Differentiation of Continuous Functions

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

1. Approximations of a first derivative
   - Forward Difference Approximation
   - Backward Difference Approximation
   - Taylor Series
   - Central Divided Difference

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Forward Difference Approximation (1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

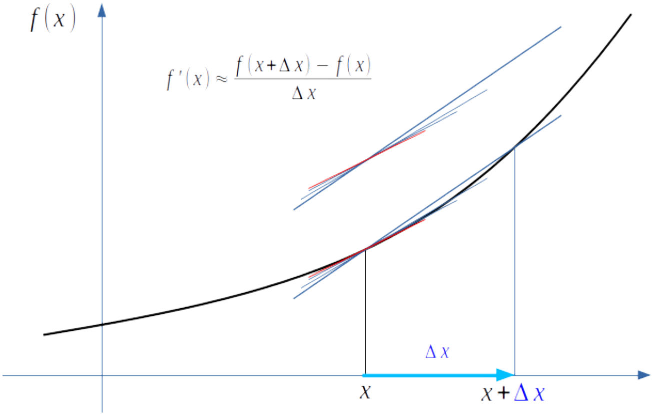

Figure: forward difference approximation

Forward Difference Approximation (3)

a forward difference approximation
as you are taking a point forward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx forwad as $x = x_{i+1}$.

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

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Backward Difference Approximation (1a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x < 0$, then $-\Delta x > 0$,

$$f'(x) \approx \frac{f(x-\Delta x) - f(x)}{-\Delta x}$$
$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (1b)

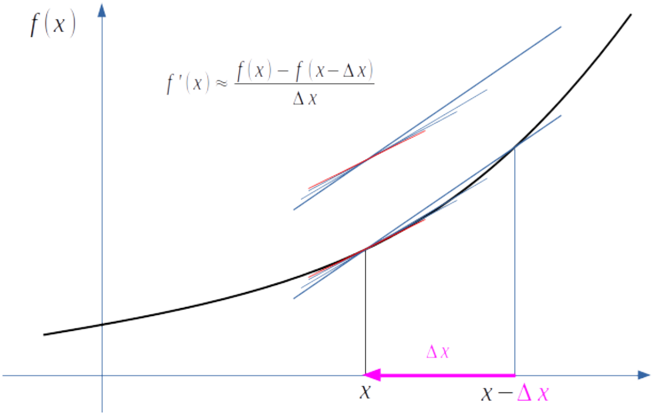

Figure: backward difference approximation (a)

Backward Difference Approximation (2a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{x - (x - \Delta x)} = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (2b)

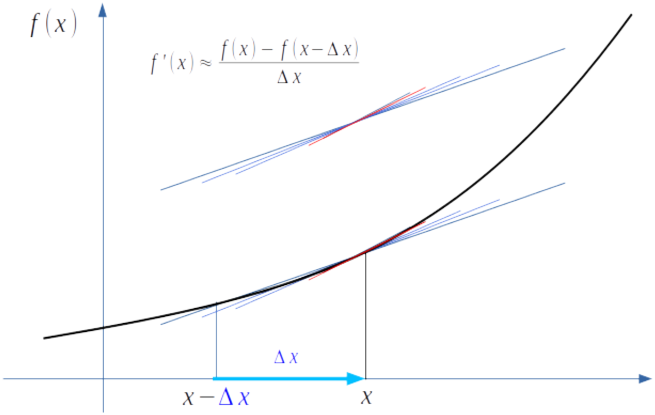

Figure: backward difference approximation (b)

Backward Difference Approximation (3)

a backward difference approximation
as you are taking a point backward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx backwad as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\ &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \end{aligned}$$

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Taylor Series (1)

the Taylor series of a function $f(x)$,
that is infinitely differentiable at a point a is the power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series (2)

If $f(x)$ is given by a convergent power series
in an open disk centred at a,
it is said to be *analytic* in this region.

Thus for x in this region,
f is given by a convergent power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Approximating the first derivative

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

- for forward difference approximatin

$$x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$$

- for forward difference approximatin

$$x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$$

Deriving Forward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots,$

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots = f'(x_i)(\Delta x)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) - \cdots = f'(x_i)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x) = f'(x_i)$$

Deriving Backward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i)(\Delta x) = f(x_i) - f(x_{i-1}) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + \frac{f''(x_i)}{2!}(\Delta x) - \cdots$$

=

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Forward and Backward Approximation

- for forward difference approximatin

$$x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

- for forward difference approximatin

$$x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Approximation Errors

- the $O(\Delta x)$ term shows that the error in the approximation is of the order of Δx
- both forward and backward difference approximation of the first derivative are accurate on the order of $O(\Delta x)$
- to get better approximations
- the Central divided difference approximation of the first derivative.

Deriving Central Divide Approximation (1)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Forward difference approximation :
Let $x_i = a$ and $x_{i+1} = x$, and substitute $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Backward difference approximation :
Let $x_i = a$ and $x_{i-1} = x$, and substitute $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Deriving Central Divide Approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \cdots$$

subtracting eq(2) from eq(1)

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O\left((\Delta x)^2\right)$$

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Central Divided Approximation

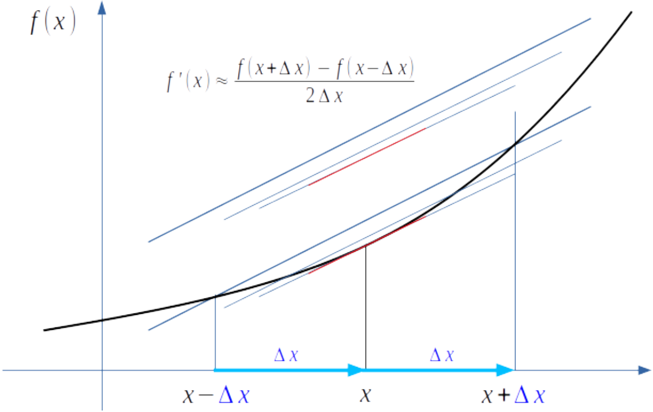

Figure: central difference approximation

Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line $\to$ the tangent line
- the slope of the tangent line

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$
$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$