z-Transform 3.Principles

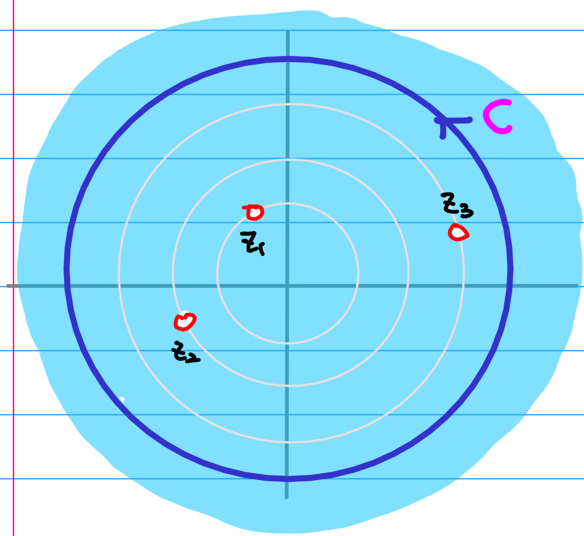
20170715

Complex Analysis in plain view

Residue Integrals Note

Laurent Series with Residue Theorem Note (H1.pdf)
Laurent Series with Applications Note (H1.pdf)
Laurent Series and z-Transform Note (H1.pdf)
Laurent Series and Geometric Series Note (H1.pdf)

Series Expansion at $z=0$



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz$$
$$= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

$n \geq 0$	$z_1, z_2, z_3, 0$
$n < 0$	z_1, z_2, z_3

* General Series Expansion at $z=0$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n z^n}$$

$$\boxed{\begin{aligned} a_n &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

* Z-transform

$$\boxed{X(z) = \sum_{k=0}^{\infty} x_k z^{-k}}$$

$$\boxed{\begin{aligned} x_n &= \frac{1}{2\pi i}\oint_C X(z) z^{n-1}\,dz \\ &= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right) \end{aligned}}$$

z - Transform	$X(z)$
Laurent Series	$f(z)$

z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z^{-1}) \longleftrightarrow x_n = a_n}$$

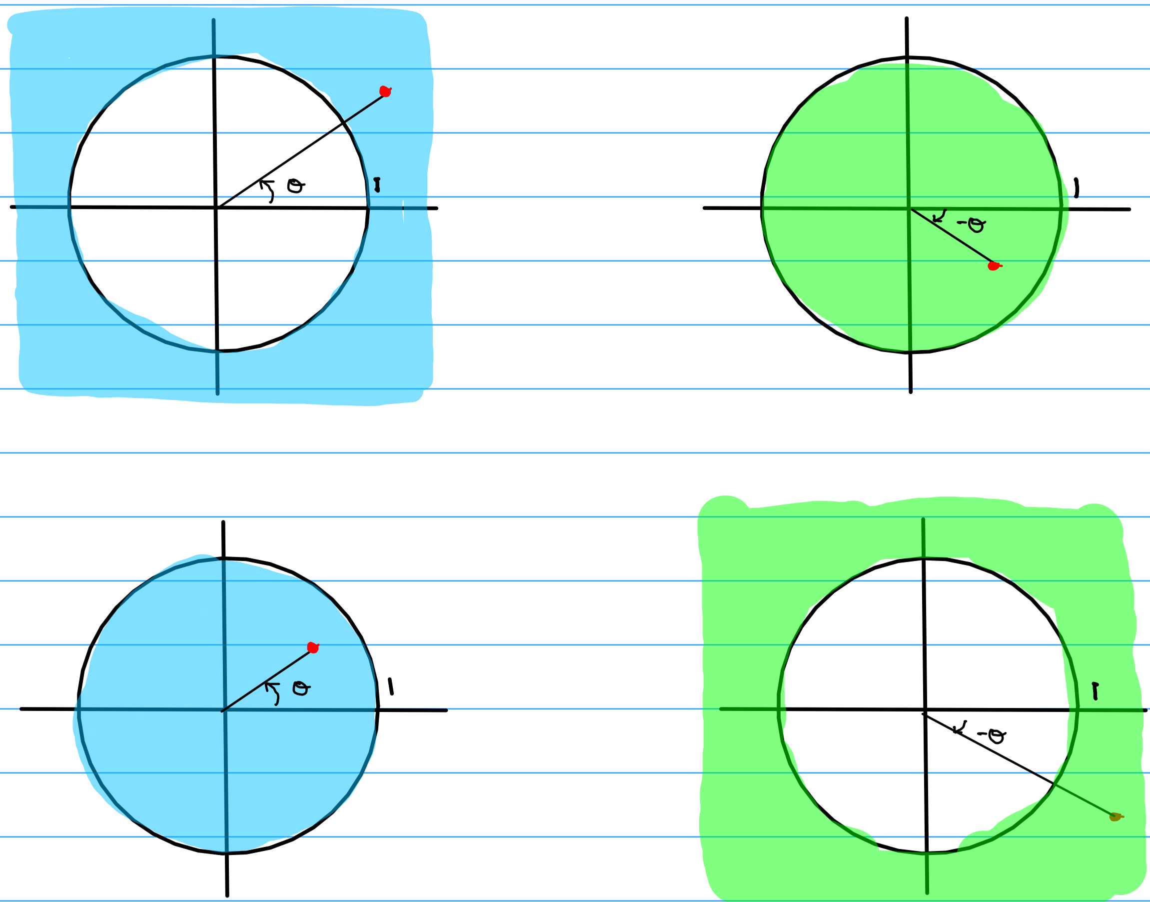
z-Transform $X(z) \longleftrightarrow x_n$

Laurent Series $f(z) \longleftrightarrow a_n$

$$\boxed{X(z) = f(z) \longleftrightarrow x_n = a_{-n}}$$

Mapping $w = \frac{1}{z}$

$z = \rho e^{j\theta}$ $\qquad\qquad$ $z^{-1} = \frac{1}{\rho} e^{-j\theta}$



- inverse magnitude
- negative phase

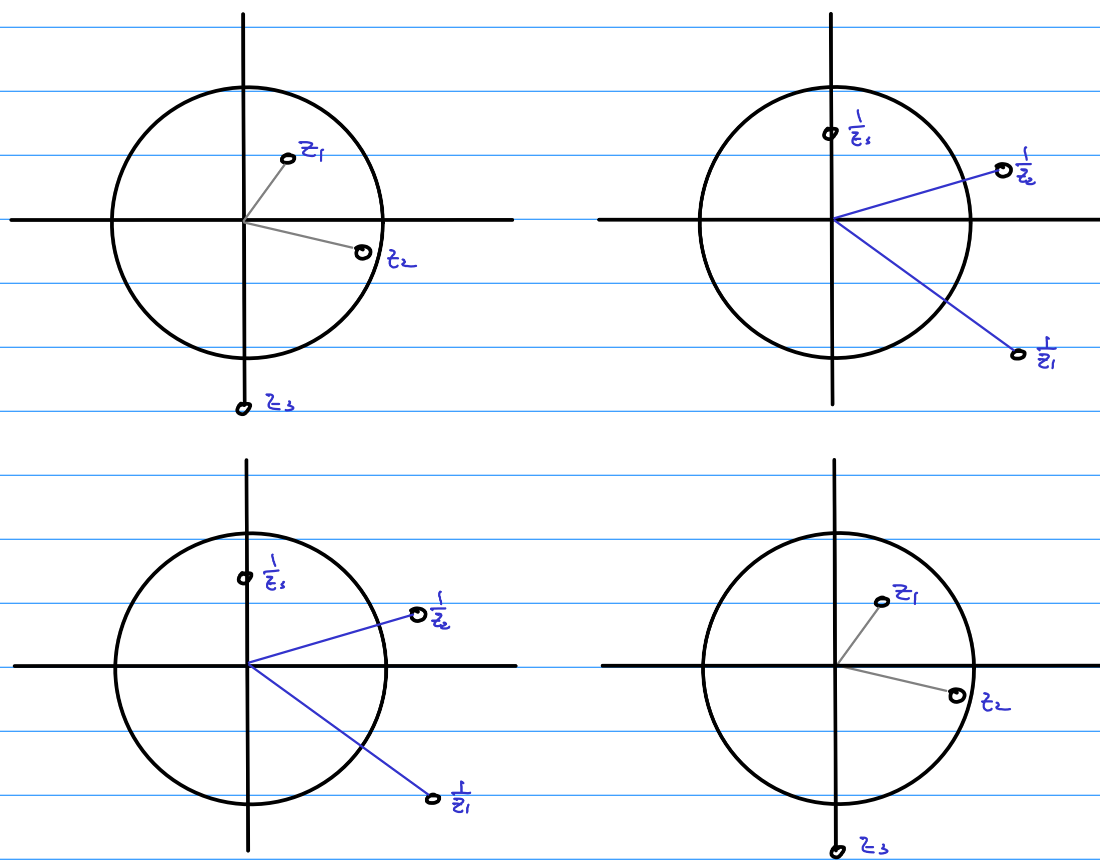
$$f(z) = \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)(z - p_3)}$$

$$f(z^{-1}) = \frac{(\frac{1}{z} - z_1)(\frac{1}{z} - z_2)}{(\frac{1}{z} - p_1)(\frac{1}{z} - p_2)(\frac{1}{z} - p_3)}$$

$$= \frac{(1 - z_1 z)(1 - z_2 z)}{(1 - p_1 z)(1 - p_2 z)(1 - p_3 z)}$$

$\frac{1}{z_1}, \frac{1}{z_2}$

$\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3}$

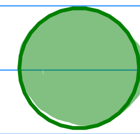


$g(z)$ with a simple pole

$b > 0$ assumed

$$g(z) = \frac{1}{1-bz} = \boxed{\frac{b^{-1}}{b^{-1}-z}}$$

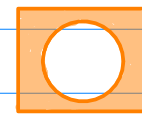
$|bz| < 1$



$|z| < \frac{1}{b}$

$$h(z) = \frac{1}{1-\frac{b}{z}} = \boxed{\frac{z}{z-b}}$$

$\left|\frac{b}{z}\right| < 1$



$|z| > b$

$$g(z^{-1}) = \frac{b^{-1}}{b^{-1}-z^{-1}} = \frac{z}{z-b} = h(z)$$

$$h(z^{-1}) = \frac{z^{-1}}{z^{-1}-b} = \frac{b^{-1}}{b^{-1}-z} = g(z)$$

$$\boxed{\begin{aligned} g(z^{-1}) &= h(z) \\ h(z^{-1}) &= g(z) \end{aligned}}$$

Infinite Sum of G. P.

Simple pole $\Rightarrow$

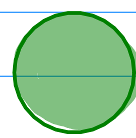
$$\frac{☁}{z - \square} \Rightarrow \frac{z}{z - \square} \Rightarrow \frac{1}{1 - \frac{\square}{z}}$$ infinite sum of G.P

$$\frac{☁}{\triangle - z} \Rightarrow \frac{\bigcirc}{\bigcirc - z} \Rightarrow \frac{1}{1 - \frac{z}{\bigcirc}}$$ infinite sum of G.P

Convergence Condition

$$\frac{b^{-1}}{b^{-1} - z} \Rightarrow$$ think this way $b^{-1} - z > 0$ $\boxed{b^{-1} > |z|}$ pole b^{-1}

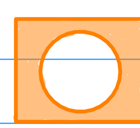
$$\boxed{\frac{b^{-1}}{b^{-1} - z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$ $|z| < p$

$$\frac{z}{z - b} \Rightarrow$$ think this way $z - b > 0$ $\boxed{|z| > b}$ pole b

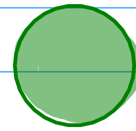
$$\boxed{\frac{z}{z - b}} = \frac{z}{z - \square}$$



$|z| > b$ $|z| > p$

Two Sequences are involved (causal, anti-causal)

$$\boxed{\frac{b^{-1}}{b^{-1}-z}} = \frac{\bigcirc}{\bigcirc - z}$$



$|z| < \frac{1}{b}$ $|z| < p$

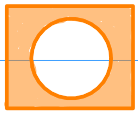
positive seq

① $(n \geq 0)$ $(bz)^0 + (bz)^1 + (bz)^2 + \cdots = \sum_{n=0}^{\infty} b^n z^n$ L.S.

negative seq

② $(n \leq 0)$ $(b^{-1}z^{-1})^0 + (b^{-1}z^{-1})^{-1} + (b^{-1}z^{-1})^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{-n}$ Z.T.

$$\boxed{\frac{z}{z-b}} = \frac{z}{z - \square}$$



$|z| > b$ $|z| > p$

positive seq

① $(n \geq 0)$ $(bz^{-1})^0 + (bz^{-1})^1 + (bz^{-1})^2 + \cdots = \sum_{n=0}^{\infty} b^n z^{-n}$ Z.T.

negative seq

② $(n \leq 0)$ $(b^{-1}z)^0 + (b^{-1}z)^{-1} + (b^{-1}z)^{-2} + \cdots = \sum_{n=0}^{-\infty} b^{-n} z^{n}$ L.S.

$\boxed{n \geq 0}$ $\boxed{n \leq 0}$ $\boxed{\text{L.S.}}$ $\boxed{\text{Z.T.}}$

$$(\quad)^0 + (\quad)^1 + (\quad)^2 + \cdots \longrightarrow (n \geq 0)$$

$$(\quad)^0 + (\quad)^{-1} + (\quad)^{-2} + \cdots \longrightarrow (n \leq 0)$$

$$\sum \ \boxed{z^n} \longrightarrow \text{L.S.}$$

$$\sum \ \boxed{z^{-n}} \longrightarrow \text{Z.T.}$$

$$(b z) = (b^{-1} z^{-1})^{-1}$$

$(n \geq 0)$

$(b z)^{n}$

$b^{n} z^{n}$ L.S.

$n = 1, 2, \cdots$

$(n \leq 0)$

$(b^{-1} z^{-1})^{n}$

$b^{-n} z^{-n}$ Z.T.

$n = -1, -2, \cdots$

$$(b z^{-1}) = (b^{-1} z)^{-1}$$

$(n \geq 0)$

$(b z^{-1})^{n}$

$b^{n} z^{-n}$ Z.T.

$n = 1, 2, \cdots$

$(n \leq 0)$

$(b^{-1} z)^{n}$

$b^{-n} z^{n}$ L.S.

$n = -1, -2, \cdots$

$$(b\,z) = (b^{-1} z^{-1})^{-1}$$

	$(n \geq 0)$		$(n \leq 0)$	
L.S.	$\sum_{n=1,2,\dots} b^{n} z^{n}$	$=$	$\sum_{n=-1,-2,\dots} b^{-n} z^{-n}$	Z.T.
	$\lvert b\,z \rvert \leqslant 1$	$=$	$\lvert b^{-1} z^{-1} \rvert^{-1} \leqslant 1$	

$$(b\,z^{-1}) = (b^{-1} z)^{-1}$$

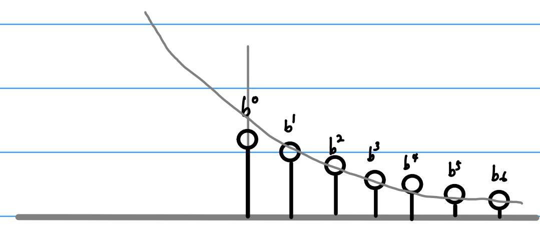
	$(n \geq 0)$		$(n \leq 0)$	
Z.T.	$\sum_{n=1,2,\dots} b^{n} z^{-n}$	$=$	$\sum_{n=-1,-2,\dots} b^{-n} z^{n}$	L.S.
	$\lvert b\,z^{-1} \rvert \leqslant 1$	$=$	$\lvert b^{-1} z \rvert^{-1} \leqslant 1$	

	$\frac{\bigcirc}{\bigcirc - z}$	$\frac{z}{z - \square}$
	pole $p = \bigcirc$	pole $p = \square$
	c.r $\left(\frac{z}{\bigcirc}\right)$	c.r $\left(\frac{\square}{z}\right)$
	r.o.c $\lvert z \rvert < \bigcirc$	r.o.c $\lvert z \rvert > \square$
$(n \geq 0)$	$\sum_{n=0}^{\infty} \left(\frac{z}{\bigcirc}\right)^{n} = \sum_{n=0}^{\infty} \left(\frac{1}{\bigcirc}\right)^{n} z^{n}$	$\sum_{n=0}^{\infty} \left(\frac{\square}{z}\right)^{n} = \sum_{n=0}^{\infty} \square^{n} z^{-n}$
$(n \leq 0)$	$\sum_{n=0}^{-\infty} \left(\frac{z}{\bigcirc}\right)^{-n} = \sum_{n=0}^{-\infty} \bigcirc^{n} z^{-n}$	$\sum_{n=0}^{-\infty} \left(\frac{\square}{z}\right)^{-n} = \sum_{n=0}^{-\infty} \left(\frac{1}{\square}\right)^{n} z^{n}$
	L.S: $b^{n} z^{n}$ $(n \geq 0)$	Z.T: $b^{n} z^{-n}$ $(n \geq 0)$
	Z.T: $b^{-n} z^{-n}$ $(n \leq 0)$	L.S: $b^{-n} z^{n}$ $(n \leq 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

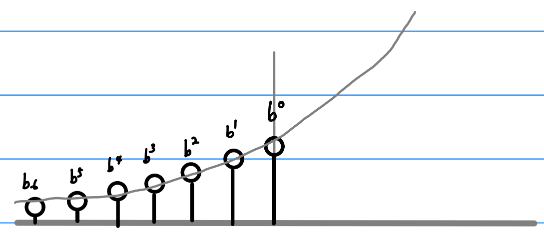
$\boxed{n \geq 0}$ $\boxed{b^n}$ & $\boxed{n \leq 0}$ $\boxed{b^{-n}}$ assumed

	$\dfrac{\bigcirc}{\bigcirc - z}$	$\dfrac{z}{z - \square}$
$(n \geq 0)$	L.S. $\left(\dfrac{z}{\bigcirc}\right)^n \Leftrightarrow b^n z^n$ $(n \geq 0)$	Z.T $\left(\dfrac{\square}{z}\right)^n \Leftrightarrow b^n z^{-n}$ $(n \geq 0)$
$(n \leq 0)$	Z.T $\left(\dfrac{z}{\bigcirc}\right)^{-n} \Leftrightarrow b^{-n} z^{-n}$ $(n < 0)$	L.S. $\left(\dfrac{\square}{z}\right)^{-n} \Leftrightarrow b^{-n} z^{n}$ $(n < 0)$
	$p = \bigcirc = b^{-1}$	$p = \square = b$

$(n \geq 0)$ n $\boxed{b^n}$



$(n \leq 0)$ $-n$ $\boxed{b^{-n}}$



$$\left(\frac{z}{\bigcirc}\right)^{n},\ \left(\frac{z}{\bigcirc}\right)^{-n},\ \left(\frac{\square}{z}\right)^{n},\ \left(\frac{\square}{z}\right)^{-n} \qquad \boxed{p}$$

$\frac{z^{+}}{\bigcirc^{-}}$ $\quad \frac{z}{p} \longrightarrow$ L.S. $(n \geq 0)$ $\quad p^{-n} z^{n}$ L.S $(n \geq 0) \longrightarrow \boxed{b^{n}}\, z^{n}$ L.S. $p = \bigcirc = b^{-1}$	$\frac{\square^{+}}{z^{-}}$ $\quad \frac{p}{z} \longrightarrow$ Z.T. $(n \geq 0)$ $\quad p^{n} z^{-n}$ Z.T. $(n \geq 0) \longrightarrow \boxed{b^{n}}\, z^{-n}$ Z.T. $p = \square = b$
$\frac{\bigcirc^{+}}{z^{-}}$ $\quad \frac{p}{z} \longrightarrow$ Z.T. $(n \leq 0)$ $\quad p^{n} z^{-n}$ Z.T. $(n \leq 0) \longrightarrow \boxed{b^{-n}}\, z^{-n}$ Z.T. $p = \bigcirc = b^{-1}$	$\frac{z^{+}}{\square^{-}}$ $\quad \frac{z}{p} \longrightarrow$ L.S. $(n \leq 0)$ $\quad p^{-n} z^{n}$ L.S. $(n \leq 0) \longrightarrow \boxed{b^{-n}}\, z^{n}$ L.S. $p = \square = b$

$\frac{z^{+}}{p^{-}}$	$\frac{p^{+}}{z^{-}}$
L.S.	Z.T.

$$p = \bigcirc = \square^{-1}$$

$$p = \square = \bigcirc^{-1}$$

	L.S. $\quad p^{-n}z^{n}$	Z.T. $\quad p^{n}z^{-n}$
$(n \geq 0)$	$\left(\frac{z}{\bigcirc}\right)^{n} \quad \left(\frac{1}{\bigcirc}\right)^{n}\boxed{z^{n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}} \quad p^{-n}z^{n}$ $z^{0},\ z^{1},\ z^{2},\ \cdots$	$\left(\frac{\square}{z}\right)^{n} \quad \square^{n}\boxed{z^{-n}}$ $\boxed{\frac{z}{z - \square}} \quad p^{n}z^{-n}$ $z^{0},\ z^{-1},\ z^{-2},\ \cdots$
$(n \leq 0)$	$\left(\frac{\square}{z}\right)^{-n} \quad \left(\frac{1}{\square}\right)^{n}\boxed{z^{n}}$ $\boxed{\frac{z}{z - \square}} \quad p^{-n}z^{n}$ $z^{0},\ z^{-1},\ z^{-2},\ \cdots$	$\left(\frac{z}{\bigcirc}\right)^{-n} \quad \bigcirc^{n}\boxed{z^{-n}}$ $\boxed{\frac{\bigcirc}{\bigcirc - z}} \quad p^{n}z^{-n}$ $z^{0},\ z^{1},\ z^{2},\ \cdots$
	$(n \geq 0) \quad p = \bigcirc = b^{-1}$ $(n \leq 0) \quad p = \square = b$	$(n \geq 0) \quad p = \square = b$ $(n \leq 0) \quad p = \bigcirc = b^{-1}$

L.S.	Z.T.
L.S.	Z.T.

$(n \geq 0)$	$(n \geq 0)$
$(n \leq 0)$	$(n \leq 0)$

p^{-n}	p^{n}
p^{-n}	p^{n}

b^{n}	b^{n}
b^{-n}	b^{-n}

$p = \bigcirc$	$p = \square$
$p = \square$	$p = \bigcirc$

$\bigcirc = b^{-1}$	$\square = b$
$\square = b$	$\bigcirc = b^{-1}$

$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

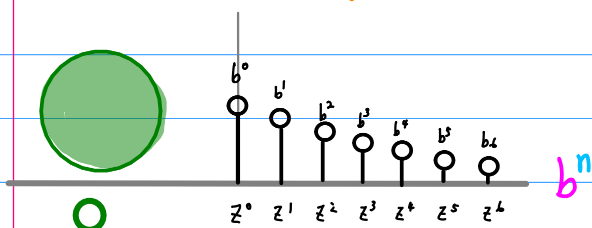
$\square^{n}$	$\square^{n}$
$\square^{-n}$	$\square^{-n}$

$\dfrac{\bigcirc}{\bigcirc - z}$	$\dfrac{z}{z - \square}$
$\dfrac{z}{z - \square}$	$\dfrac{\bigcirc}{\bigcirc - z}$

$\dfrac{\square^{-1}}{\square^{-1} - z}$	$\dfrac{z}{z - \square}$
$\dfrac{z}{z - \square}$	$\dfrac{\square^{-1}}{\square^{-1} - z}$

L.S.: $a_n z^n$ Z.T.: $x_n z^{-n}$

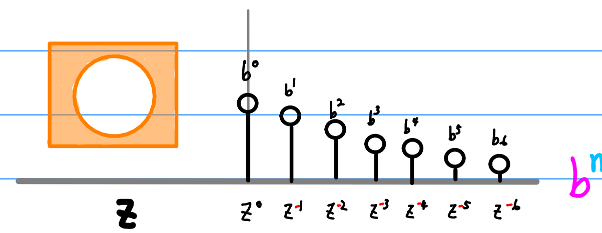
$p^{-n} z^{n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

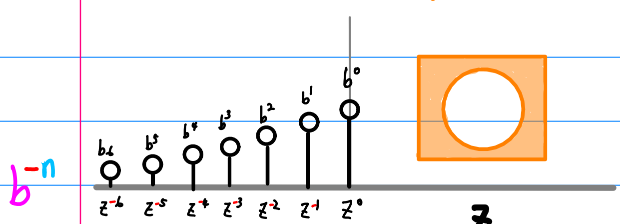
$p^{n} z^{-n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

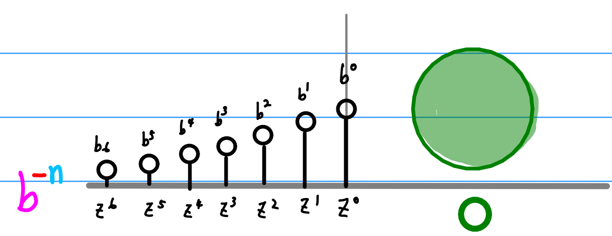
$p^{-n} z^{n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

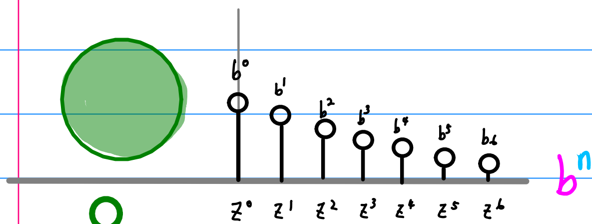
$p^{n} z^{-n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

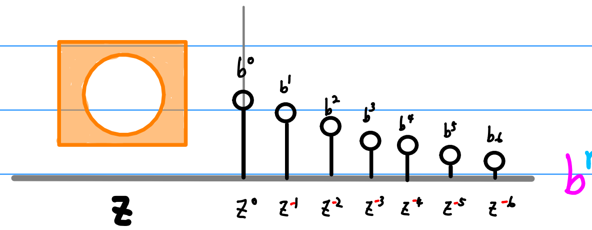
$b^{n} z^{n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

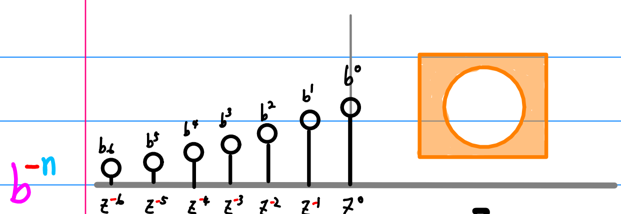
$b^{n} z^{-n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

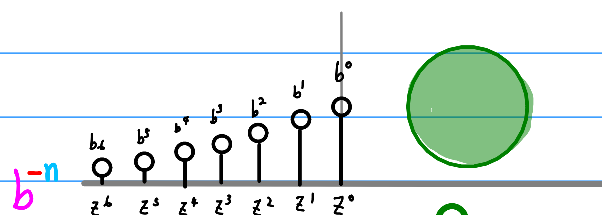
$b^{-n} z^{n}$



$$\frac{z}{z - \square}$$

$$\frac{z}{z - b}$$

$b^{-n} z^{-n}$



$$\frac{\bigcirc}{\bigcirc - z}$$

$$\frac{b^{-1}}{b^{-1} - z}$$

Z.T.

$p z^{-1} = b z^{-1}$ $(n \geq 0)$

$$\sum_k (b z^{-1})^k = \frac{z}{z-b}$$

$(k = n \geq 0)$

$|b z^{-1}| < 1$

$p = b$

$p z^{-1} = b^{-1} z^{-1}$ $(n < 0)$

$$\sum_k (b z)^k = \frac{b^{-1}}{b^{-1} - z}$$

$(k = -n > 0)$

$|b z| < 1$

$p = b^{-1}$

L.S.

$p^{-1} z = b z$ $(n \geq 0)$

$$\sum_k (b z)^k = \frac{b^{-1}}{b^{-1} - z}$$

$(k = n \geq 0)$

$|b z| < 1$

$p = b^{-1}$

$p^{-1} z = b^{-1} z$ $(n < 0)$

$$\sum_k (b z^{-1})^k = \frac{z}{z-b}$$

$(k = -n > 0)$

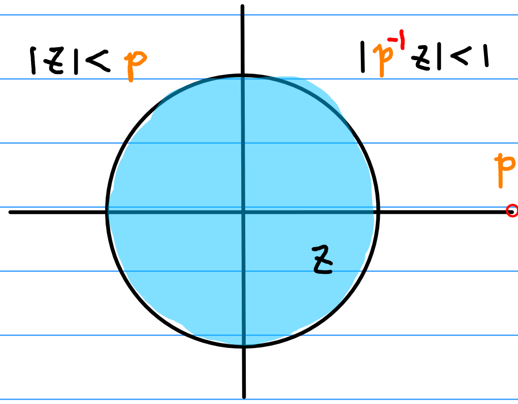
$|b z^{-1}| < 1$

$p = b$

L.S. | Z.T.

causal $(n \geq 0)$

L.S.

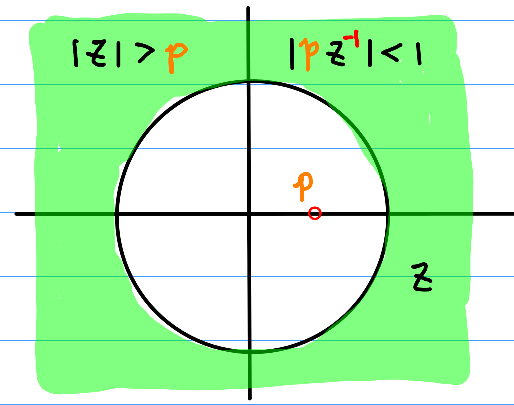


$$f(z) = \frac{p}{p-z} = \sum_{n=0}^{\infty} p^{-n} z^{n}$$

$$= \sum_{n=0}^{\infty} b^{n} z^{n}$$

$$\boxed{p = b^{-1}}$$

Z.T.



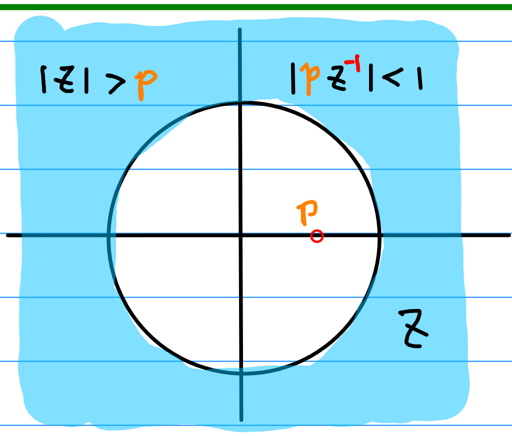
$$X(z) = \frac{z}{z-p} = \sum_{n=0}^{\infty} p^{n} z^{-n}$$

$$= \sum_{n=0}^{\infty} b^{n} z^{-n}$$

$$\boxed{p = b}$$

anticausal $(n \leq 0)$

L.S.

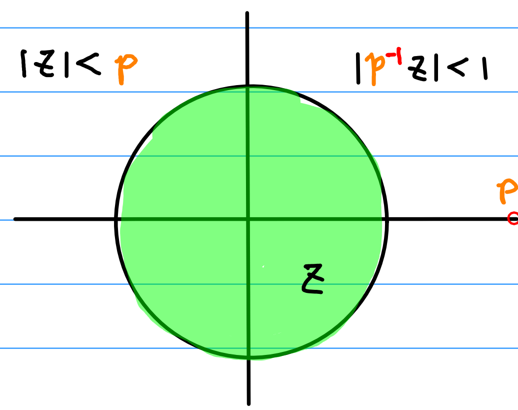


$$f(z) = \frac{z}{z-p} = \sum_{n=0}^{\infty} p^{n} z^{-n}$$

$$= \sum_{n=0}^{-\infty} p^{-n} z^{n} = \sum_{n=0}^{-\infty} b^{-n} z^{n}$$

$$\boxed{p = b}$$

Z.T.



$$X(z) = \frac{p}{p-z} = \sum_{n=0}^{\infty} p^{-n} z^{n}$$

$$= \sum_{n=0}^{-\infty} p^{n} z^{-n} = \sum_{n=0}^{\infty} b^{-n} z^{-n}$$

$$\boxed{p = b^{-1}}$$

$$a_n = \left(\tfrac{1}{2}\right)^n \quad (n \geqslant 0)$$
$$= p^{-n} \quad (n \geqslant 0) \quad p = 2$$
$$f(z) = \frac{2}{2 - z}$$

$$x_n = \left(\tfrac{1}{2}\right)^n \quad (n \geqslant 0)$$
$$= p^n \quad (n \geqslant 0) \quad p = \tfrac{1}{2}$$
$$X(z) = \frac{z}{z - 0.5}$$

$$a_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leqslant 0)$$
$$= p^{-n} \quad (n \leqslant 0) \quad p = \tfrac{1}{2}$$
$$f(z) = \frac{z}{z - 0.5}$$

$$x_n = \left(\tfrac{1}{2}\right)^{-n} \quad (n \leqslant 0)$$
$$= p^n \quad (n \leqslant 0) \quad p = 2$$
$$X(z) = \frac{2}{2 - z}$$

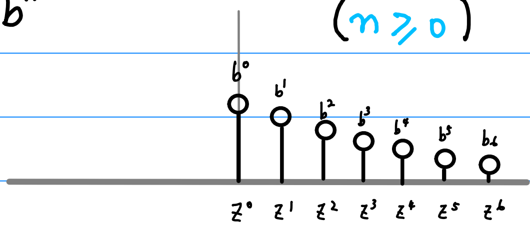
$$a_n = b^n \quad (n \geqslant 0)$$
$$= p^{-n} \quad (n \geqslant 0) \quad p = b^{-1}$$
$$f(z) = \frac{b^{-1}}{b^{-1} - z}$$

$$x_n = b^n \quad (n \geqslant 0)$$
$$= p^n \quad (n \geqslant 0) \quad p = b$$
$$X(z) = \frac{z}{z - b}$$

$$a_n = b^{-n} \quad (n \leqslant 0)$$
$$= p^{-n} \quad (n \leqslant 0) \quad p = b$$
$$f(z) = \frac{z}{z - b}$$

$$x_n = b^{-n} \quad (n \leqslant 0)$$
$$= p^n \quad (n \leqslant 0) \quad p = b^{-1}$$
$$X(z) = \frac{b^{-1}}{b^{-1} - z}$$

b^n $\quad (n \geq 0)$



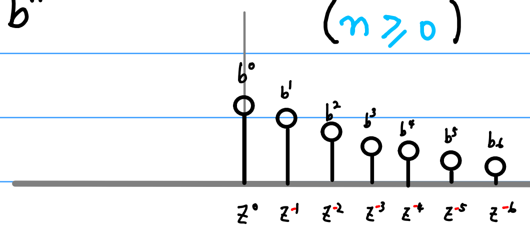
$$X(z^{-1}) = \frac{z^{-1}}{z^{-1} - 0.5} \qquad |z| < 2$$

$$f(z) = \frac{2}{2-z} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

$$a_n = \left(\frac{1}{2}\right)^n$$

$$= p^{-n} \qquad \boxed{p = 2}$$

b^n $\quad (n \geq 0)$



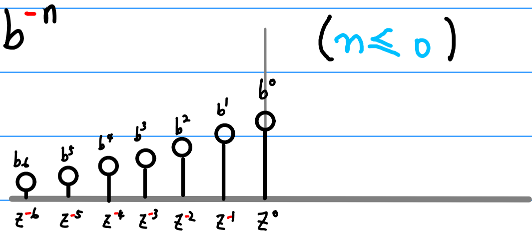
$$|z| > \frac{1}{2}$$

$$X(z) = \frac{z}{z - 0.5} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$x_n = \left(\frac{1}{2}\right)^n$$

$$= p^n \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $\quad (n \leq 0)$



$$X(z^{-1}) = \frac{2}{2 - z^{-1}} \qquad |z| > \frac{1}{2}$$

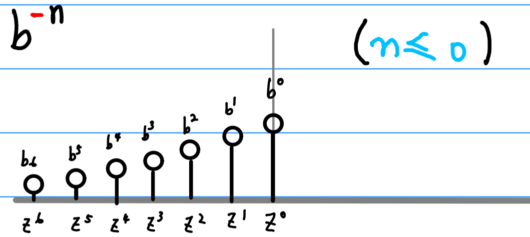
$$f(z) = \frac{z}{z - 0.5} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$a_n = \left(\frac{1}{2}\right)^{-n}$$

$$= p^{-n} \qquad \boxed{p = \frac{1}{2}}$$

b^{-n} $\quad (n \leq 0)$



$$|z| < 2$$

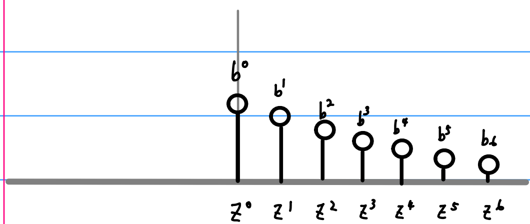
$$X(z) = \frac{2}{2 - z} = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} z^{-n}$$

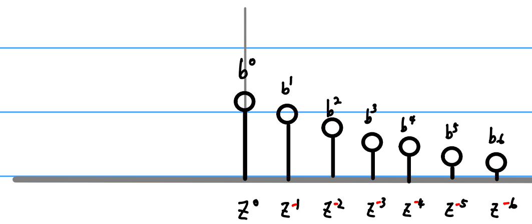
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n$$

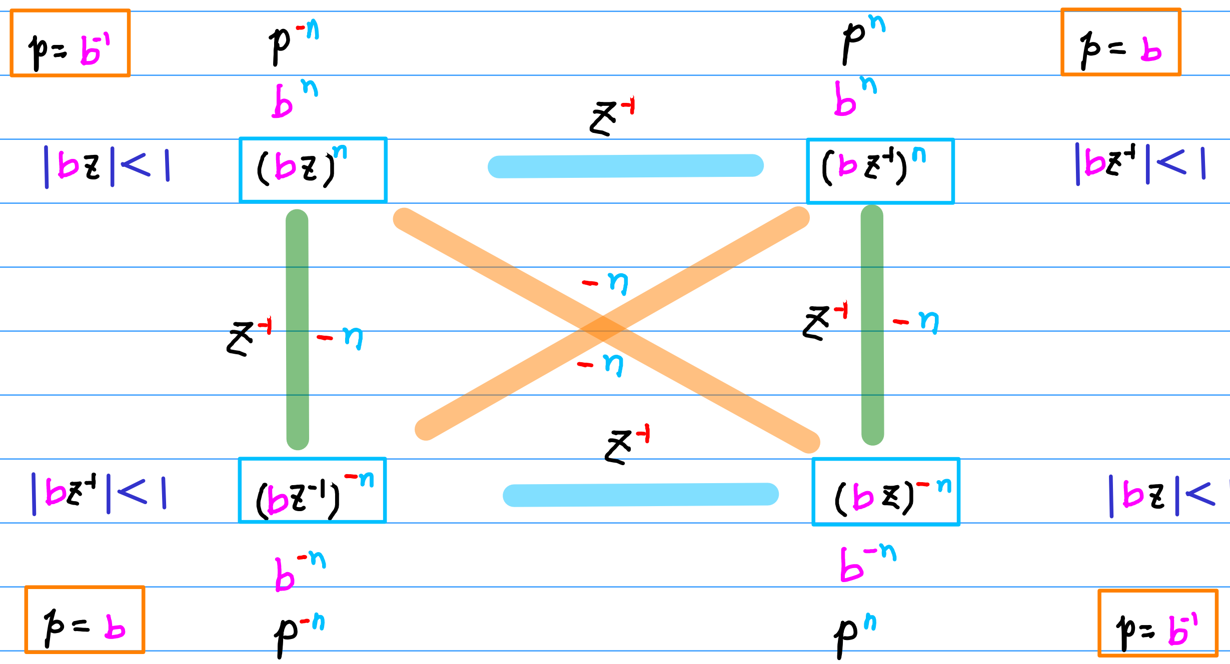
$$x_n = \left(\frac{1}{2}\right)^{-n}$$

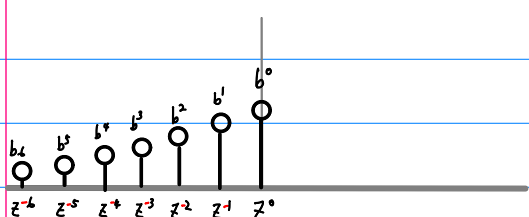
$$= p^n \qquad \boxed{p = 2}$$

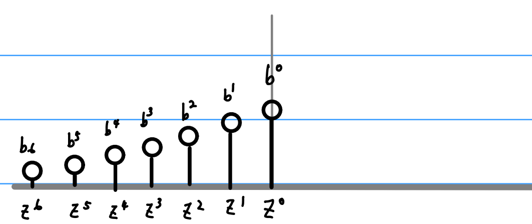
$$\boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}} \qquad \boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}}$$



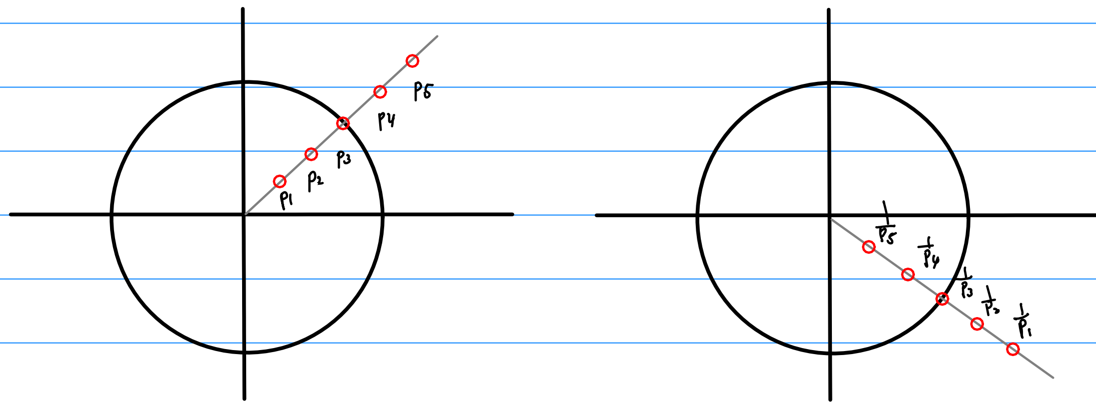


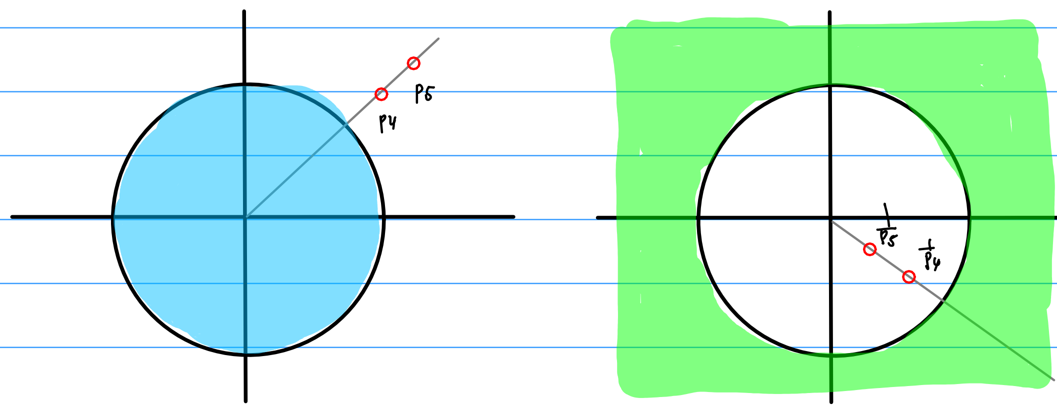






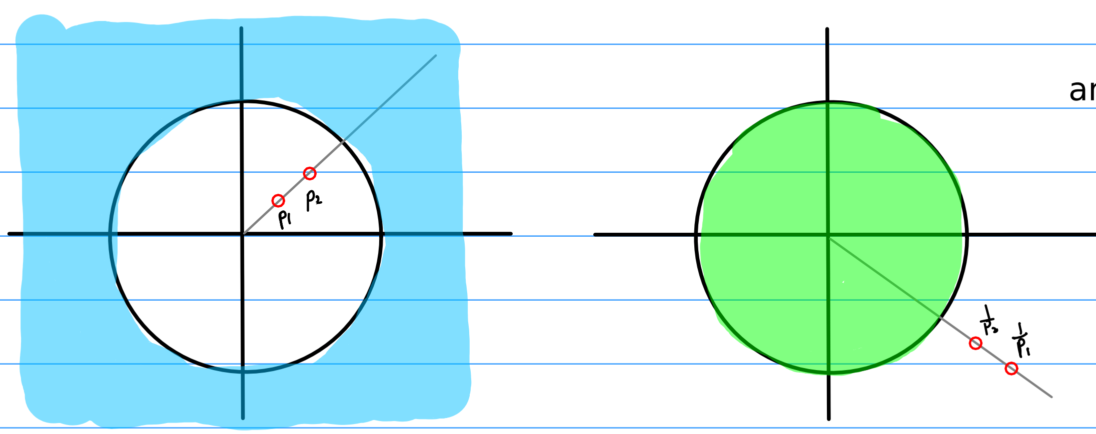
$$\boxed{\frac{1}{1-(bz^{-1})} = \frac{z}{z-b}} \qquad \boxed{\frac{1}{1-(bz)} = \frac{b^{-1}}{b^{-1}-z}}$$





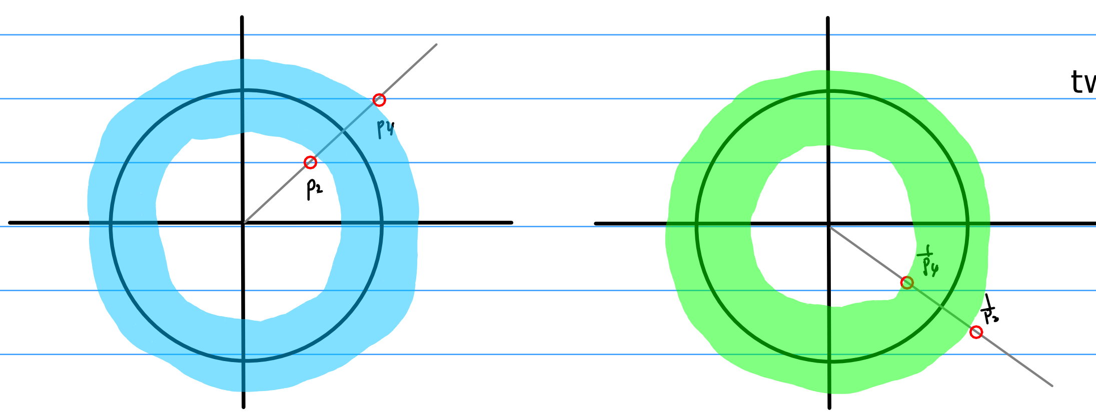
causal

$n \geq 0$

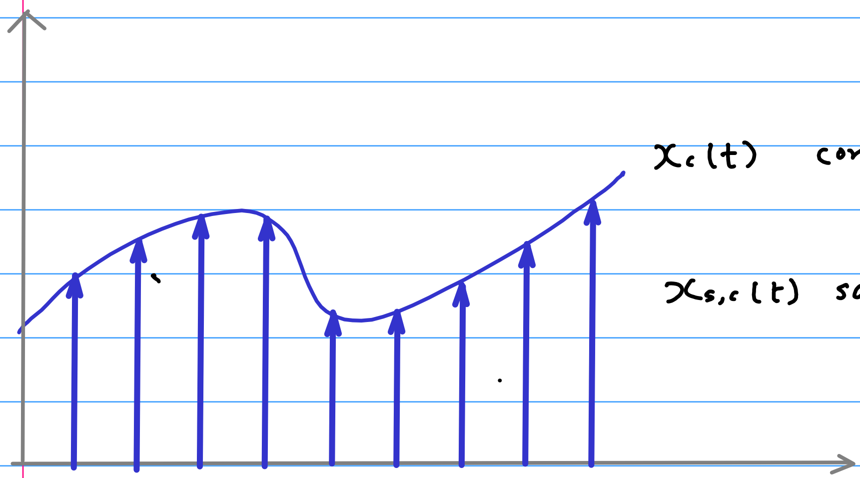


anticausal

$n < 0$



two-sided



$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \boxed{\sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)}\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t)\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-sn\Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \Big|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z) \Big|_{z = e^{s\Delta t}}$$

$$X_{s.c}(s) = \mathcal{L}\{x_{s.c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s.c}(t)$. an impulse train
whose coefficients are given by $x[n] = x_c(n\Delta t)$

z - transform : a special Laurent series

$$\boxed{z_m = 0} \quad \boxed{a_{-n}^{\{0\}} = h(n)} \quad \boxed{n \to -n}$$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}}$$

Time Reversal ⟵ Laplace Transform

the transform functions

$X(s) = \int$ over negative powers e^{-st} for $t > 0$

$X(z) = \int$ over negative powers z^{-n} for $n > 0$

the time expansion functions

$x(t) = \int$ over negative powers e^{-st} for $t > 0$

$x[n] = \int$ over negative powers z^{-n} for $n > 0$

Time Reversal ⟵ z^{-1} : unit delay , char eq (modes in z^k)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Stable System : $h[n]$ must be asbsolutely summable

$$|e^{j\omega n}| = 1$$
$$|z^n| \qquad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \qquad \text{asolutely summable}$$
$$= \sum_{n=-\infty}^{+\infty} |h[n]\, e^{-j\omega n}|$$
$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n} \right|$$
$$= \left| H(z) \big|_{z=e^{j\omega}} \right|$$
$$\infty > \left| H(z) \big|_{z=e^{j\omega}} \right|$$

a stable system,
$H(z)$ must converge on the unit circle $|z| = 1$
ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z)\Big|_{|z|=1} = H(e^{j\hat{\omega}})$$ DTFT of $h[n]$

discrete all stable sequence must have convergent DTFTs

continuous all stable signal must have convergent CTFTs

$C \leftarrow$ unit circle $z = e^{j\hat{\omega}}$

ZT^{-1} $DTFT^{-1}$ identical formulas

$h[n]$ causal

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=0}^{+\infty} h[n]\, z^{-n} \qquad n \in [0, \infty)$$

for finite values of n,

each term must be finite as long as $z \neq 0$

For the sum to converge,

$h[n]\, z^{-n}$ must vanish as $n \to \infty$

$|z| > r_h \qquad z_h = r_h e^{j\theta}$

z_h^n is the largest magnitude geometrically increasing component

$n^m z_e^n$: the most general term for impulse responses

$n \to \infty \qquad z_e^n$ dominant over n^m for finite m

geometric components — as poles

$$\mathcal{Z}\{z_\ell^n u[n]\} = \frac{1}{1-\left(\frac{z_\ell}{z}\right)} = \frac{z}{z-z_\ell}$$

ROC of a causal sequence $h[n]$

outside the radius of the largest magnitude pole of H(

ROC of a causal signal $h(t)$

to the right of the rightmost pole of $H_c(s)$

if $h[n]$ is a stable, causal sequence,

the unit circle must be included in the ROC

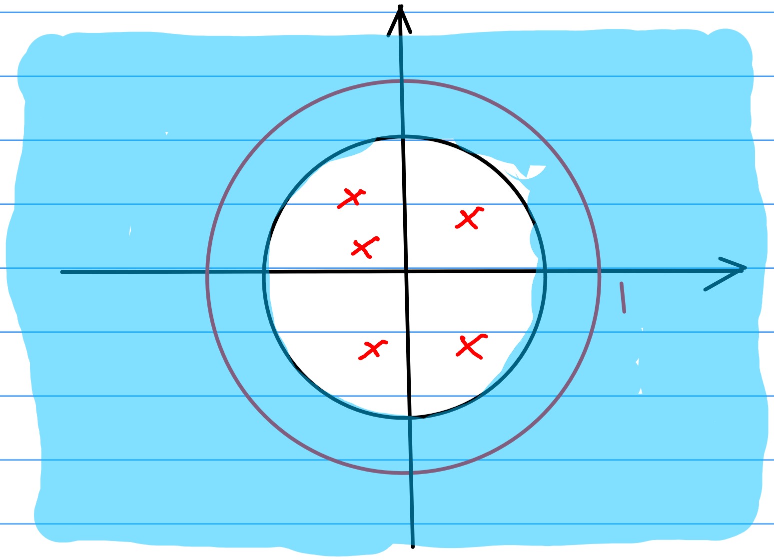
- **Causal** $h[n]$

ROC: outside of a circle

- **Stable** $h[n]$

all poles inside the unit circle

ROC circle must be smaller than the unit circle



$\Rightarrow$ all the geometric components of $h[n]$: modes must decay with increasing n

all the poles of $H(z)$ must be within the unit circle

all the poles of $H_c(s)$ must be in the left half plane

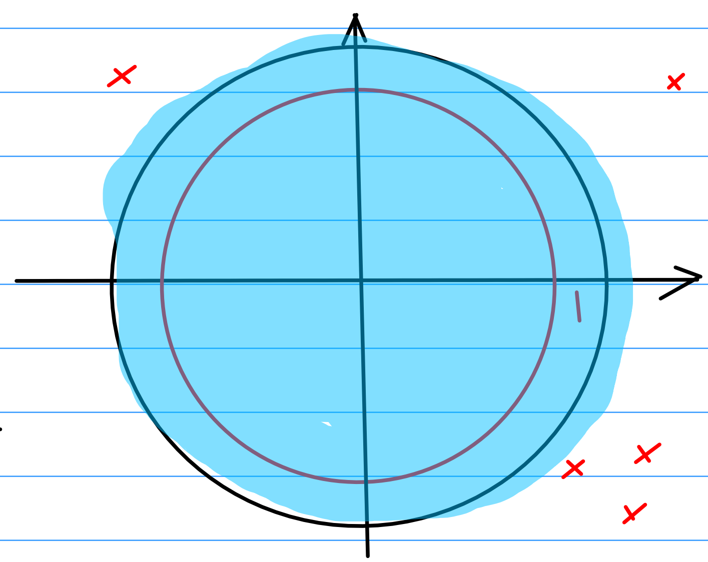
- **anti-Causal** $h[n]$

  ROC: in side of a circle

- **Stable** $h[n]$

  all poles outside the unit circle

  ROC circle must be larger than the unit circle



$\Rightarrow$ all the geometric components of $h[n]$: modes must decay with decreasing n

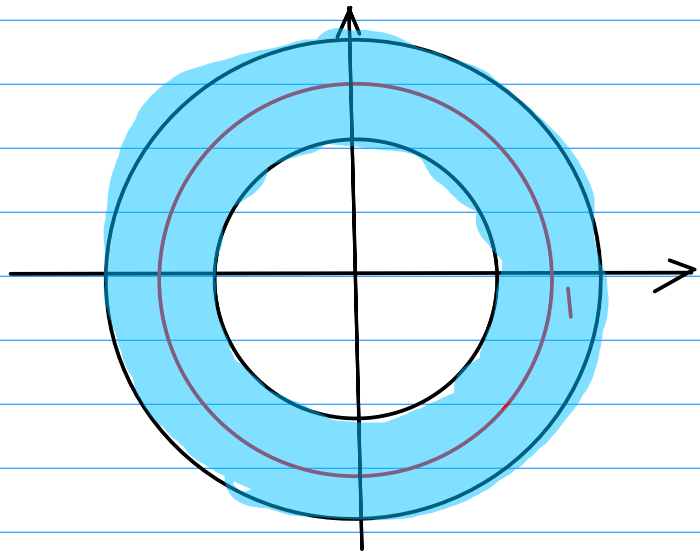
- bi-Causal $h[n]$

  $h_c[n] + h_{ac}[n]$

  outside   inside

  max mag < min mag

  Overlapped ROC



- Stable $h[n]$

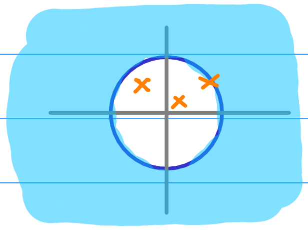
  all poles outside

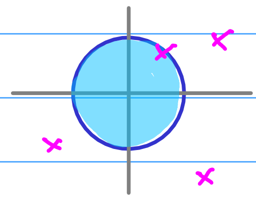
  the unit circle

  ROC circle must include the unit circle

◉ bi-Causal $h[n]$

$$h[n] = \underbrace{h_c[n]}_{\text{causal comp.}} + \underbrace{h_{ac}[n]}_{\text{anti-causal comp}}$$

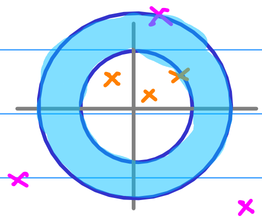




outside a circle — inside a circle

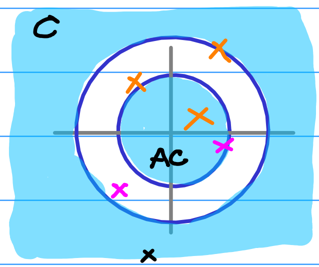
max mag < min mag

Overlapped ROC



max mag > min mag

non-overlapping ROC



◦ Stable $h[n]$

all poles <u>outside</u> the large circle
<u>inside</u> the small circle

ROC circle must include the <u>unit circle</u>

only one annulus include the <u>unit circle</u>

only one stable sequence

Existence of the z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

the existence of the z-transform is guaranteed if

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z^n|} < \infty \quad \text{for some } |z|$$

any signal $x[n]$ that grows no faster than
an exponential signal r_0^n, for some r_0
satisfies the above condition

if $|x[n]| \leq r_0^n$ for some r_0

then $|X(z)| \leq \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \dfrac{1}{1 - \frac{r_0}{|z|}} \quad |z| > r_0$

therefore $X(z)$ exists for $|z| > r_0$

Almost all practical signal satisfy this condition
$|x[n]| \leq r_0^n$ for some r_0
and z-transformable

Some signal models (e.g. r^{n^2}) grows faster than
the exponential signal r_0^n (for any r_0)
and do not satisfy this condition
and are not z-transformable

Such signals are of little practical or theoretical interest
Even such signals over a finite interval are z-transformable

Region of Convergence

Laplace Transform $Ae^{\alpha t}u(t)$ $\alpha > 0$

z - Transform $A\alpha^n u[n]$ $|\alpha| > 0$

DTFT (X)

$$X(z) = A \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = A \sum_{n=0}^{\infty} \alpha^n z^{-n} = A \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

converge $\left|\frac{\alpha}{z}\right| < 1$ $|z| > |\alpha|$

open exterior of
a circle of radius $|\alpha|$

the sum of a geometric series

$$X(z) = A \frac{1}{1-\frac{\alpha}{z}} = \frac{A}{1-\alpha z^{-1}} = A \frac{z}{z-\alpha} \qquad |z| > |\alpha|$$

DTFT

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega} n}$$

DTFT

DTFT of the unit sequence $u[n]$

$$X(e^{-j\hat{\omega}n}) = \sum_{n=-\infty}^{+\infty} u[n] e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} e^{-j\hat{\omega}n}$$

not converge

$\hat{\omega} = 0$ $\quad \sum_{n=0}^{\infty} 1^n$ $\quad$ diverge

$\hat{\omega} = \pi$ $\quad \sum_{n=0}^{\infty} (-1)^n$ $\quad$ oscillates

$\hat{\omega} = \frac{\pi}{2}$ $\quad \sum_{n=0}^{\infty} (j)^n$

The DTFTs of some commonly used functions do not exist in the strict sense.

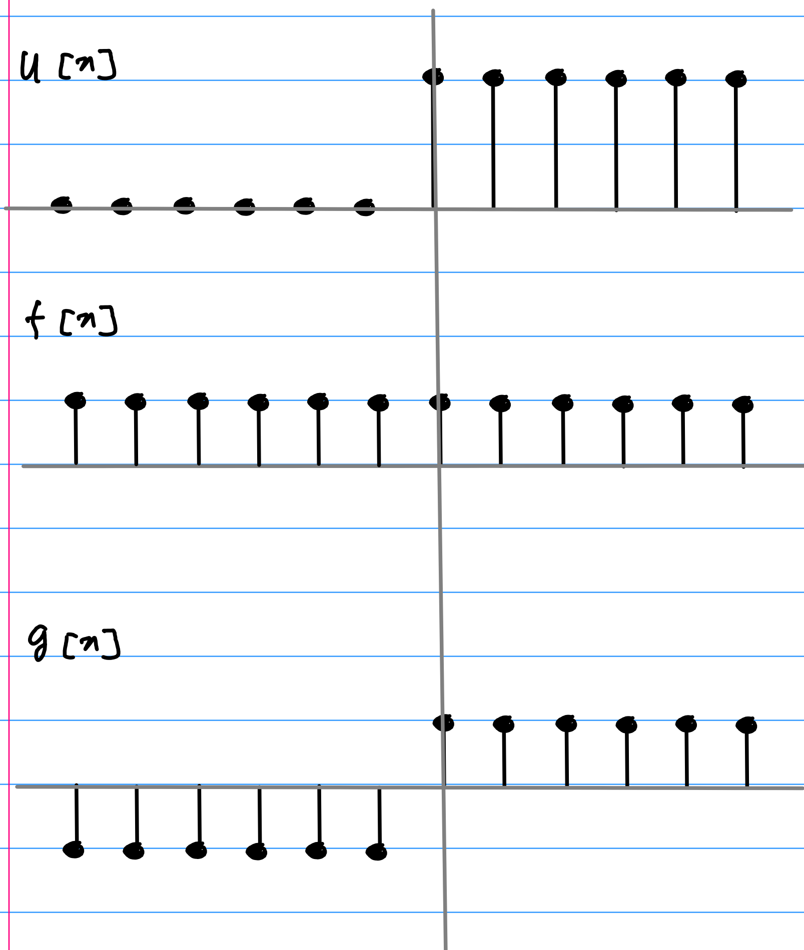
But even though the DTFT does not exist, the z-transform does exist.

$$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$|z| > 1$ $\quad X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

$X(z) = \frac{z}{z-1}$ $\quad$ pole $z=1$, zero $z=0$

$X(z) = \frac{1}{1-z^{-1}}$ $\quad$ useful when a system is synthesized from a z-domain transfer function http://math.stackexchange.com/questions/1772285/dtft-of-the-unit-step-function



$$f[n] = \frac{1}{2} \qquad -\infty < n < \infty$$

$$g[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

$$u[n] = f[n] + g[n]$$

$$\delta[n] = g[n] - g[n-1]$$

$$1 = G(e^{j\hat{\omega}}) - e^{-j\hat{\omega}} G(e^{j\hat{\omega}})$$

$$G(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}}$$

$$F(e^{j\hat{\omega}}) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$ (impulse train)

$$U(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Discrete Time Exponential γ^n

continuous time exponential $e^{\lambda t}$

$$e^{\lambda t} = \gamma^t \qquad (e^{\lambda})^t = \gamma^t$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$$e^{-0.3t} = (0.7408)^t$$

$$4^t = e^{1.386t}$$

continuous time analysis $e^{\lambda t}$

discrete time analysis γ^n

$$e^{\lambda n} = \gamma^n \qquad (e^{\lambda})^n = \gamma^n$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$e^{\lambda n}$

exponentially grows if $Re\,\lambda > 0$ (λ ~~in~~ RHP)

exponentially decays if $Re\,\lambda < 0$ (λ in LHP)

oscillates or constant if $Re\,\lambda = 0$ (λ in imag axis)

the location of λ in the complex plain indicates whether

① $e^{\lambda t}$ will grow exponentially

② $e^{\lambda t}$ will decay exponentially

③ $e^{\lambda t}$ will oscillates with constant amplitude

constant signal : oscillation with zero frequency

$e^{j\Omega n}$ $\quad \lambda = j\Omega$ imaginary axis

constant amplitude oscillating signal

$e^{j\Omega n} = (e^{j\Omega})^n = \gamma^n \qquad \gamma = e^{j\Omega} \qquad |\gamma| = 1$

$\lambda = j\Omega$ imaginary axis $\rightarrow |\gamma| = 1$ unit circle

if γ lies on the unit circle,

γ^n oscillates with constant amplitude

the imaginary axis in the λ plane

the unit circle in the γ plane

$e^{\lambda n}$ $\quad \lambda = a + jb$ in the LHP $(a<0)$

exponentially decaying

$\gamma = e^{\lambda} = e^{a+jb} = e^{a} e^{jb}$

$|\gamma| = |e^{\lambda}| = |e^{a}| \cdot |e^{jb}| = |e^{a}| = e^{a}$

$|\gamma| = e^{a} < 1$ inside the unit circle

γ^{n} : exponentially decaying

$|\gamma| = e^{a} > 1$ outside the unit circle

γ^{n} : exponentially growing

λ-plane		r-plane
the imaginary axis	$\longrightarrow$	the unit circle
the LHP	$\longrightarrow$	inside of the unit circle
the RHP	$\longrightarrow$	outside of the unit circle