inv(z), neg(n) sym(n)

(8)

$\bar{g}(z) = \dfrac{z}{1-az}$	$\lvert z \rvert < a^{-1}$
$a_n = a^{n-1}$	$(n \geq 1)$

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$
$\bar{f}(z)$	$\bar{f}(z^{-1})$
$\bar{g}(z^{-1})$	$\bar{g}(z)$

(1)

$a_n = -a^{n+1}$	$(n \geq 0)$
$f(z) = -\dfrac{a}{1-az}$	$\lvert z\rvert < a^{-1}$

neg(n) sym(n)

inv(z) inv(z)

(2)

$a_n = -\left(\frac{1}{a}\right)^{n-1}$	$(n < 1)$
$f(z^{-1}) = -\dfrac{a}{1-az^{-1}}$	$\lvert z\rvert > a$

inv(a) inv(a)
inv(a)

inv(a) inv(a)
inv(a)

(5)

$\bar{f}(z) = -\dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = -\left(\frac{1}{a}\right)^{n+1}$	$(n \geq 0)$

inv(z) inv(z)

neg(n) sym(n)

(6)

$\bar{f}(z^{-1}) = -\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = -a^{n-1}$	$(n < 1)$

(3)

$a_n = a^{n+1}$	$(n < 0)$
$g(z^{-1}) = \dfrac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$

neg(n) sym(n)

inv(z) inv(z)

(4)

$a_n = \left(\frac{1}{a}\right)^{n-1}$	$(n \geq 1)$
$g(z) = \dfrac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$

inv(a) inv(a)
inv(a)

inv(a) inv(a)
inv(a)

(7)

$\bar{g}(z^{-1}) = \dfrac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = \left(\frac{1}{a}\right)^{n+1}$	$(n < 0)$

inv(z) inv(z)

neg(n) sym(n)

(8)

$\bar{g}(z) = \dfrac{z}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = a^{n-1}$	$(n \geq 1)$

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$a^{-1}f(z)$ $\quad a^{-1}f(z^{-1})$

$zg(z^{-1})$ $\quad z^{-1}g(z)$

$a\bar{f}(z)$ $\quad a\bar{f}(z^{-1})$

$z\bar{g}(z^{-1})$ $\quad z^{-1}\bar{g}(z)$

a unit nominator

(1)

$a_n = -a^n \quad (n \geq 0)$

$a^{-1}f(z) = -\dfrac{1}{1-az} \quad |z| < a^{-1}$

neg(n) sym(n)

inv(z) inv(z)

(2)

$a_n = -\left(\frac{1}{a}\right)^n \quad (n < 1)$

$a^{-1}f(z^{-1}) = -\dfrac{1}{1-az^{-1}} \quad |z| > a$

(1) ↔ (3): sign, inv(a,z) comp(z); sign comp(n)

(2) ↔ (4): sign, inv(a,z) comp(z); sign comp(n)

(3)

$zg(z^{-1}) = \dfrac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$

$a_n = a^n \quad (n < 1)$

inv(z) inv(z)

neg(n) sym(n)

(4)

$z^{-1}g(z) = \dfrac{1}{1-a^{-1}z} \quad |z| < a$

$a_n = \left(\frac{1}{a}\right)^n \quad (n \geq 0)$

(5)

$a_n = -\left(\frac{1}{a}\right)^n \quad (n \geq 0)$

$a\bar{f}(z) = -\dfrac{1}{1-a^{-1}z} \quad |z| < a$

neg(n) sym(n)

inv(z) inv(z)

(6)

$a_n = -a^n \quad (n < 1)$

$a\bar{f}(z^{-1}) = -\dfrac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$

(5) ↔ (7): sign, inv(a,z) comp(z); sign comp(n)

(6) ↔ (8): sign, inv(a,z) comp(z); sign comp(n)

(7)

$z\bar{g}(z^{-1}) = \dfrac{1}{1-az^{-1}} \quad |z| > a$

$a_n = \left(\frac{1}{a}\right)^n \quad (n < 1)$

inv(z) inv(z)

neg(n) sym(n)

(8)

$z^{-1}\bar{g}(z) = \dfrac{1}{1-az} \quad |z| < a^{-1}$

$a_n = a^n \quad (n \geq 0)$

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$a^{-1}f(z)$	$a^{-1}f(z^{-1})$
$z\,g(z^{-1})$	$z^{-1}g(z)$
$a\bar{f}(z)$	$a\bar{f}(z^{-1})$
$z\,\bar{g}(z^{-1})$	$z^{-1}\bar{g}(z)$

a unit nominator

①

$a_n = -a^n$	$(n \geq 0)$
$a^{-1}f(z) = -\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$

① ⟷ ②: neg(n) sym(n) / inv(z) inv(z)

②

$a_n = -(\frac{1}{a})^n$	$(n < 1)$
$a^{-1}f(z^{-1}) = -\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$

① ⟷ ⑤: inv(a) inv(a) inv(a)

② ⟷ ⑥: inv(a) inv(a) inv(a)

⑤

$a\bar{f}(z) = -\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a_n = -(\frac{1}{a})^n$	$(n \geq 0)$

⑤ ⟷ ⑥: inv(z) inv(z) / neg(n) sym(n)

⑥

$a\bar{f}(z^{-1}) = -\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a_n = -a^n$	$(n < 1)$

③

$a_n = a^n$	$(n < 1)$
$z\,g(z^{-1}) = \dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$

③ ⟷ ④: neg(n) sym(n) / inv(z) inv(z)

④

$a_n = (\frac{1}{a})^n$	$(n \geq 0)$
$z^{-1}g(z) = \dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$

③ ⟷ ⑦: inv(a) inv(a) inv(a)

④ ⟷ ⑧: inv(a) inv(a) inv(a)

⑦

$z\,\bar{g}(z^{-1}) = \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$a_n = (\frac{1}{a})^n$	$(n < 1)$

⑦ ⟷ ⑧: inv(z) inv(z) / neg(n) sym(n)

⑧

$z^{-1}\bar{g}(z) = \dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$a_n = a^n$	$(n \geq 0)$

	simple pole models original expression				simple pole models with a unit nominator	
1	$a_n = -a^{n+1}$	$(n \geq 0)$	$\cdot\, a^{-1}$	id	$a_n = -a^{n}$	$(n \geq 0)$
	$f(z) = -\dfrac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$\cdot\, a^{1}$	id	$a^{-1}f(z) = -\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
2	$a_n = -\left(\frac{1}{a}\right)^{n-1}$	$(n < 1)$	$\cdot\, a^{-1}$	id	$a_n = -\left(\frac{1}{a}\right)^{n}$	$(n < 1)$
	$f(z^{-1}) = -\dfrac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$\cdot\, a^{-1}$	id	$a^{-1}f(z^{-1}) = -\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
3	$g(z^{-1}) = \dfrac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\cdot\, z$	id	$z\,g(z^{-1}) = \dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
	$a_n = a^{n+1}$	$(n < 0)$	$\cdot\, a^{-1}$	S(c(n))	$a_n = a^{n}$	$(n < 1)$
4	$g(z) = \dfrac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\cdot\, z^{-1}$	id	$z^{-1}g(z) = \dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
	$a_n = \left(\frac{1}{a}\right)^{n-1}$	$(n \geq 1)$	$\cdot\, a^{-1}$	S(c(n))	$a_n = \left(\frac{1}{a}\right)^{n}$	$(n \geq 0)$
5	$a_n = -\left(\frac{1}{a}\right)^{n+1}$	$(n \geq 0)$	$\cdot\, a$	id	$a_n = -\left(\frac{1}{a}\right)^{n}$	$(n \geq 0)$
	$\bar{f}(z) = -\dfrac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\cdot\, a$	id	$a\bar{f}(z) = -\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
6	$a_n = -a^{n-1}$	$(n < 1)$	$\cdot\, a$	id	$a_n = -a^{n}$	$(n < 1)$
	$\bar{f}(z^{-1}) = -\dfrac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$\cdot\, a$	id	$a\bar{f}(z^{-1}) = -\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
7	$\bar{g}(z^{-1}) = \dfrac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$\cdot\, z$	id	$z\bar{g}(z^{-1}) = \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
	$a_n = \left(\frac{1}{a}\right)^{n+1}$	$(n < 0)$	$\cdot\, a$	s(c(n))	$a_n = \left(\frac{1}{a}\right)^{n}$	$(n < 1)$
8	$\bar{g}(z) = \dfrac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$\cdot\, z^{-1}$	id	$z^{-1}\bar{g}(z) = \dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
	$a_n = a^{n-1}$	$(n \geq 1)$	$\cdot\, a$	s(c(n))	$a_n = a^{n}$	$(n \geq 0)$

simple pole models with a unit nominator

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$a^{-1} f(z)$, $z\, g(z^{-1})$, $a\, \bar{f}(z)$, $z\, \bar{g}(z^{-1})$

① $a^{-1} f(z)$

$-\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$-a^{n}$	$u(n)$

② $a^{-1} f(z^{-1})$

$-\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^{-n}$	$u(-n)$

③ $z\, g(z^{-1})$

$\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
a^{n}	$u(-n)$

④ $z^{-1} g(z)$

$\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
a^{-n}	$u(n)$

⑤ $a\, \bar{f}(z)$

$-\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$-a^{-n}$	$u(n)$

⑥ $a\, \bar{f}(z^{-1})$

$-\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$-a^{n}$	$u(-n)$

⑦ $z\, \bar{g}(z^{-1})$

$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
a^{-n}	$u(-n)$

⑧ $z^{-1} \bar{g}(z)$

$\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
a^{n}	$u(n)$

simple pole models original expressions

(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

$f(z)$, $g(z^{-1})$, $\bar{f}(z)$, $\bar{g}(z^{-1})$

① $f(z)$

$a \times -\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$-a^{n+1}$	$u(n)$

② $f(z^{-1})$

$a \times -\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$-a^{-n+1}$	$u(-n)$

③ $g(z^{-1})$

$z^{-1} \times \dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
a^{n+1}	$u(-n-1)$

④ $g(z)$

$z \times \dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
a^{-n+1}	$u(n-1)$

⑤ $\bar{f}(z)$

$a^{-1} \times -\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$-a^{-n-1}$	$u(n)$

⑥ $\bar{f}(z^{-1})$

$a^{-1} \times -\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$-a^{n-1}$	$u(-n)$

⑦ $\bar{g}(z^{-1})$

$z^{-1} \times \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
a^{-n-1}	$u(-n-1)$

⑧ $\bar{g}(z)$

$z \times \dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
a^{n-1}	$u(n-1)$

Shift Left / Shift Right

①

$a \times -\dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
$a \times -a^{n}$	$u(n)$

②

$a \times -\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
$a \times -a^{-n}$	$u(-n)$

③ shift left

$z^{-1} \times \dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
a^{n}	$u(-n)$

④ shift right

$z \times \dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
a^{-n}	$u(n)$

⑤

$a^{-1} \times -\dfrac{1}{1-a^{-1}z}$	$\lvert z\rvert < a$
$a^{-1} \times -a^{-n}$	$u(n)$

⑥

$a^{-1} \times -\dfrac{1}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$
$a^{-1} \times -a^{n}$	$u(-n)$

⑦ shift left

$z^{-1} \times \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > a$
a^{-n}	$u(-n)$

⑧ shift right

$z \times \dfrac{1}{1-az}$	$\lvert z\rvert < a^{-1}$
a^{n}	$u(n)$

shift left: $a^{n} \xrightarrow{n \leftarrow n+1} a^{n+1}$

shift left: $u(-n) \xrightarrow{n \leftarrow n+1} u(-n-1)$

shift right: $a^{-n} \xrightarrow{n \leftarrow n-1} a^{-n+1}$

shift right: $u(n) \xrightarrow{n \leftarrow n-1} u(n-1)$

shift left: $a^{-n} \xrightarrow{n \leftarrow n+1} a^{-n-1}$

shift left: $u(-n) \xrightarrow{n \leftarrow n+1} u(-n-1)$

shift right: $a^{n} \xrightarrow{n \leftarrow n-1} a^{n-1}$

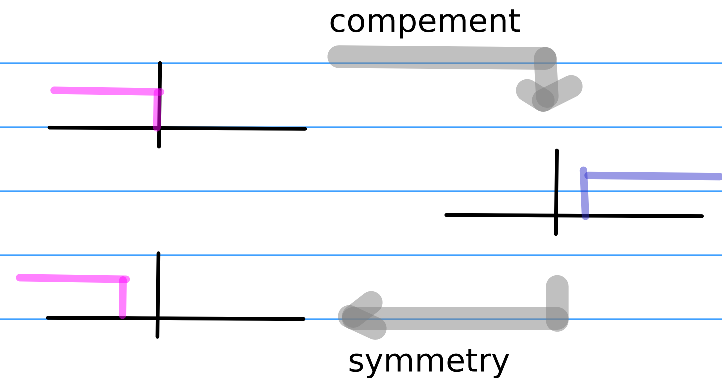
shift right: $u(n) \xrightarrow{n \leftarrow n-1} u(n-1)$

shift left $\quad n \leftarrow n+1$ $\qquad$ Comp(Symm(n))

$a^{n} \longrightarrow a^{n+1}$

$a^{-n} \longrightarrow a^{-n-1}$

$u(-n) \longrightarrow u(-n-1)$

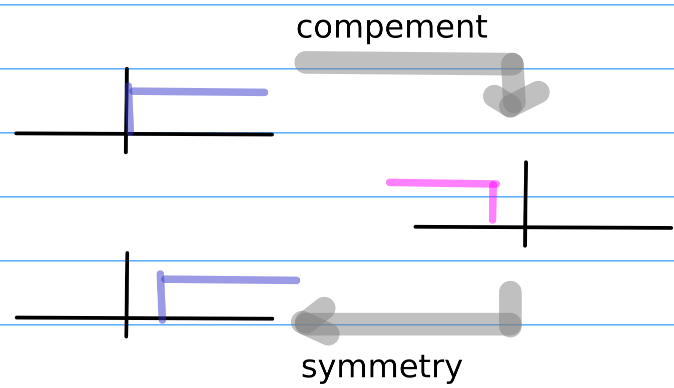


shift right $\quad n \leftarrow n-1$ $\qquad$ Comp(Symm(n))

$a^{-n} \longrightarrow a^{-n+1}$

$a^{n} \longrightarrow a^{n-1}$

$u(n) \longrightarrow u(n-1)$



Comp(Symm(n)) = Symm(Comp(n))

simple pole models
from the unit nominator to original expression

$-\frac{1}{1-az}$ $\quad a$ $\qquad\qquad -\frac{1}{1-az^{-1}}$ $\quad a$

$\frac{1}{1-a^{-1}z^{-1}}$ $\quad z^{-1}$ $\qquad\qquad \frac{1}{1-a^{-1}z}$ $\quad z$

$-\frac{1}{1-a^{-1}z}$ $\quad a^{-1}$ $\qquad\qquad -\frac{1}{1-a^{-1}z^{-1}}$ $\quad a^{-1}$

$\frac{1}{1-az^{-1}}$ $\quad z^{-1}$ $\qquad\qquad \frac{1}{1-az}$ $\quad z$

Geometric series of the unit nominator expressions

$-\frac{1}{1-az} = -(a^0z^0 + a^1z^1 + a^2z^2 + \cdots)$ $-a^n$ $u(n)$	$-\frac{1}{1-az^{-1}} = -(a^0z^0 + a^1z^{-2} + a^2z^{-2} + \cdots)$ $-a^{-n}$ $u(-n)$
$\frac{1}{1-a^{-1}z^{-1}} = +(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots)$ a^n $u(-n)$	$\frac{1}{1-a^{-1}z} = +(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots)$ a^{-n} $u(n)$
$-\frac{1}{1-a^{-1}z} = -(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots)$ $-a^{-n}$ $u(n)$	$-\frac{1}{1-a^{-1}z^{-1}} = -(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots)$ $-a^n$ $u(-n)$
$\frac{1}{1-az^{-1}} = +(a^0z^0 + a^1z^{-2} + a^2z^{-2} + \cdots)$ a^{-n} $u(-n)$	$\frac{1}{1-az} = +(a^0z^0 + a^1z^1 + a^2z^2 + \cdots)$ a^n $u(n)$

Geometric series of the original expressions

$a \cdot \frac{-1}{1-az} = -(a^1 z^0 + a^2 z^1 + a^3 z^2 + \cdots)$ $-a^{n+1}$ $\quad u(n)$	$a \cdot \frac{-1}{1-az^{-1}} = -(a^1 z^0 + a^2 z^{-1} + a^3 z^{-2} + \cdots)$ $-a^{-n+1}$ $\quad u(-n)$
$z^{-1} \cdot \frac{1}{1-a^{-1}z^{-1}} = +(a^0 z^{-1} + a^{-1} z^{-2} + a^{-2} z^{-3} + \cdots)$ a^{n+1} $\quad u(-n-1)$	$z \cdot \frac{1}{1-a^{-1}z} = +(a^0 z^1 + a^{-1} z^2 + a^{-2} z^3 + \cdots)$ a^{-n+1} $\quad u(n-1)$
$a^{-1} \cdot \frac{-1}{1-a^{-1}z} = -(a^{-1} z^0 + a^{-2} z^1 + a^{-3} z^2 + \cdots)$ $-a^{-n-1}$ $\quad u(n)$	$a^{-1} \cdot \frac{-1}{1-a^{-1}z^{-1}} = -(a^{-1} z^0 + a^{-2} z^{-1} + a^{-3} z^{-2} + \cdots)$ $-a^{n-1}$ $\quad u(-n)$
$z^{-1} \cdot \frac{1}{1-az^{-1}} = +(a^0 z^{-1} + a^1 z^{-2} + a^2 z^{-3} + \cdots)$ a^{-n-1} $\quad u(-n-1)$	$z \cdot \frac{1}{1-az} = +(a^0 z^1 + a^1 z^2 + a^2 z^3 + \cdots)$ a^{n-1} $\quad u(n-1)$

Relations between the unit nominator and original expressions (1)

$$-\frac{1}{1-az} = -(a^0z^0 + a^1z^1 + a^2z^2 + \cdots)$$

$-a^{n}$ $u(n)$

$$-\frac{1}{1-az^{-1}} = -(a^0z^0 + a^1z^{-2} + a^2z^{-2} + \cdots)$$

$-a^{-n}$ $u(-n)$

$$a \cdot \frac{-1}{1-az} = -(a^1z^0 + a^2z^1 + a^3z^2 + \cdots)$$

$-a^{n+1}$ $u(n)$

$$a \cdot \frac{-1}{1-az^{-1}} = -(a^1z^0 + a^2z^{-1} + a^3z^{-2} + \cdots)$$

$-a^{-n+1}$ $u(-n)$

$$-a^{n} \xrightarrow{a\times} -a^{n+1} \qquad u(n) \rightarrow u(n)$$

$$-a^{-n} \xrightarrow{a\times} -a^{-n+1} \qquad u(-n) \rightarrow u(-n)$$

$$-\frac{1}{1-a^{-1}z} = -(a^0z^0 + a^{-1}z^1 + a^{-2}z^2 + \cdots)$$

$-a^{-n}$ $u(n)$

$$-\frac{1}{1-a^{-1}z^{-1}} = -(a^0z^0 + a^{-1}z^{-1} + a^{-2}z^{-2} + \cdots)$$

$-a^{n}$ $u(-n)$

$$a^{-1} \cdot \frac{-1}{1-a^{-1}z} = -(a^{-1}z^0 + a^{-2}z^1 + a^{-3}z^2 + \cdots)$$

$-a^{-n-1}$ $u(n)$

$$a^{-1} \cdot \frac{-1}{1-a^{-1}z^{-1}} = -(a^{-1}z^0 + a^{-2}z^{-1} + a^{-3}z^{-2} + \cdots)$$

$-a^{n-1}$ $u(-n)$

$$-a^{-n} \xrightarrow{a^{-1}\times} -a^{-n-1} \qquad u(n) \rightarrow u(n)$$

$$-a^{n} \xrightarrow{a^{-1}\times} -a^{n-1} \qquad u(-n) \rightarrow u(-n)$$

Relations between the unit nominator and original expressions (2)

$$\frac{1}{1-a^{-1}z^{-1}} = +\left(a^0 z^0 + a^{-1} z^{-1} + a^{-2} z^{-2} + \cdots\right) \qquad \frac{1}{1-a^{-1}z} = +\left(a^0 z^0 + a^{-1} z^{1} + a^{-2} z^{2} + \cdots\right)$$

a^{n} $\quad u(-n)$ $\qquad a^{-n}$ $\quad u(n)$

$$z^{-1}\cdot\frac{1}{1-a^{-1}z^{-1}} = +\left(a^0 z^{-1} + a^{-1} z^{-2} + a^{-2} z^{-3} + \cdots\right) \qquad z\cdot\frac{1}{1-a^{-1}z} = +\left(a^0 z^{1} + a^{-1} z^{2} + a^{-2} z^{3} + \cdots\right)$$

a^{n+1} $\quad u(-n-1)$ $\qquad a^{-n+1}$ $\quad u(n-1)$

$$a^{n} \xrightarrow{n \leftarrow n+1} a^{n+1} \qquad u(-n) \xrightarrow{n \leftarrow n+1} u(-n-1) \qquad a^{-n} \xrightarrow{n \leftarrow n-1} a^{-n+1} \qquad u(n) \xrightarrow{n \leftarrow n-1} u(n-1)$$

$$\frac{1}{1-a z^{-1}} = +\left(a^0 z^0 + a^1 z^{-2} + a^2 z^{-2} + \cdots\right) \qquad \frac{1}{1-a z} = +\left(a^0 z^0 + a^1 z^1 + a^2 z^2 + \cdots\right)$$

a^{-n} $\quad u(-n)$ $\qquad a^{n}$ $\quad u(n)$

$$z^{-1}\cdot\frac{1}{1-a z^{-1}} = +\left(a^0 z^{-1} + a^1 z^{-2} + a^2 z^{-3} + \cdots\right) \qquad z\cdot\frac{1}{1-a z} = +\left(a^0 z^1 + a^1 z^2 + a^2 z^3 + \cdots\right)$$

a^{-n-1} $\quad u(-n-1)$ $\qquad a^{n-1}$ $\quad u(n-1)$

$$a^{-n} \xrightarrow{n \leftarrow n+1} a^{-n-1} \qquad u(-n) \xrightarrow{n \leftarrow n+1} u(-n-1) \qquad a^{n} \xrightarrow{n \leftarrow n-1} a^{n-1} \qquad u(n) \xrightarrow{n \leftarrow n-1} u(n-1)$$

v(n) : a range selection expression

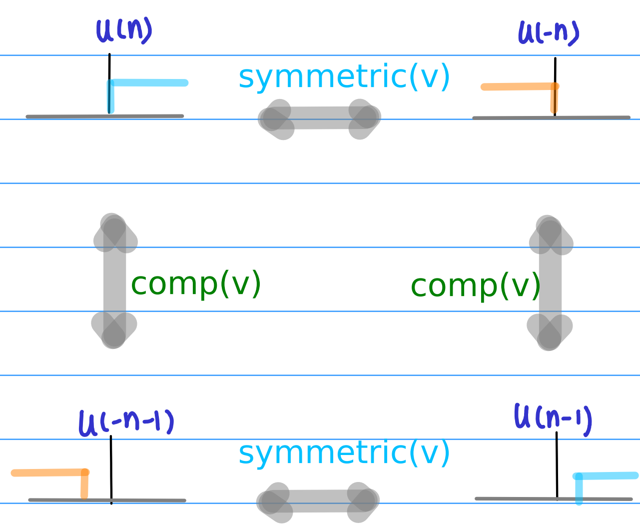
v(n) $\in$ {u(n), u(-n-1), u(n), u(-n-1)}

unit step function to denote a range

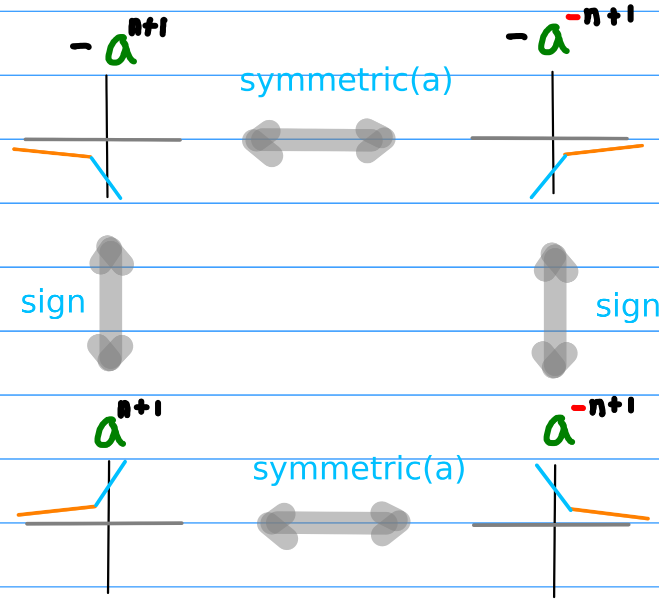
complementary and symmetric regions

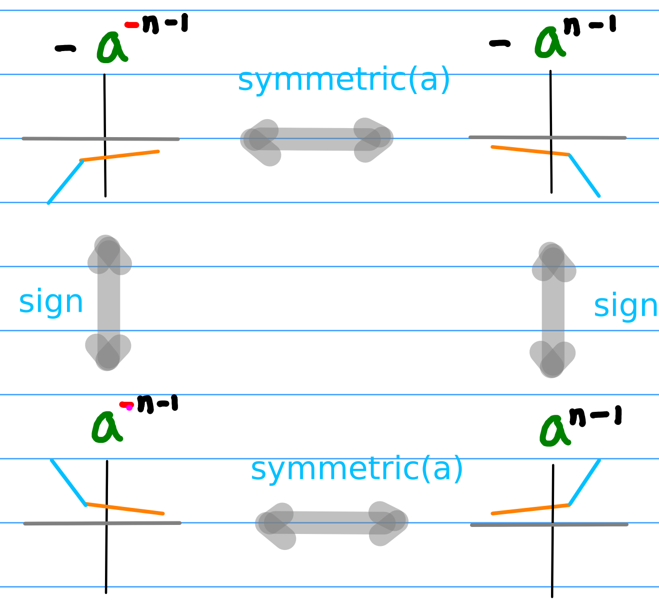
$$\begin{pmatrix} \boxed{\begin{matrix} u(n) \\ u(-n-1) \end{matrix}} & \boxed{\begin{matrix} u(-n) \\ u(n-1) \end{matrix}} \end{pmatrix}$$

complementary regions | complementary regions

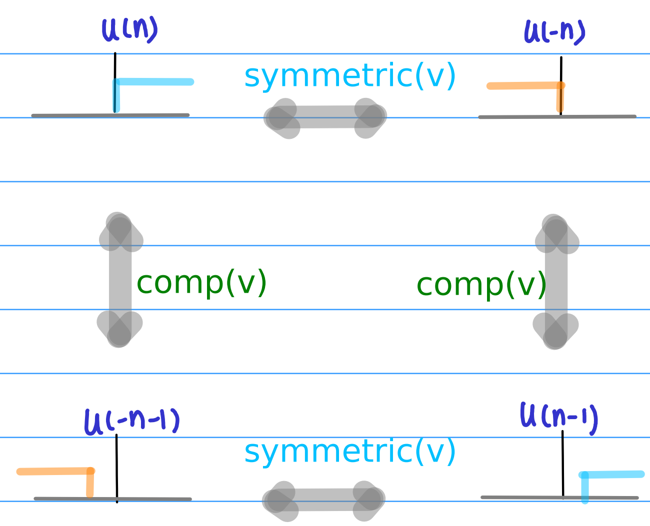


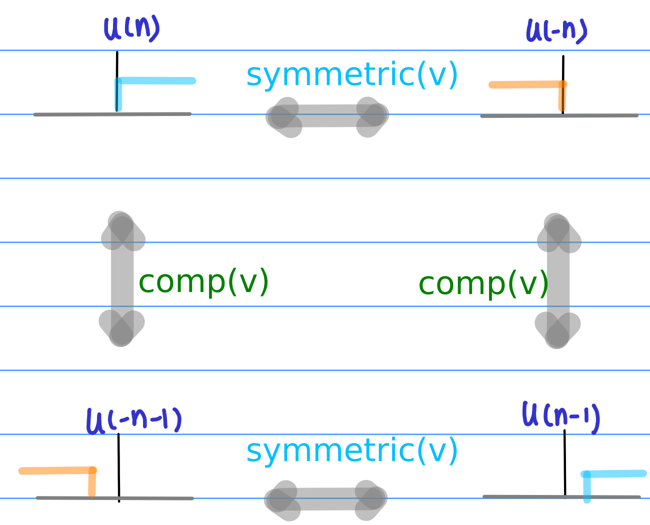
Power Selection

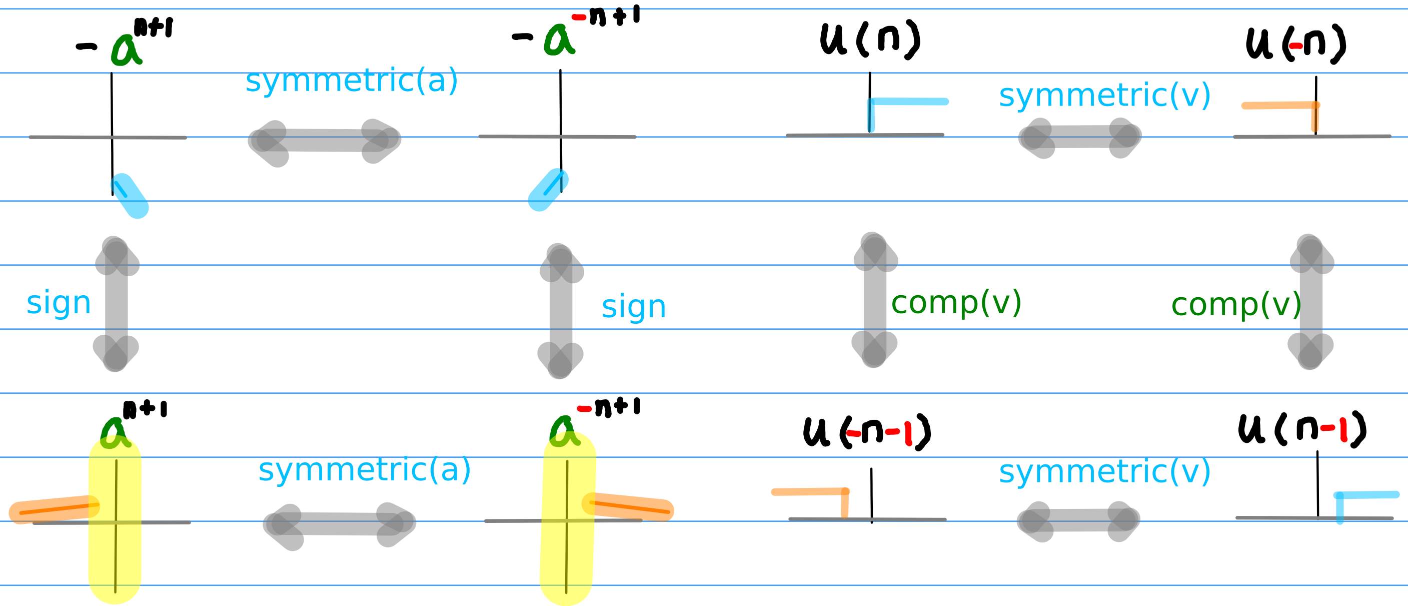


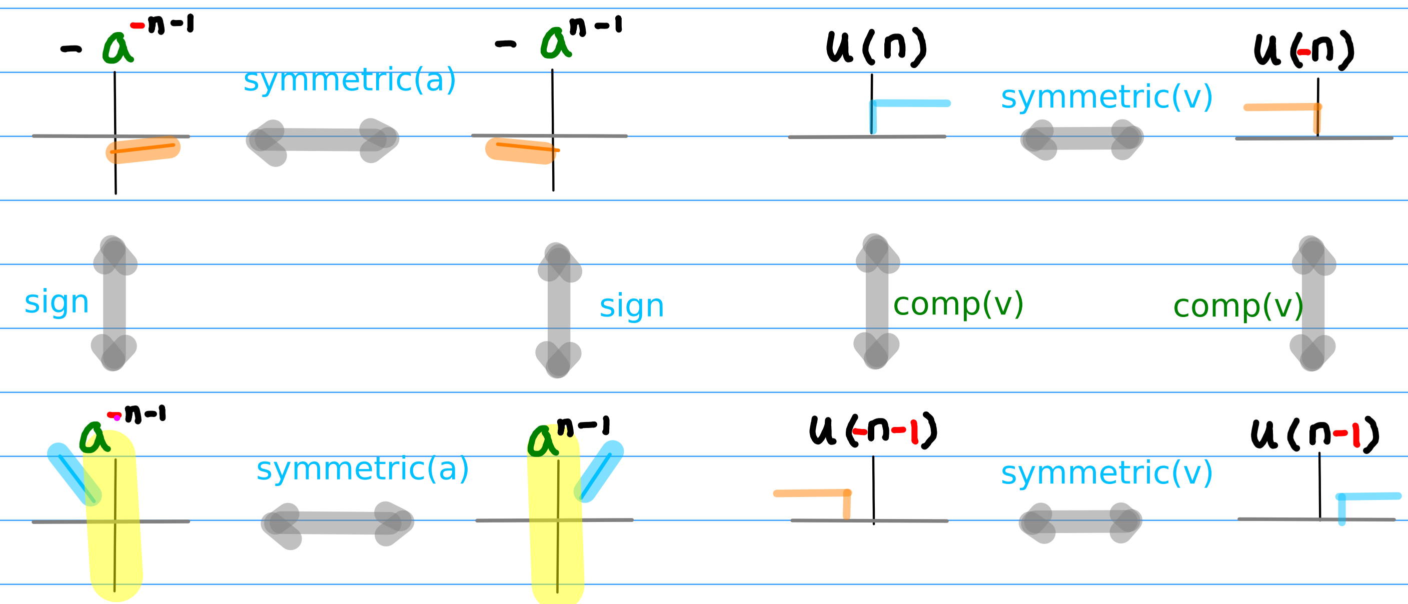


Range Selection

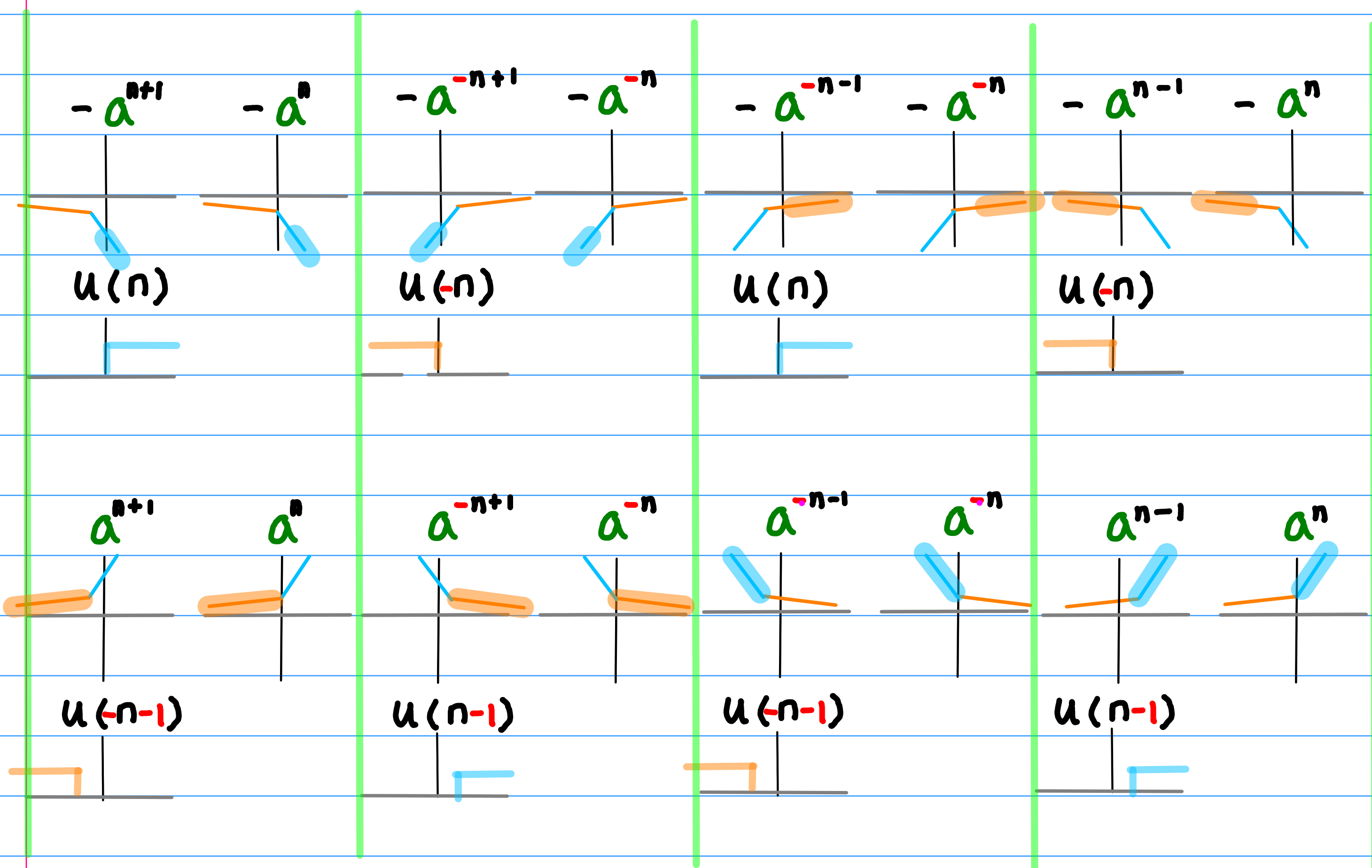


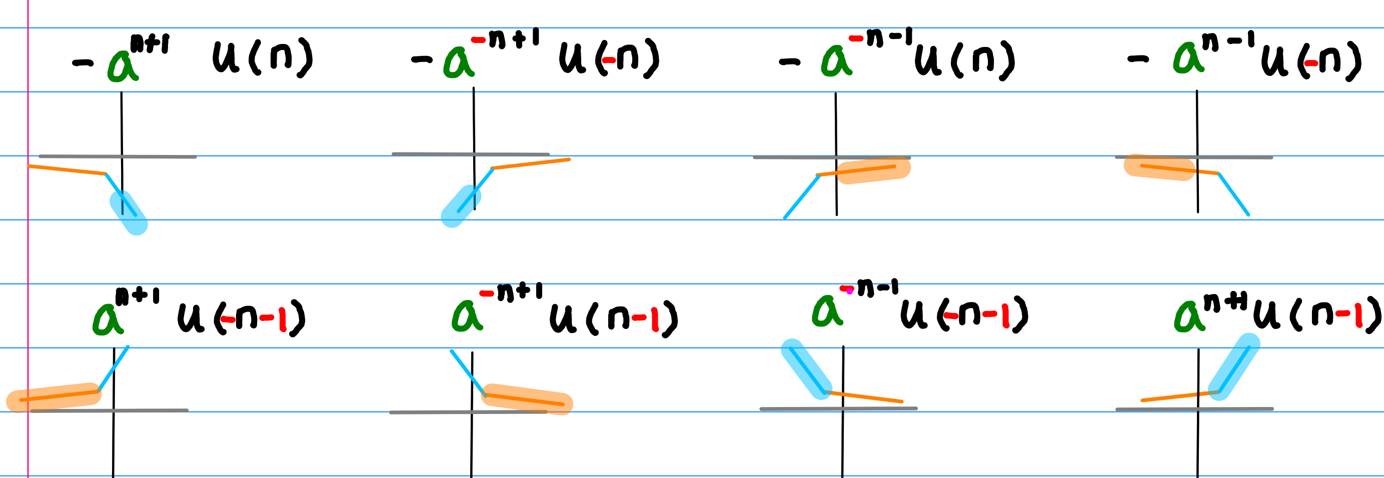



$-a^{n+1}$
symmetric(a)
$-a^{-n+1}$
$u(n)$
symmetric(v)
$u(-n)$
sign
sign
comp(v)
comp(v)
a^{n+1}
symmetric(a)
a^{-n+1}
$u(-n-1)$
symmetric(v)
$u(n-1)$


$-a^{-n-1}$
symmetric(a)
$-a^{n-1}$
$u(n)$
symmetric(v)
$u(-n)$
sign
sign
comp(v)
comp(v)
a^{-n-1}
symmetric(a)
a^{n-1}
$u(-n-1)$
symmetric(v)
$u(n-1)$

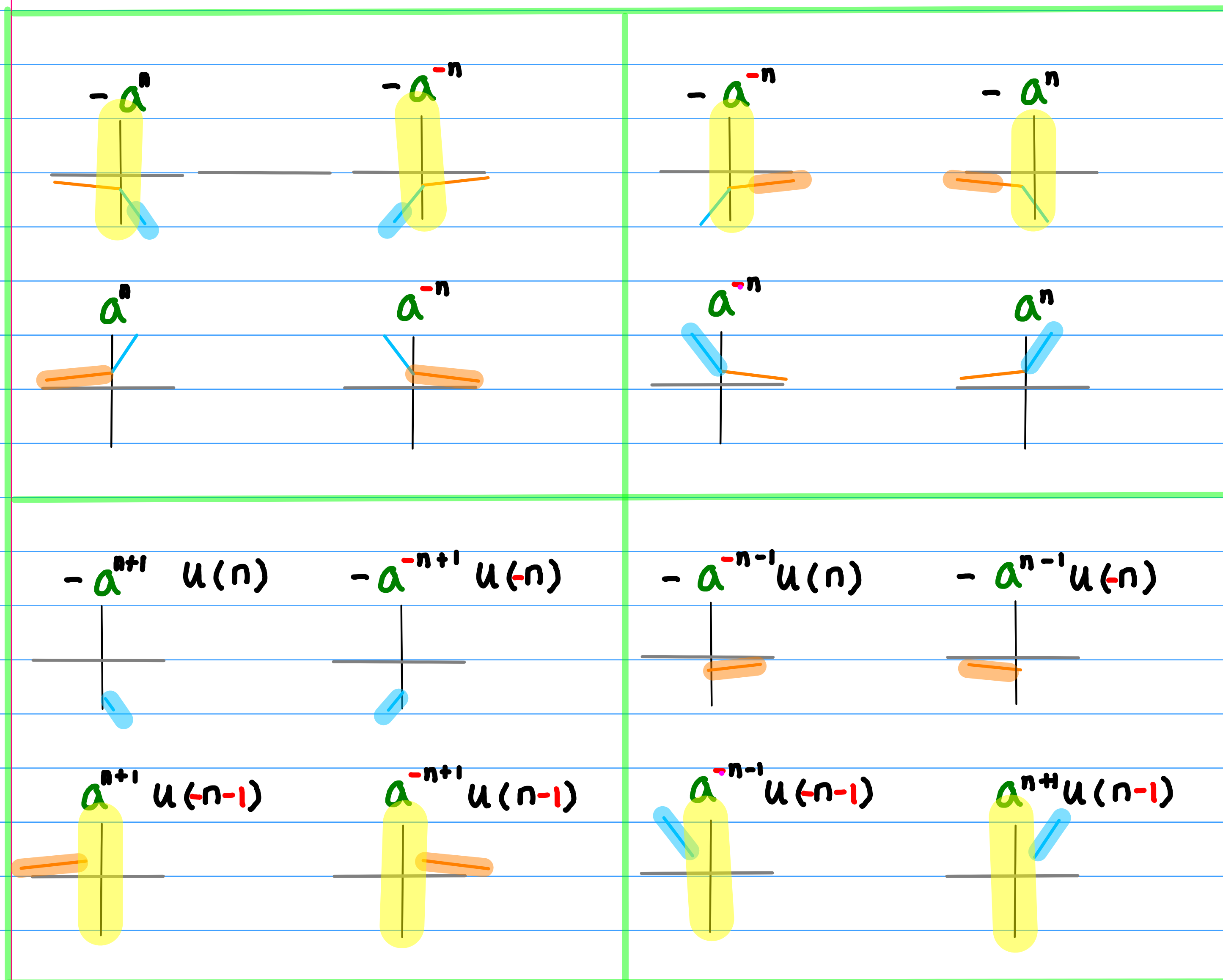
$-a^{n+1}$	$\boxed{u(n)}$	$-a^{-n+1}$	$\boxed{u(-n)}$	$-a^{-n-1}$	$\boxed{u(n)}$	$-a^{n-1}$	$\boxed{u(-n)}$
a^{n+1}	$\boxed{u(-n-1)}$	a^{-n+1}	$\boxed{u(n-1)}$	a^{-n-1}	$\boxed{u(-n-1)}$	a^{n+1}	$\boxed{u(n-1)}$



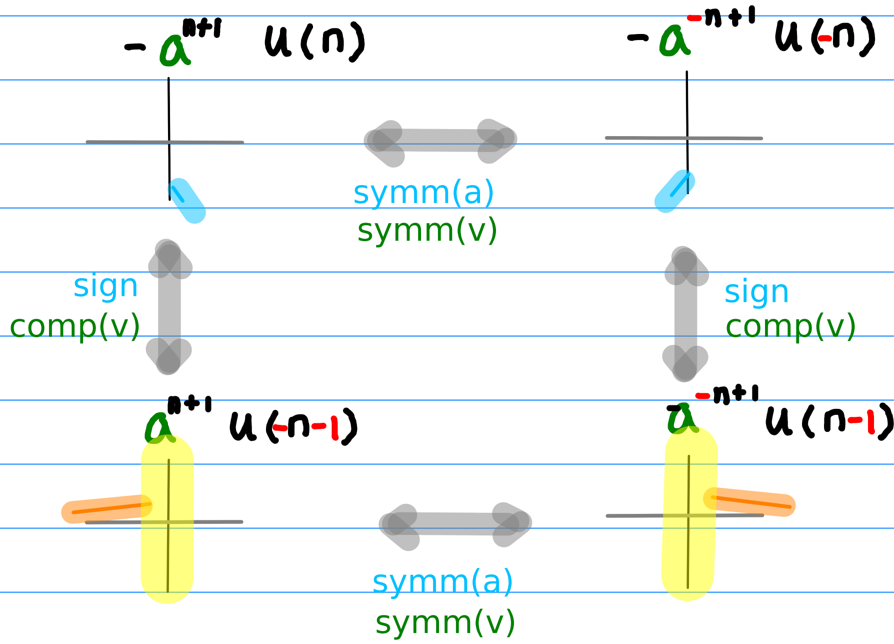


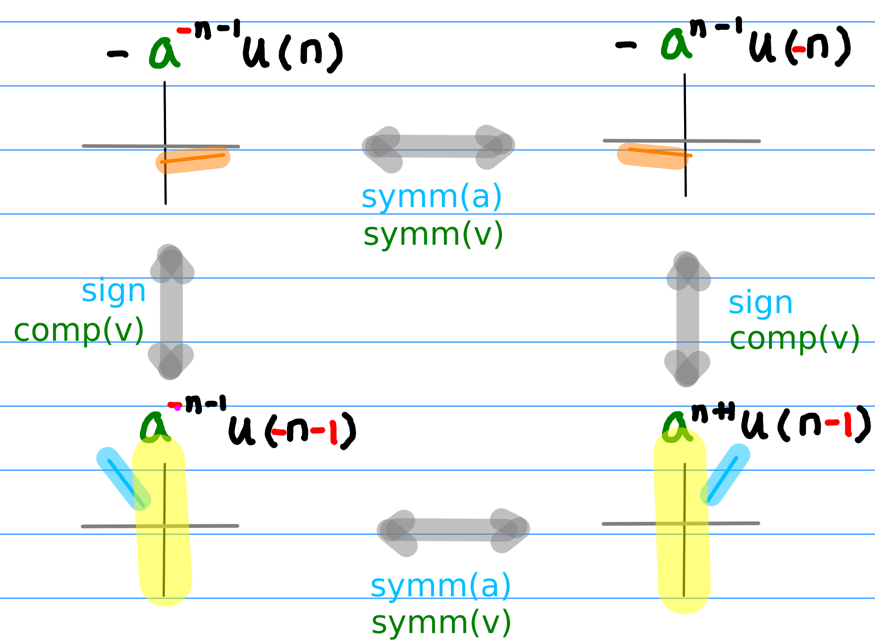
$$-a^{n+1}\ \boxed{u(n)} \qquad -a^{-n+1}\ \boxed{u(-n)} \qquad -a^{-n-1}\ \boxed{u(n)} \qquad -a^{n-1}\ \boxed{u(-n)}$$

$$a^{n+1}\ \boxed{u(-n-1)} \qquad a^{-n+1}\ \boxed{u(n-1)} \qquad a^{-n-1}\ \boxed{u(-n-1)} \qquad a^{n-1}\ \boxed{u(n-1)}$$

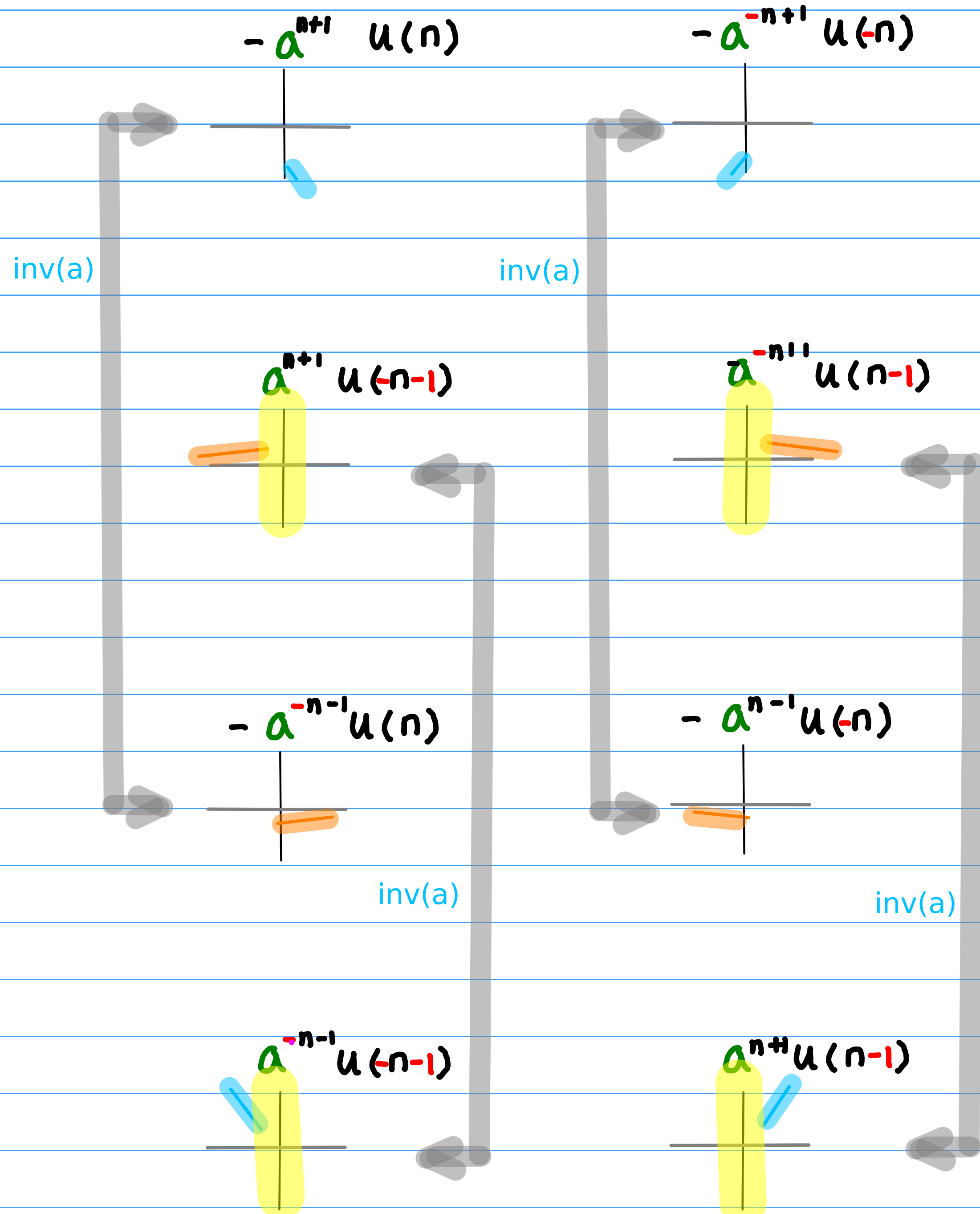


$$\begin{matrix} -a^{n+1} \\ a^{n+1} \end{matrix} \boxed{\begin{matrix} u(n) \\ u(-n-1) \end{matrix}} \quad \begin{matrix} -a^{-n+1} \\ a^{-n+1} \end{matrix} \boxed{\begin{matrix} u(-n) \\ u(n-1) \end{matrix}} \quad \begin{matrix} -a^{-n-1} \\ a^{-n-1} \end{matrix} \boxed{\begin{matrix} u(n) \\ u(-n-1) \end{matrix}} \quad \begin{matrix} -a^{n-1} \\ a^{n-1} \end{matrix} \boxed{\begin{matrix} u(-n) \\ u(n-1) \end{matrix}}$$





$-a^{n+1}$	$u(n)$	$-a^{-n+1}$	$u(-n)$	$-a^{-n-1}$	$u(n)$	$-a^{n-1}$	$u(-n)$
a^{n+1}	$u(-n-1)$	a^{-n+1}	$u(n-1)$	a^{-n-1}	$u(-n-1)$	a^{n-1}	$u(n-1)$



$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{-\frac{a}{1-az}} \boxed{|z|<a^{-1}} \arrow[r, leftrightarrow, "\substack{\mathrm{inv}(z)\\ \mathrm{inv}(z)}"'] \arrow[d, leftrightarrow, "\substack{\mathrm{id}\\ \mathrm{comp}(z)}"'] & \boxed{-\frac{a}{1-az^{-1}}} \boxed{|z|>a} \arrow[d, leftrightarrow, "\substack{\mathrm{sign}\\ \mathrm{comp}(v)}"] \\
\boxed{\frac{z^{-1}}{1-a^{-1}z^{-1}}} \boxed{|z|>a^{-1}} \arrow[r, leftrightarrow, "\substack{\mathrm{inv}(z)\\ \mathrm{inv}(z)}"'] & \boxed{-\frac{z}{1-a^{-1}z}} \boxed{|z|<a}
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{-\frac{a^{-1}}{1-a^{-1}z}} \boxed{|z|<a} \arrow[r, leftrightarrow, "\substack{\mathrm{inv}(z)\\ \mathrm{inv}(z)}"'] \arrow[d, leftrightarrow, "\substack{\mathrm{sign}\\ \mathrm{comp}(z)}"'] & \boxed{-\frac{a^{-1}}{1-a^{-1}z^{-1}}} \boxed{|z|>a^{-1}} \arrow[d, leftrightarrow, "\substack{\mathrm{sign}\\ \mathrm{comp}(z)}"] \\
\boxed{\frac{z^{-1}}{1-az^{-1}}} \boxed{|z|>a} \arrow[r, leftrightarrow, "\substack{\mathrm{inv}(z)\\ \mathrm{inv}(z)}"'] & \boxed{\frac{z}{1-az}} \boxed{|z|<a^{-1}}
\end{tikzcd}
$$

Geometric Series Expression $h(a, z)$

Region of Convergence Expression $R(a, z)$

$$\begin{pmatrix} +1 \\ -1 \end{pmatrix} \times \begin{pmatrix} a \\ a^{-1} \end{pmatrix} \times \begin{pmatrix} z \\ z^{-1} \end{pmatrix}$$

$h(a, z)$	$R(a, z)$	
$-\dfrac{a}{1-\boxed{az}}$	$\lvert z\rvert < a^{-1}$	$\lvert az\rvert < 1$
$-\dfrac{a}{1-\boxed{az^{-1}}}$	$\lvert z\rvert > a$	$\lvert az^{-1}\rvert < 1$
$\dfrac{z^{-1}}{1-\boxed{a^{-1}z^{-1}}}$	$\lvert z\rvert > a^{-1}$	$\lvert a^{-1}z^{-1}\rvert < 1$
$\dfrac{z}{1-\boxed{a^{-1}z}}$	$\lvert z\rvert < a$	$\lvert a^{-1}z\rvert < 1$
$-\dfrac{a^{-1}}{1-\boxed{a^{-1}z}}$	$\lvert z\rvert < a$	$\lvert a^{-1}z\rvert < 1$
$-\dfrac{a^{-1}}{1-\boxed{a^{-1}z^{-1}}}$	$\lvert z\rvert > a^{-1}$	$\lvert a^{-1}z^{-1}\rvert < 1$
$\dfrac{z^{-1}}{1-\boxed{az^{-1}}}$	$\lvert z\rvert > a$	$\lvert az^{-1}\rvert < 1$
$\dfrac{z}{1-\boxed{az}}$	$\lvert z\rvert < a^{-1}$	$\lvert az\rvert < 1$