# Random Process Background (1B)

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Based on
Probability, Random Variables and Random Signal Principles,
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# Outline

1. Borel Sets
   - Measurable Space
   - Topological Space II
   - Borel Sets
2. Stochatic Process

# Outline

# Mathematical objects (1)

- a **mathematical object** is
  an abstract concept arising in mathematics.
- an **mathematical object** is anything
  that has been (or could be) formally defined,
  and with which one may do
  - deductive reasoning
  - mathematical proofs

https://en.wikipedia.org/wiki/Mathematical_object

# Mathematical objects (2)

- typically, a **mathematical object**
  - can be a value that can be assigned to a variable
  - therefore can be involved in formulas

https://en.wikipedia.org/wiki/Mathematical_object

# Mathematical objects (3)

- commonly encountered **mathematical objects** include
  - numbers
  - sets
  - functions
  - expressions
  - geometric objects
  - transformations of other mathematical objects
  - spaces

https://en.wikipedia.org/wiki/Mathematical_object

# Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
  - for example, the followings are considered as **mathematical objects** in **proof theory**.
    - theorems
    - proofs
    - theories

https://en.wikipedia.org/wiki/Mathematical_object

# Structure (1)

- a **structure** is a set
  endowed with some *additional features* on the set
  - an *operation*
  - *relation*
  - *metric*
  - *topology*
- often, the *additional features* are
  attached or related to the set,
  so as to provide it with some
  *additional meaning* or *significance*.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Structure (2)

- A partial list of possible **structures** are
  - measures
  - algebraic structures (groups, fields, etc.)
  - topologies
  - metric structures (geometries)
  - orders
  - events
  - equivalence relations
  - differential structures
  - categories.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Space (1)

- A **space** consists of
  selected mathematical objects that are treated as points,
  and selected relationships between these points.
  - the *nature* of the points can vary widely:
    for example, the points can be
    - elements of a set
    - functions on another space
    - subspaces of another space
  - It is the relationships between points
    that define the *nature* of the **space**.

https://en.wikipedia.org/wiki/Space_(mathematics)

# Space (2)

- *modern mathematics* uses many types of **spaces**, such as
  - Euclidean spaces
  - linear spaces
  - topological spaces
  - Hilbert spaces
  - probability spaces
- *modern mathematics* does not define the notion of **space** itself.

https://en.wikipedia.org/wiki/Space_(mathematics)

# Space (3)

- a **space** is
  a set (or a universe) with some added features
- it is not always clear
  whether a given mathematical object should be considered
  as a geometric **space**, or an algebraic **structure**
- a general definition of **structure** embraces
  all common types of **space**

https://en.wikipedia.org/wiki/Space_(mathematics)

# Mathematical space (1)

- A **mathematical space** is, informally,
  a collection of mathematical objects
  under consideration.

- The **universe** of mathematical objects within a **space**
  are *precisely* defined entities
  whose rules of *interaction* come baked
  into the rules of the **space**.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
  - A **mathematical set** is also a collection of objects
  - but these objects are being pulled from a **space** (or **universe**) of objects where the rules and definitions have already been agreed upon

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
  - a **mathematical set** has no internal structure,
  - a **space** usually has some internal structure.

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Mathematical space (4)

- having some internal structure
  could mean a variety of things,
  but typically it involves
  - *interactions* and *relationships*
    between elements of the **space**
  - *rules* on how to *create* and *define*
    *new* elements of the **space**

https://www.localmaxradio.com/questions/what-is-a-mathematical-space

# Measurable space (1)

- A **measurable space** is
  any space with a sigma-algebra
  which can then be equipped with a measure
  - collection of subsets of the space
    following certain rules with a way
    to assign sizes to those sets.

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

# Measurable space (2)

- Intuitively,
  certain sets belonging to a measurable space
  can be given a size in a *consistent way*.

  *consistent way* means that certain axioms are met:
  - the empty set is given a size of zero
  - if a measurable set is contained inside another one,
    then its size is less than or equal to
    the size of the containing set
  - the size of a disjoint union of sets is
    the sum of the individual sets' sizes

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

# The set of all real numbers

- In the set of all real numbers, one has the natural Euclidean metric; that is, a function which *measures* the distance between two real numbers: $d(x, y) = |x - y|$.

https://en.wikipedia.org/wiki/Open_set

# All points close to a real number $x$

- Therefore, given a real number $x$, one can speak of the set of all points **close** to that real number $x$; that is, **within** $\varepsilon$ of $x$.
- In essence, points **within** $\varepsilon$ of $x$ ***approximate*** $x$ to an accuracy of degree $\varepsilon$.
- Note that $\varepsilon > 0$ always, but as $\varepsilon$ becomes *smaller* and *smaller*, one obtains points that approximate x to a *higher* and *higher* degree of accuracy.

https://en.wikipedia.org/wiki/Open_set

# The points within $\varepsilon$ of $x$

- For example, if $x = 0$ and $\varepsilon = 1$,
  the points within $\varepsilon$ of $x$ are precisely
  the points of the interval $(-1, 1)$;
- However, with $\varepsilon = 0.5$,
  the points within $\varepsilon$ of $x$ are precisely
  the points of $(-0.5, 0.5)$.
- Clearly, these points approximate $x$
  to a *greater* degree of accuracy than when $\varepsilon = 1$.

https://en.wikipedia.org/wiki/Open_set

# without a concrete Euclidean metric

- The previous examples shows, for the case $x = 0$, that one may **approximate** $x$ to *higher* and *higher* degrees of accuracy by defining $\varepsilon$ to be *smaller* and *smaller*.
- In particular, sets of the form $(-\varepsilon, \varepsilon)$ give us a lot of information about points **close** to $x = 0$.
- Thus, rather than speaking of a concrete Euclidean metric, one may *use* **sets** to *describe* points **close** to $x$.

https://en.wikipedia.org/wiki/Open_set

# Different collections of sets containing 0

- This innovative idea has far-reaching consequences; in particular, by defining

  different collections of sets containing 0
  (distinct from the sets $(-\varepsilon, \varepsilon)$),
  one may find different results
  regarding the distance
  between 0 and other real numbers.

https://en.wikipedia.org/wiki/Open_set

# A set for measuring distance

- For example, if we were to define $\boldsymbol{R}$ as the *only* such set for "*measuring distance*", all points are close to 0
- since there is only one possible degree of accuracy one may achieve in **approximating** 0: being a member of $\boldsymbol{R}$.

https://en.wikipedia.org/wiki/Open_set

# The measure as a binary condition

- Thus, we find that in some sense, every real number is distance 0 away from 0.
- It may help in this case to think of the measure as being a binary condition:
  - all things in $\boldsymbol{R}$ are equally close to 0,
  - while any item that is not in $R$ is not close to 0.

https://en.wikipedia.org/wiki/Open_set

# Probability space

- A **probability space** is simply
  a measurable space equipped with a probability measure.
- A probability measure has the special property
  of giving the entire **space** a size of **1**.
  - this then implies that the size
    of any disjoint union of sets
    (the sum of the **sizes** of the sets)
    in the **probability space**
    is less than or equal to **1**

https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have

# Euclidean space definition (1)

- A subset $U$ of the **Euclidean n-space** $\mathbb{R}^n$ is open

  if, for every point $x$ in $U$,
  there exists a positive real number $\varepsilon$
  (depending on $x$)

  such that any point in $\mathbb{R}^n$
  whose Euclidean distance from $x$ is smaller than $\varepsilon$

  belongs to $U$

https://en.wikipedia.org/wiki/Open_set

# Euclidean space definition (2)

- Equivalently, a subset $U$ of $\mathbb{R}^n$ is open

  if every point in $U$ is
  the center of an open ball contained in $U$

- An example of a subset of $\mathbb{R}$ that is not open is

  the closed interval $[0,1]$,
  since neither $0-\varepsilon$ nor $1+\varepsilon$ belongs to $[0,1]$
  for any $\varepsilon > 0$, no matter how small.

https://en.wikipedia.org/wiki/Open_set

# Metric space definition (1)

- A subset $U$ of a **metric space** $(M, d)$ is called open

  if, for any point $x$ in $U$, there exists a real number $\varepsilon > 0$
  such that any point $y \in M$ satisfying $d(x, y) < \varepsilon$ belongs to $U$.

- Equivalently, $U$ is open
  if every point in $U$
  has a neighborhood contained in $U$.

- This generalizes the **Euclidean space** example,
  since **Euclidean space** with the **Euclidean distance**
  is a **metric space**.

https://en.wikipedia.org/wiki/Open_set

# Metric space definition (2)

- formally, a **metric space** is an ordered pair $(M, d)$
  where $M$ is a set and
  d is a metric on $M$,
  i.e., a function

$$d : M \times M \to \mathbb{R}$$

  satisfying the following axioms for all points $x, y, z \in M$:
  - $d(x,x) = 0.$
  - if $x \neq y$, then $d(x,y) > 0.$
  - $d(x,y) = d(y,x).$
  - $d(x,z) \leq d(x,y) + d(y,z).$

https://en.wikipedia.org/wiki/Open_set

# Metric space definition (3)

- satisfying the following axioms for all points $x, y, z \in M$:
  - the distance from a point *to itself* is zero:
  - (Positivity) the distance between two distinct points is always positive:
  - (Symmetry) the distance from $x$ to $y$ is always the same as the distance from $y$ to $x$:
  - (Triangle inequality) you can arrive at $z$ from $x$ by taking a detour through $y$, but this will not make your journey any faster than the shortest path.
- If the metric $d$ is *unambiguous*, one often refers by abuse of notation to "the **metric space** $M$".

https://en.wikipedia.org/wiki/Open_set

# Outline

# Definition via Open Sets (1)

- A topology $\tau$ on a set $X$ is

  a set of subsets of $X$ with the *properties* below.

  - a topology $\tau$ on a set $X$ : a set of subsets of $X$
  - members of $\tau$ : subsets of $X$
- each member of $\tau$ is called an open set.
- $X$ together with $\tau$ is called a **topological space**

https://en.wikipedia.org/wiki/Open_set

# Definition via Open Sets (2)

- topology $\tau$ : a set of subsets of $X$ has the *properties* below
  - $X \in \tau$ and $\varnothing \in \tau$
  - any union of sets in $\tau$ belong to $\tau$ :
    any union of subsets of $X$ belong to $\tau$ :
    if $\{U_i : i \in I\} \subseteq \tau$ then

$$\bigcup_{i \in I} U_i \in \tau$$

  - any finite intersection of sets in $\tau$ belong to $\tau$
    any finite intersection of subsets of $X$ belong to $\tau$ :
    if $U_1, \ldots, U_n \in \tau$ then

$$U_1 \cap \cdots \cap U_n \in \tau$$

https://en.wikipedia.org/wiki/Open_set

# Definition via Open Sets (3)

- Infinite intersections of open sets need not be open.
- For example, the intersection of all intervals of the form $(-1/n, 1/n)$, where $n$ is a positive integer, is the set $\{0\}$ which is not open in the real line.
- A metric space is a topological space, whose topology consists of the collection of all subsets that are unions of open balls.
- There are, however, topological spaces that are not metric spaces.

https://en.wikipedia.org/wiki/Open_set

# Definition via Open Sets (4)

- A **topology** on a set $X$ may be defined as a collection $\tau$ of subsets of $X$, called open sets and satisfying the following axioms:
  - The empty set and $X$ itself belong to $\tau$ .
  - any arbitrary (finite or infinite) union of members of $\tau$ belongs to $\tau$ .
  - the intersection of any finite number of members of $\tau$ belongs to $\tau$ .

https://en.wikipedia.org/wiki/Topological_space

# Definition via Open Sets (5)

- As this definition of a topology is the most commonly used, the set $\tau$ of the open sets is commonly called a **topology** on $X$.
- A subset $C \subseteq X$ is said to be closed in $(X, \tau)$ if its complement $X \setminus C$ is an open set.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Neighborhoods (1)

- This axiomatization is due to Felix Hausdorff.
- Let $X$ be a set;
- the elements of $X$ are usually called points, though they can be any mathematical object.
- We allow $X$ to be empty.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Neighborhoods (2)

- Let $\mathscr{N}$ be a function assigning to each $x$ (point) in $X$ a non-empty collection $\mathscr{N}(x)$ of subsets of $X$.
- The elements of $\mathscr{N}(x)$ will be called neighbourhoods of $x$ with respect to $\mathscr{N}$ (or, simply, neighbourhoods of $x$).
- The function $\mathscr{N}$ is called a neighbourhood topology if *the axioms* below are satisfied; and
- then $X$ with $\mathscr{N}$ is called a topological space.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Neighborhoods (3)

1. If $N$ is a neighbourhood of $x$ (i.e., $N \in \mathcal{N}(x)$), then $x \in N$. In other words, each point belongs to every one of its neighbourhoods.
2. If $N$ is a subset of $X$ and includes a neighbourhood of $x$, then $N$ is a neighbourhood of $x$.
   i.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of $x$.
3. The intersection of two neighbourhoods of $x$ is a neighbourhood of $x$.
4. Any neighbourhood $\mathcal{N}$ of $x$ includes a neighbourhood $\mathcal{M}$ of $x$ such that $\mathcal{N}$ is a neighbourhood of each point of $M$.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Neighborhoods (4)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory, that of linking together the neighbourhoods of different points of $X$.
- A standard example of such a system of neighbourhoods is for the real line $\mathbb{R}$, where a subset $N$ of $\mathbb{R}$ is defined to be a neighbourhood of a real number $x$ if it includes an open interval containing $x$.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Neighborhoods (5)

- Given such a structure, a subset $U$ of $X$ is defined to be **open** if $U$ is a neighbourhood of all points in $U$.
- The **open sets** then satisfy the axioms given below.
- Conversely, when given the **open sets** of a topological space, the neighbourhoods satisfying the above axioms can be recovered by defining $N$ to be a neighbourhood of $x$ if $N$ includes an open set $U$ such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

# Definition via Closed Sets (1)

- Using **de Morgan's laws**,
  the above axioms defining **open sets**
  become axioms defining **closed sets**:
- The empty set and $X$ are **closed**.
  - The intersection of any collection of **closed sets**
    s also **closed**.
  - The union of any finite number of **closed sets**
    is also **closed**.
- Using these axioms, another way to define a **topological space**
  is as a set $X$ together with a collection $\tau$ of **closed subsets** of $X$.
  Thus the **sets** in the topology $\tau$ are the **closed sets**,
  and their complements in $X$ are the **open sets**.

https://en.wikipedia.org/wiki/Open_set

# Homeomorphism (1)

- a **homeomorphism**
  (from Greek ὅμοιος (homoios) 'similar, same',
  and μορφή (morphē) 'shape, form',
  named by Henri Poincaré), **topological isomorphism**,
  or **bicontinuous function** is
  a bijective and continuous function
  between topological spaces
  that has a continuous inverse function.

https://en.wikipedia.org/wiki/Homeomorphism

# Homeomorphism (2)

- **Homeomorphisms** are the isomorphisms in the category of topological spaces – the mappings that preserve all the topological properties of a given space.
- Two spaces with a **homeomorphism** between them are called homeomorphic, and from a topological viewpoint they are the same.

https://en.wikipedia.org/wiki/Homeomorphism

# Homeomorphism (3)

- Very roughly speaking,
  a **topological space** is a geometric object,
  and the **homeomorphism** is
  a *continuous* stretching and bending
  of the object into a *new* shape.

https://en.wikipedia.org/wiki/Homeomorphism

# Homeomorphism (4)

- Thus, a *square* and a *circle* are homeomorphic to each other, but a *sphere* and a *torus* are not.
- However, this description can be misleading.
- Some *continuous deformations* are not homeomorphisms, such as the *deformation* of a *line* into a *point*.
- Some homeomorphisms are not *continuous deformations*, such as the homeomorphism between a *trefoil knot* and a *circle*.

https://en.wikipedia.org/wiki/Homeomorphism

# Outline

# Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** $X$) are:
  - the $\{\}$ and $X$ are in the **set**.
  - if $A$ is in the **set**, $complement(A)$ is in the **set**.
  - for any **sets** $E_i$ in the set,
    $\bigcup_i E_i$ is in the **set** (for countable $i$).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

# Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do probability on.
  - for example, we can assign ratios of areas and length, so the measure on such a set $X$ tells something about the probability of its subsets.
  - we can find the probability of subsets $A$ and $B$ because we know their ratios with respect to a set $X$ ;
  - we also know that
    - (the measure of) their complements are defined, and
    - their unions and intersections are defined,
    - so we know how to find the probability of things in this set $X$.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

# Sigma algebra (3)

- The **sigma algebra** which contains the standard topology on R (that is, *all* open sets on R) is called the **Borel Sigma Algebra**, and the elements of this set are called **Borel sets**.
- What this gives us, is the set of sets on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

# Sigma algebra (4)

- The set of Lebesgue measurable sets is
  the set of **Borel sets**,
  along with (union) all the sets
  which differ from a Borel set
  by a set of measure 0.
- More intuitively, it is all the sets
  we can normally measure,
  plus a bunch of stuff
  that doesn't affect our ideas of area or volume
  (think about the border of the circle above).

https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7

# Borel Sets (1-1)

- a **Borel set** is any set in a topological space
  that can be formed from open sets
  (or, equivalently, from closed sets)
  through the operations of
  - countable union,
  - countable intersection, and
  - relative complement.

https://en.wikipedia.org/wiki/Borel_set

# Borel Sets (1-2)

- For a **topological space X**,
  the collection of all Borel sets on X forms a σ-algebra,
  known as the **Borel algebra** or **Borel σ-algebra**.
- The B**orel algebra** on **X** is the smallest **σ-algebra**
  containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

# Borel Sets (1-3)

- **Borel sets** are important in measure theory,
  since any measure defined on the open sets of a space,
  or on the closed sets of a space,
  must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets
  is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

# Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **$\sigma$-algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- . . . And again and again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

# Borel Sets (3-1)

1. Start with finite unions of closed-open intervals.
   These sets are completely elementary, and they form an algebra.
2. Adjoin countable unions and intersections of elementary sets.
   What you get already includes open sets and closed sets, intersections of an open set and a closed set, and so on.
   Thus you obtain an algebra, that is still not a σ-algebra.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

# Borel Sets (3)

3. Again, adjoin countable unions and intersections to 2.
   Observe that you get a strictly larger class, since a countable intersection of countable unions of intervals is not necessarily included in 2.
   Explicit examples of sets in 3 but not in 2 include $F_\sigma$ sets, like, say, the set of *rational numbers*.
4. And do the same again.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

# Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of $\sigma$-algebra, you should include it as well - if you want, as step $\infty$

https://math.stackexchange.com/questions/220248/understanding-borel-sets

# Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated $\sigma$ -algebra.

https://math.stackexchange.com/questions/220248/understanding-borel-sets

# Stochastic Process (1)

In probability theory and related fields,
a **stochastic** (/stoʊˈkæstɪk/) or **random** process is
a mathematical object usually defined as
a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokházomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

https://en.wikipedia.org/wiki/Stochastic
https://en.wiktionary.org/wiki/stochastic

# Stochastic Process (2)

The definition of a **stochastic process** varies,
but a **stochastic process** is *traditionally* defined as
a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process**
are considered synonyms and are used interchangeably,
without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used
while instead of "**index set**", sometimes
the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process (3)

The term **random function** is also used to refer to
a **stochastic** or **random process**,
though sometimes it is only used
when the stochastic process takes real values.

This term is also used when the **index sets** are
**mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process**
are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field**
when the **index set** is $n$-dimensional **Euclidean space** $\mathbb{R}^n$
or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process (4)

A **stochastic process** can be denoted,
by $\{X(t)\}_{t\in T}$, $\{X_t\}_{t\in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as $X$ or $X(t)$,
although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or $X_t$ are used to refer to
the **random variable** with the **index** $t$,
and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write,
for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process Definition (1)

A **stochastic process** is defined as
a collection of **random variables**
defined on a common **probability space** $(\Omega, \mathscr{F}, P)$,

- $\Omega$ is a **sample space**,
- $\mathscr{F}$ is a $\sigma$ **-algebra**,
- $P$ is a **probability measure**;
- the **random variables**, indexed by some set $T$,
- all take values in the same **mathematical space** $S$, which must be **measurable** with respect to some $\sigma$ -algebra $\Sigma$

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process Definition (2)

In other words, for a given **probability space** $(\Omega, \mathscr{F}, P)$
and a **measurable space** $(S, \Sigma)$,
a **stochastic process** is a
**collection** of $S$-valued **random variables**,
which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process Definition (3)

Historically, in many problems from the natural sciences
a point $t \in T$ had the meaning of time,
so $X(t)$ is a **random variable** representing
a value observed at time $t$.

A **stochastic process** can also be written as $\{X(t,\omega) : t \in T\}$
to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

# Stochastic Process Definition (4)

There are other ways to consider a stochastic process,
with the above definition being considered the traditional one.

For example, a stochastic process
can be interpreted or defined as
a $S^T$-valued **random variable**,
where $S^T$ is the space of all the possible functions
from the set $T$ into the space $S$.

However this alternative definition as
a "**function-valued random variable**"
in general requires additional regularity assumptions
to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

# Index set (1)

The set $T$ is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set $T$ the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

# Index set (2)

In addition to these sets, the index set $T$ can be another set with a **total order** or a more general set, such as the Cartesian plane $R^2$ or $n$-dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

# State space

The **mathematical space** $S$ of a **stochastic process**
is called its **state space**.

This mathematical space can be defined
using integers, real lines, $n$-dimensional Euclidean spaces,
complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements
that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

# Sample function (1)

A **sample function** is a single **outcome** of a **stochastic process**,
so it is formed by taking a single possible value
of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t,\omega) : t \in T\}$ is a **stochastic process**,
then for any point $\omega \in \Omega$ , the mapping $X(\cdot,\omega) : T \to S$,
is called a **sample function**, a **realization**, or,
particularly when $T$ is interpreted as time,
a **sample path** of the **stochastic process** $\{X(t,\omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

# Sample function (2)

This means that for a fixed $\omega \in \Omega$ ,
there exists a **sample function**
that maps the **index set** $T$ to the **state space** $S$.

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process