

Temporal Characteristics of Random Processes

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Oct 25, 2022

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 The concepts of the random process

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a particular value : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, \dots, s_n\}$
- an **outcome** (an element of S) : s

$$s \rightarrow X(s)$$

$$x = X(s)$$

$$s \rightarrow x$$

Understanding Random Variables (1)

random variables are used to quantify outcomes of a random occurrence, and therefore, can take on many values.

Random variables are required to be measurable and are typically real numbers.

For example, the letter X may be designated to represent the sum of the resulting numbers after three dice are rolled.

In this case, X could be 3 ($1 + 1 + 1$), 18 ($6 + 6 + 6$), or somewhere between 3 and 18, since the highest number of a die is 6 and the lowest number is 1.

<https://www.investopedia.com/terms/r/random-variable.asp>

Understanding Random Variables (2)

A random variable is different from an algebraic variable. The variable in an algebraic equation is an unknown value that can be calculated.

The equation $10 + x = 13$ shows that we can calculate the specific value for x which is 3.

On the other hand, a random variable has a set of values, and any of those values could be the resulting outcome as seen in the example of the dice above.

<https://www.investopedia.com/terms/r/random-variable.asp>

Understanding Random Variables (3)

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Understanding Random Variables (4)

A typical example of a random variable is the outcome of a coin toss. Consider a probability distribution in which the outcomes of a random event are not equally likely to happen. If the random variable Y is the number of heads we get from tossing two coins, then Y could be 0, 1, or 2. This means that we could have no heads, one head, or both heads on a two-coin toss.

<https://www.investopedia.com/terms/r/random-variable.asp>

Formal definition of a random variable

A random variable X is a measurable function $X: \Omega \rightarrow E$ from a set of possible outcomes Ω to a measurable space E .

The technical axiomatic definition requires Ω to be a sample space of a probability triple (Ω, \mathcal{F}, P)

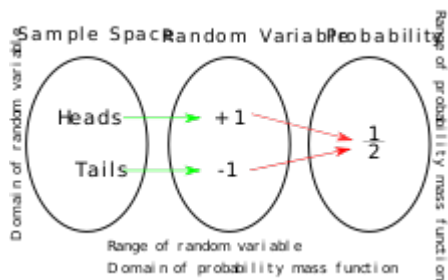
A random variable is often denoted by capital roman letters such as X, Y, Z, T .

The probability that X takes on a value in a measurable set $S \subseteq E$ is written as

$$P(X \in S) = P(\{\omega \in \Omega \mid X(\omega) \in S\})$$

https://en.wikipedia.org/wiki/Random_variable

Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

https://en.wikipedia.org/wiki/Random_variable

Probability Space (1)

In probability theory, a probability space or a probability triple (Ω, \mathcal{F}, P) is a mathematical construct that provides a formal model of a random process or "experiment".

For example, one can define a probability space which models the throwing of a die

https://en.wikipedia.org/wiki/Probability_space

Probability Space (2)

A probability space consists of three elements

A sample space, Ω , which is the set of all possible outcomes.

An event space, which is a set of events \mathcal{F} ,

an event being a set of outcomes in the sample space.

A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

https://en.wikipedia.org/wiki/Probability_space

Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be $\{1, 2, 3, 4, 5, 6\}$. For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as $\{5\}$ ("the die lands on 5"), as well as complex events such as $\{2, 4, 6\}$ ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example, $\{5\}$ would be mapped to $1/6$, and $\{2, 4, 6\}$ would be mapped to $3/6 = 1/2$.

https://en.wikipedia.org/wiki/Probability_space

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t, s)$$

assigning a **time function** to every **outcome** s_i

$$s_i \rightarrow x(t, s_i)$$

Random Process (2)

A random process

the family of such **time functions** is called a **random process**

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a random variable X is a rule which assigns a number to every outcome e of an experiment.

The random variable is a function $X(e)$ that maps the set of experiment outcomes to the set of numbers.

A random process is a rule that maps every outcome e of an experiment to a function $X(t, e)$.

A random process is usually conceived of as a function of time,

but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

The function $X(u, v, e)$ would be a function whose value depended on the location (u, v) and the outcome e ,

and could be used in representing random variations in an image.

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents a family or ensemble of **time functions**

$X(t, s)$ represents

- a **single time function** $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a **sample function**,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a **random process** $X(t)$
for a given **outcome** s_j

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t)$$

$$s_1 \longrightarrow x_1(t)$$

$$X(t, s_2) = x_2(t)$$

$$s_2 \longrightarrow x_2(t)$$

...

...

$$X(t, s_n) = x_n(t)$$

$$s_n \longrightarrow x_n(t)$$

$S = \{s_1, s_2, s_3, \dots, s_n\}$ a sample space

$X(t) = \{x_1(t), x_2(t), x_3(t), \dots, x_n(t)\}$ a random process

Random variables with time

a **random process** $X(t, s)$ represents a **single time function** when t is a variable and s is fixed at an outcome

a random process $X(t, s)$ represents a **single random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i)$$

random variable

$$X(t, s) = X(t)$$

random process

An alphabet

the **alphabet** of $X(t)$

the set of its possible values

- the values of **time** t for which a **random process** is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a **random process** is defined
 - continuous time
 - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
 - continuous alphabet
 - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
 - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
 - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
 - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
 - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic** if **future values** of any sample function cannot be predicted exactly from **observed past values**
- A process is **deterministic** if **future values** of any sample function can be predicted from **observed past values**

Deterministic Random Process Example (1)

$$X(t) = A \cos(\omega_0 t + \Theta)$$

A , Θ , or ω_0 (or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i}t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \quad \implies \quad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a **family of functions**. Imagine a giant strip chart recording in which each pen is identified with a different θ . This family of functions is traditionally called an **ensemble**.
- A **single function** $X(t, \theta_k)$ is selected by the **outcome** θ_k . This is just a **time function** that we could call $X_k(t)$. Different **outcomes** give us different **time functions**.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a **random variable**. Its value depends on the **outcome** θ .
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a **number**.

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

