Complex Functions (1A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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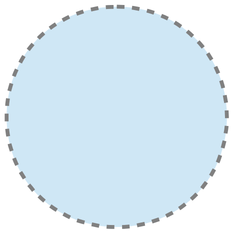
Concepts of Sets

An **open** set S

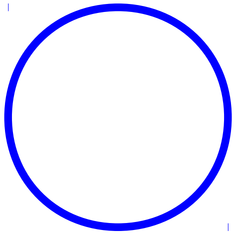
Every point of S has a neighborhood
consisting entirely of points that belong to S

{ points in the interior of a circle }



A **boundary point** set S

A point every neighborhood of which contains
both points that belong to S
and points that do *not* belong to S

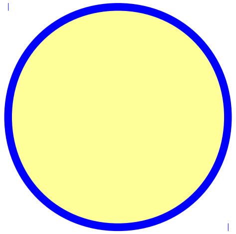


The **boundary of** a set S

The set of all boundary points of a set S

An **closed** set

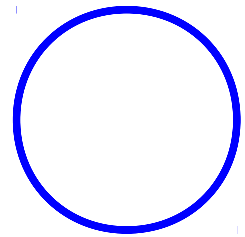
If its complement set is open



Neighborhood

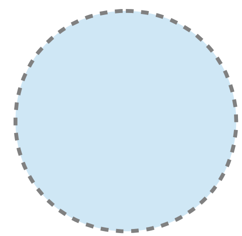
A **circle** of radius ρ and center a

$$|z - a| = \rho$$



Open circular disk

$$|z - a| < \rho$$

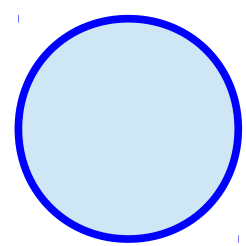


A **neighborhood** of a

$\rho-neighborhood\ of\ a$

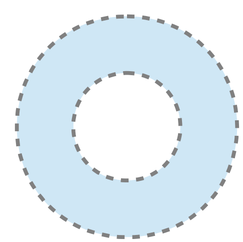
Closed circular disk

$$|z - a| \leq \rho$$



Open annulus

$$\rho_1 < |z - a| < \rho_2$$

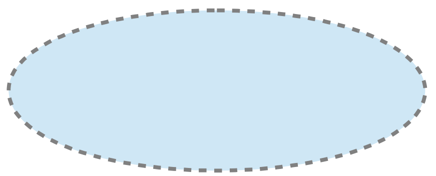


Domain and Region

A **connected** set S

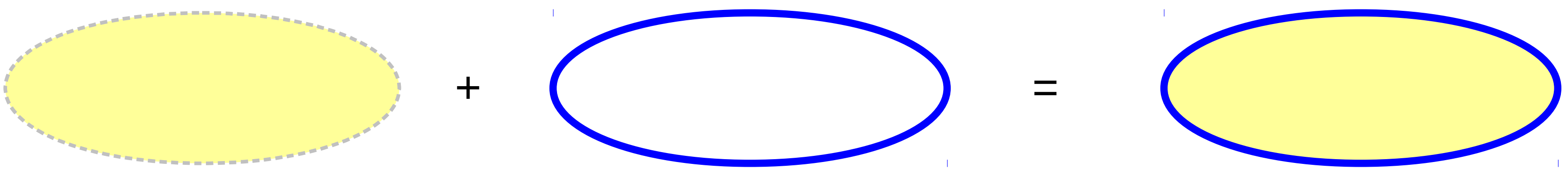
Any two of its points can be joined by a broken line
of finitely many straight-line segments
all of whose points belong to S

An **open connected** set S : a **domain**



An **open connected** set S **+**

some or all of its boundary points : a **region**



Derivatives

the complex function f is defined in a neighborhood of a point z_0

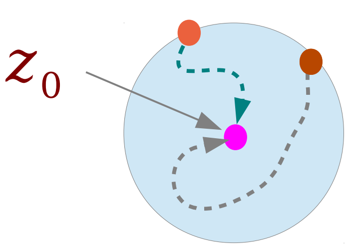
Derivative (function) of f

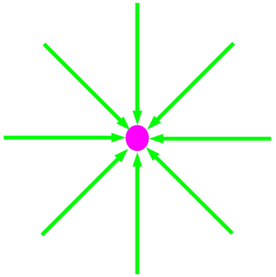
$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

Derivative of f at z_0

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Δz can approach zero
from any convenient direction





If the limit exists
f is said to be **differentiable** at z_0

a unique derivative

complex differentiable ➡ A neighborhood property
"holomorphic"

Analytic Functions

a complex function f is **analytic** at a point z_0

$\triangleq$

f **differentiable** at z_0 and
f **differentiable** at every point in some *neighborhood* of z_0

holomorphic

a neighborhood property

a complex function f is **analytic** in a domain D

$\triangleq$

f **analytic** at every point z_0 in the domain D

a complex function f is an **entire** function

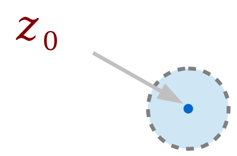
$\triangleq$

f **analytic** at every point z_0 in the entire complex plane

polynomial functions

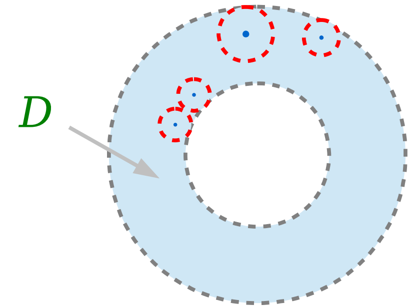
Analytic Functions – a neighborhood property

a complex function f is **analytic** at a point z_0



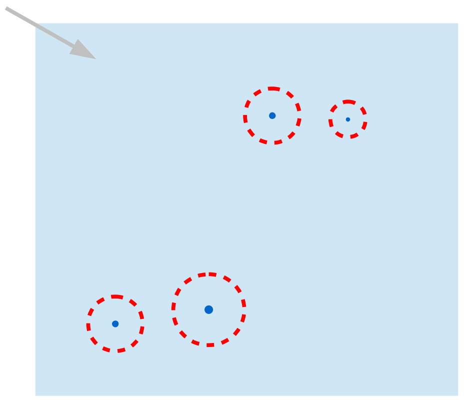
always can find some neighborhood where the function is complex differentiable at each point

a complex function f is **analytic** in a domain D



a complex function f is an **entire** function

the whole complex plane



Analytic Function Examples

A complex function can be differentiable *at a point* z_0 but differentiable *nowhere else* → not an **analytic** function

$$f(z) = |z|^2 = z\bar{z} = x^2 + y^2$$

$$f'(0) = \lim_{\Delta z \to 0} \frac{(\Delta x)^2 + (\Delta y)^2}{\Delta x + i\Delta y} = 0$$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - x^2 - y^2}{\Delta x + i\Delta y}$$

$$\begin{cases} \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} & (\Delta y = 0) \\ \lim_{\Delta y \to 0} \frac{(y+\Delta y)^2 - y^2}{i\Delta y} & (\Delta x = 0) \end{cases}$$

differentiable *at zero but differentiable nowhere else*

~~**analytic**~~

$$f(z) = z^2$$

differentiable *everywhere in the complex plane*

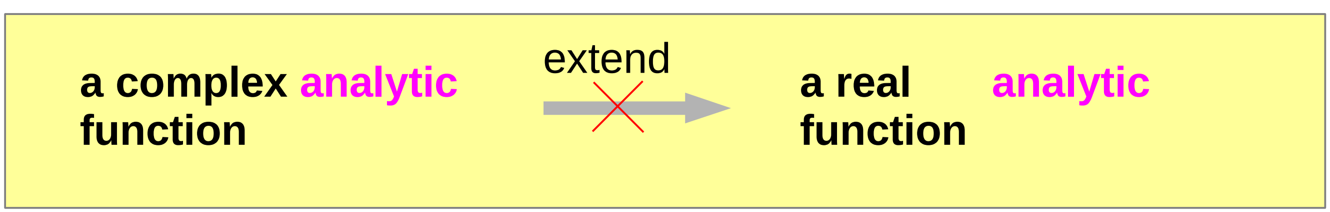
analytic *everywhere* → **entire function**

Extending Complex Analytic Functions

If a **complex** analytic **function** is defined in an open ball around a point x_0, its power series expansion at x_0 is convergent in the whole ball
(analyticity of holomorphic functions).

The corresponding statement for **real** analytic **functions** (with open interval of the real line) is not true in general;

an example for $x_0 = 1$ and a ball of radius exceeding 1, since the power series $f(x) = 1 - x^2 + x^4 - x^6 \dots$ diverges for $|x| > 1$.



The Radius of Convergence ≤ 1

$$f(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

Differentiable everywhere in the real line

$f(x) \Leftrightarrow \dfrac{1}{1+x^2}$ Converge when $|x| < 1$

Differentiable at 0 and at every point in an open set (-1, +1)

Differentiable everywhere

$$f(z) = 1 - z^2 + z^4 - z^6 + \cdots + (-1)^n z^{2n} + \cdots$$

entire function

Differentiable everywhere in the complex plane

$f(z) \Leftrightarrow \dfrac{1}{1+z^2}$ Converge when $|z| < 1$

rational function

Not differentiable at +i, –i

Differentiable at 0 and at every point in an open set a ball of radius ≤ 1

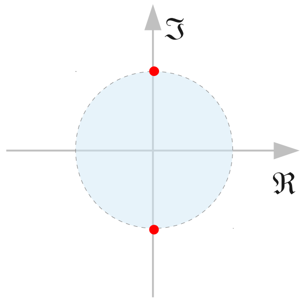
$$|z| < R, \quad R \le 1$$

The Radius of Convergence > 1

$$f(z) = 1 - z^2 + z^4 - z^6 + \cdots + (-1)^n z^{2n} + \cdots$$

complex

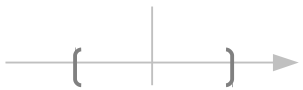
$f(z)$ converges to $\dfrac{1}{1+z^2}$ for $|z| < 1$



$$f(x) = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

real

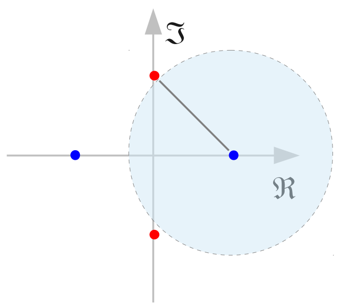
$f(x)$ converges to $\dfrac{1}{1+x^2}$ for $|x| < 1$



$$f(z) = \sum_{n=0}^{\infty} a_i (z-1)^i$$

complex

$f(z)$ converges to $\dfrac{1}{1+z^2}$ for $|z-1| < \sqrt{2}$

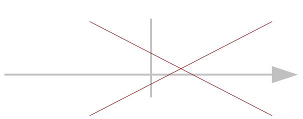


the radius of convergence

The distance from the center to the nearest isolated singularity

$|z-1| < \sqrt{2}$

➡ No corresponding case for real analytic function

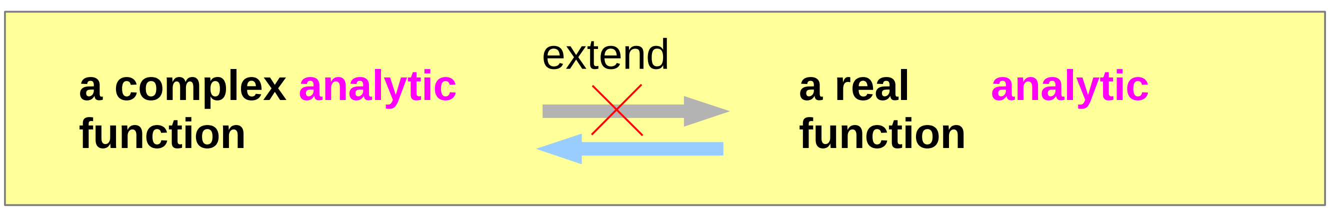


Extending Real Analytic Functions

Any **real analytic function** on **some open set** on the real line can be extended to a **complex analytic function** on **some open set** of the complex plane.

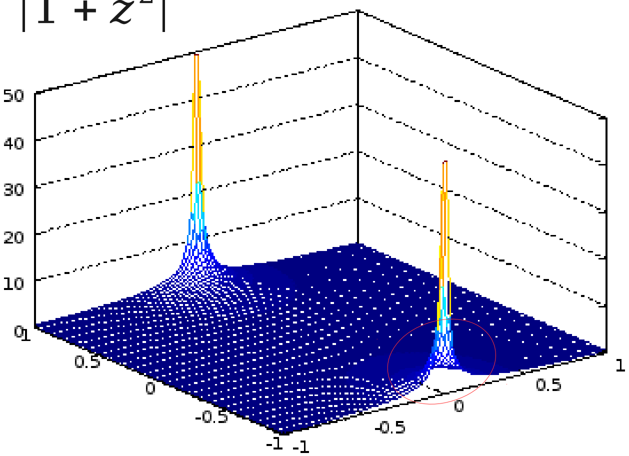
However, not every **real analytic function** defined on the **whole** real line can be extended to a **complex analytic** defined on the **whole** complex plane.

$f(x) = 1 - x^2 + x^4 - x$.. is a counterexample,
as it is not defined for $x = \pm i$.

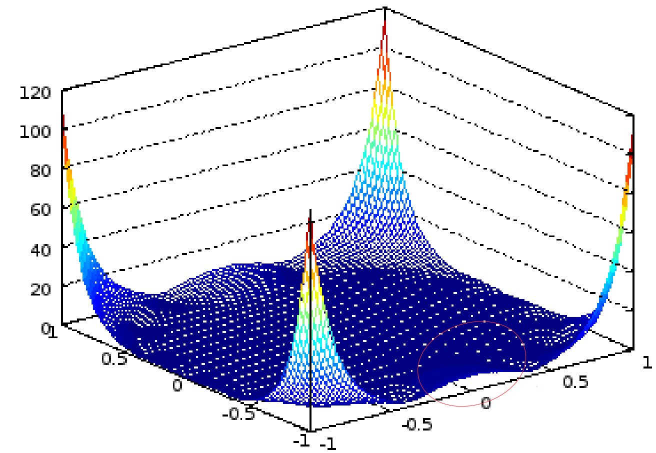


Examples of Complex Analytic Functions

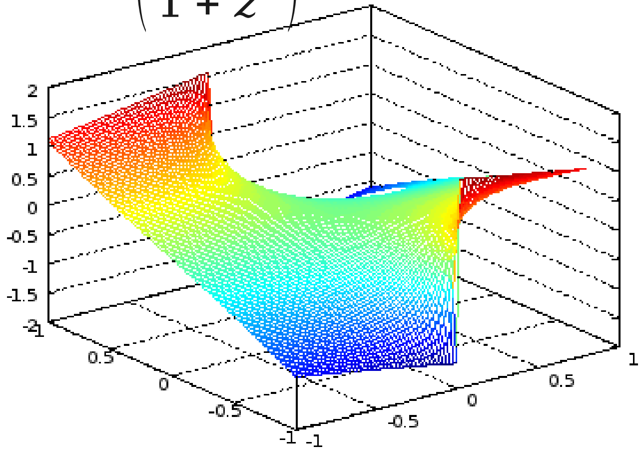
$$\left|\frac{1}{1+z^2}\right|$$



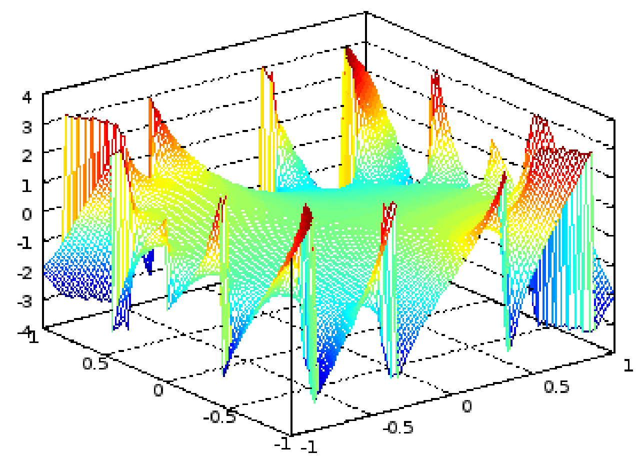
$$\left|1-z^2+z^4-z^6+z^8-z^{10}+z^{12}-z^{14}\right|$$



$$arg\left(\frac{1}{1+z^2}\right)$$



$$arg\left(1-z^2+z^4-z^6+z^8-z^{10}+z^{12}-z^{14}\right)$$



Analyticity and Differentiability

Any real analytic function ⟹ Infinitely differentiable / Smooth

Any real analytic function ⟸̸ Infinitely differentiable / Smooth

There exist a non-analytic smooth function

Any complex analytic function ⟹ Infinitely differentiable / Smooth

Any complex analytic function ⟸ Infinitely differentiable / Smooth

Any complex analytic function ⟹ Any differentiable function in an open set

Any complex analytic function ⟸ Any differentiable function in an open set

Any complex analytic function ≡ Any complex holomorphic function

Other Definitions of Analyticity

A function **f(z)** is **analytic** (or **regular** or **holomorphic** or **monogenic**) in a **region** if it has a (unique) derivative at every point of the region.

The statement **f(z)** is **analytic** at a point $z=z_0$ means that **f(z)** has a derivative at every point inside some small circle about $z=z_0$.

Isolated points and curves are not regions; a region must be two dimensional

M. L. Boas, “Mathematical methods in the physical sciences”

A function **f(z)** is said to be **analytic** in a domain D if **f(z)** is defined and differentiable at all points of D.

The function f(z) is said to be **analytic** at a point z=z0 in D if **f(z)** is **analytic** in a neighborhood of z0.

Also, by an **analytic** function we mean a function that is analytic in some domain

E. Kreyszig, “Advanced Engineering Mathematics”

General Definitions of Analyticity

A function that is locally given by a **convergent power series**.

A function is **analytic** if and only if its **Taylor series** about x_0 *converges* to the function in some neighborhood for every x_0 in its domain.

real analytic functions ➡ infinitely differentiable

complex analytic functions ➡ infinitely differentiable

Complex analytic functions exhibit properties that do not hold generally for real analytic functions.

Complex Analytic Functions

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

$$\Delta f = f(z+\Delta z) - f(z)$$
$$\Delta z = \Delta x + i\Delta y$$

complex differentiable

$f(z)$: **analytic** in a region ⟺ $f(z)$ has a (unique) derivative at every point of the region

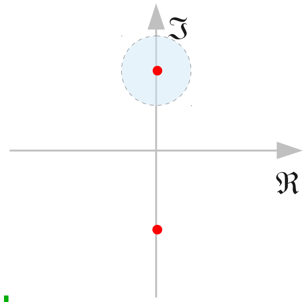
$f(z)$: **analytic** at a point $z = a$ ⟺ $f(z)$ has a (unique) derivative at every point of some small circle about $z = a$

Singular Point

Regular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is analytic

Singular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is not analytic

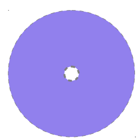
Isolated Singular point of $f(z)$ $\triangleq$ a point at which $f(z)$ is analytic everywhere else inside some small circle about the singular point



Isolated Singularity

Isolated Singularity of $f(z)$
z=z0

$\triangleq$ If z=z0 has a neighborhood *without further singularities* of f(z)



$\triangleq$ There exists some *deleted neighborhood* or *punctured open disk* of z0 throughout which f(z) is analytic

$$0 < |z - z_0| < R$$

$\tan(z)$	$\pm\pi/2,\ \pm 3\pi/2,\ \cdots$	**Isolated Singularity**
$\tan(1/z)$	0	**Non-isolated Singularity**

Non-isolated Singularity

Cluster points: limit points of isolated singularities. If they are all poles, despite admitting Laurent series expansions on each of them, no such expansion is possible at its limit

$$f(z) = \tan(1/z)$$

simple poles $\quad z_n = \frac{1}{(\pi/2 + n\pi)}$

$$\lim_{n \to 0} z_n = 0$$

Every punctured disk centered at 0 has an infinite number of singularities. No Laurent expansion

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Natural boundaries: non-isolated set (e.g. a curve) which functions can not be analytically continued around (or outside them if they are closed curves in the Riemann sphere).

$$f(z) = Ln\ z$$

the branch point 0

and the negative axis

Every neighborhood of z0 contains at least one singularity of f(z) other than z0

Being analytic means (1)

$$f(z) = u(x,y) + i\,v(x,y)$$

: **differentiable** at a point

$$z = x + iy$$

$$\frac{\partial u}{\partial x} = + \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y}$$

Δz can approach zero from any convenient direction z_0

a unique derivative

Necessary condition for analyticity

$$f(z) = u(x,y) + i\,v(x,y) \qquad f(z) = u(x,y) + i\,v(x,y)$$

Being analytic means (2)

$f(z) = u(x,y) + i v(x,y)$: **differentiable** at a point $z = x + iy$

➡ $f'(z)$ exists

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \qquad \Delta z = \Delta x + i \Delta y$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta z}$$

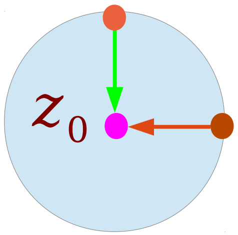
horizontal approach

$\Delta z \to 0$ ➡ $\Delta x \to 0$, $\Delta y = 0$

vertical approach

$\Delta z \to 0$ ➡ $\Delta y \to 0$, $\Delta x = 0$

must have the same $f'(z)$



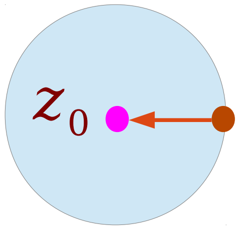
Being analytic means (3)

horizontal approach $\Delta z \to 0 \Rightarrow \Delta x \to 0 \quad \Delta y = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad = \frac{\partial f}{\partial x}$$

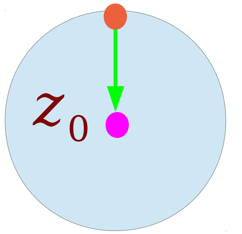


vertical approach $\Delta z \to 0 \Rightarrow \Delta y \to 0 \quad \Delta x = 0$

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x+\Delta x, y+\Delta y) + i v(x+\Delta x, y+\Delta y) - u(x,y) - i v(x,y)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

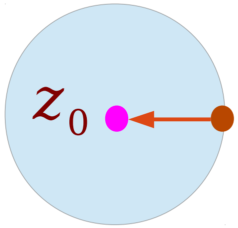
$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad = -i \frac{\partial f}{\partial y}$$



Being analytic means (4)

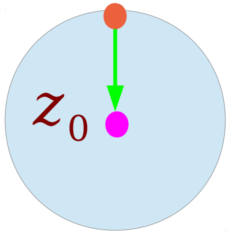
horizontal approach $\Delta z \to 0 \Rightarrow \Delta x \to 0 \quad \Delta y = 0$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x}$$



vertical approach $\Delta z \to 0 \Rightarrow \Delta y \to 0 \quad \Delta x = 0$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial f}{\partial y}$$



$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \Rightarrow$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Cauchy-Riemann equations

Being analytic means (5)

$f(z) = u(x,y) + i\,v(x,y)$: **analytic** in a region R

derivatives of all orders at points inside region

$f'(z_0),\ f''(z_0),\ f^{(3)}(z_0),\ f^{(4)}(z_0),\ f^{(5)}(z_0),\ \cdots$

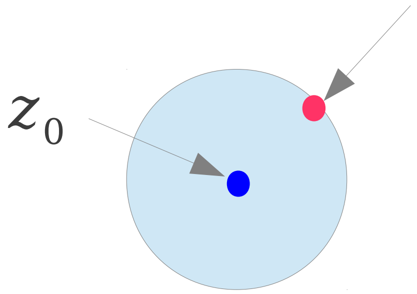
Infinitely differentiable

Smooth

Taylor series expansion about any point z_0 inside the region

The power series **converges** inside the circle about z_0

This circle extends to the nearest singular point



A function that is locally given by a **convergent power series**.

Being analytic means (6)

$f(z) = u(x,y) + i\,v(x,y)$: **analytic** in a region R

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

satisfy **Laplace's equation** in the region

harmonic functions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

satisfy **Laplace's equation** in
simply connected region

$u(x,y),\ v(x,y)$

real and **imaginary** part of
an **analytic** function $f(z)$

The necessary and sufficient conditions

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a domain D

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$f(z) = u(x,y) + i v(x,y)$: **analytic** in a domain D

$\Leftarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$u(x,y),\ v(x,y)$: **continuous** on in a domain D

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$: **continuous** on in a domain D

To Be Analytic (1)

if the real functions u(x,y) and v(x,y) are **continuous**

and have **continuous** first order partial derivatives

in a neighborhood of z,

and if u(x,y) and v(x,y) satisfy

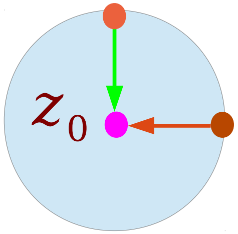
the **Cauchy-Riemann equations** at the point z,

then the complex function f(z) = u(x,y) + iv(x,y)

is **differentiable** at z

and f'(z) is as belows.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$



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