Inversion Integration (H.1)

20160420

Based on

1. Transforms and Application Primer
   for Engineers with Examples and MATLAB

   Alexander D. Poularikas

2. Transforms and Application Handbook

   Alexander D. Poularikas

AB : arc of a circle $|z| = R$ $\theta_1 \leqslant \theta \leqslant \theta_2$

$$\lim_{R \to \infty} z\,W(z) = k$$

$$\lim_{R \to \infty} \int_{AB} W(z)\,dz = jk(\theta_2 - \theta_1)$$

AB : arc of a circle $|z - z_0| = r$ $\theta_1 \leqslant \theta \leqslant \theta_2$

$$\lim_{R \to \infty} (z - z_0)\, W(z) = k$$

$$\lim_{R \to \infty} \int_{AB} W(z)\, dz = jk(\theta_2 - \theta_1)$$

max value of $w(z)$ along a path C $= M$

max value of the integral of $w(z)$ along a path C $= Ml$

where the length of a path $C = l$

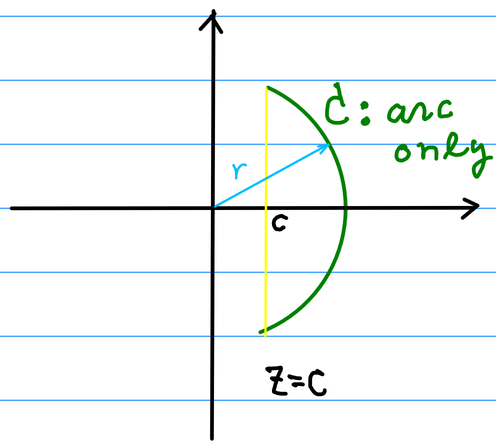
$$\left| \int_c w(z)\, dz \right| \leqslant Ml$$

Jordan's Lemma

If $\quad t < 0$

$\quad$ as $z \to \infty \qquad f(z) \to 0$

Then $\quad$ as $r \to \infty \qquad \int_C e^{tz} f(z)\, dz \to 0$



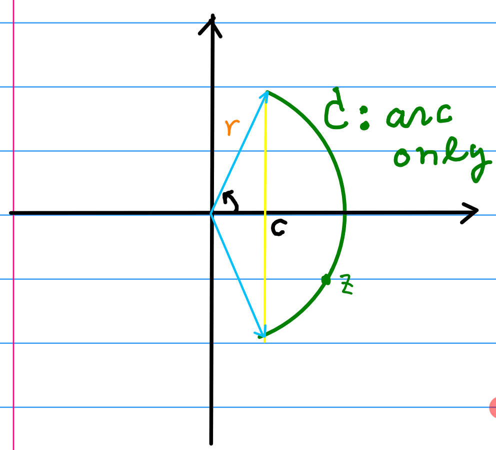
$\boxed{t < 0} \quad \boxed{c > 0}$

$$\boxed{\lim_{z \to \infty} f(z) = 0}$$

$$\boxed{\lim_{z \to \infty} e^{tz} = 0}$$

$$\Rightarrow \quad \boxed{\lim_{r \to \infty} \int_C e^{tz} f(z)\, dz = 0}$$

Proof (I) $\boxed{t<0}$ $\boxed{c>0}$



$c > 0 \rightarrow \quad 0 < \arg(z) < \pi$

$\text{length}(C) \leq \pi$

$$\boxed{\begin{array}{l} |f(z)| < \varepsilon \quad \text{for} \quad |z| > r_0 \\ z = re^{j\theta} = r(\cos\theta + j\sin\theta) \end{array}}$$

$$\boxed{\left| \int_C e^{tz} f(z)\, dz \right| < \frac{\pi\varepsilon}{|t|}}$$

$$\left| \int_C e^{tz} f(z)\, dz \right| = \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{tr(\cos\theta + j\sin\theta)} f(re^{j\theta})\, jre^{j\theta}\, d\theta \right|$$

$$\leq \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{tr\cos\theta} e^{jtr\sin\theta}\, \varepsilon r\, je^{j\theta}\, d\theta \right|$$

$$\leq \varepsilon r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{tr\cos\theta}| \cdot |e^{jtr\sin\theta}| \cdot |je^{j\theta}|\, d\theta$$

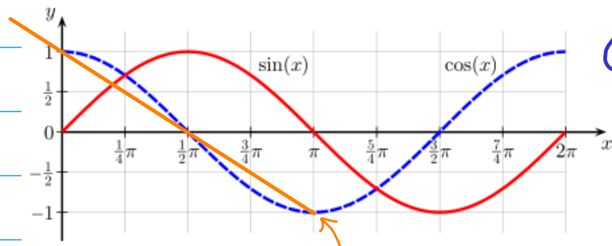
$$\leq \varepsilon r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{tr\cos\theta}\, d\theta \leq 2\varepsilon r \int_0^{\frac{\pi}{2}} e^{tr(1-2\theta/\pi)}\, d\theta$$

$$= 2\varepsilon r \left[-\frac{\pi}{2tr} e^{tr(1-2\theta/\pi)} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\varepsilon r \pi}{|t| r} (1 - e^{rt}) < \frac{\pi\varepsilon}{|t|}$$

$$\left| \int_C e^{tz} f(z)\, dz \right| \leq \varepsilon r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{tr\cos\theta}| \cdot |e^{jtr\sin\theta}| \cdot |je^{j\theta}|\, d\theta < \frac{\pi\varepsilon}{|t|}$$

wikipedia.org



$\cos(\theta)$

Slope $= -\frac{2}{\pi}$

y-intercept $= 1$

$y = -\frac{2}{\pi}\theta + 1$

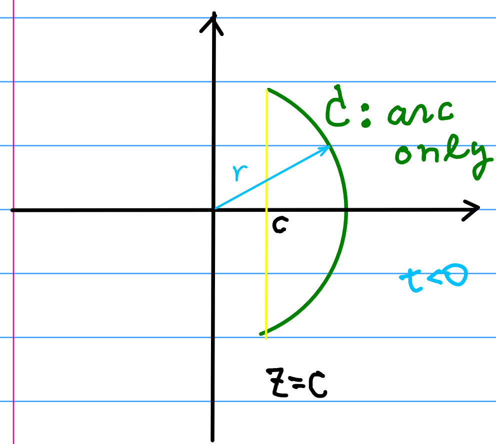
$$0 < \theta < \frac{\pi}{2} \rightarrow \quad 1 - \frac{2}{\pi}\theta \leq \cos(\theta)$$

$$t\left(1 - \frac{2}{\pi}\theta\right) \geq t\cos(\theta) \qquad \boxed{t<0}$$

$$\varepsilon r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{tr\cos\theta}\, d\theta \leq 2\varepsilon r \int_0^{\frac{\pi}{2}} e^{tr(1-2\theta/\pi)}\, d\theta$$

$$= 2\varepsilon r \left[-\frac{\pi}{2tr} e^{tr(1-2\theta/\pi)} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\varepsilon r \pi}{|t| r}\left(1 - e^{rt}\right) < \frac{\pi\varepsilon}{|t|}$$



$\boxed{t<0}$ $\boxed{c>0}$

$$\boxed{\lim_{z\to\infty} f(z) = 0}$$

$$\boxed{\lim_{r\to\infty} \int_C e^{tz} f(z)\, dz = 0}$$

$$\boxed{\lim_{z\to\infty} e^{tz} = 0}$$

$|f(z)| < \varepsilon$ for $|z| > r_0$

$z = re^{j\theta} = r(\cos\theta + j\sin\theta)$

$$\left| \int_C e^{tz} f(z)\, dz \right| < \varepsilon r \pi$$

Jordan's Lemma summary

$t < 0$

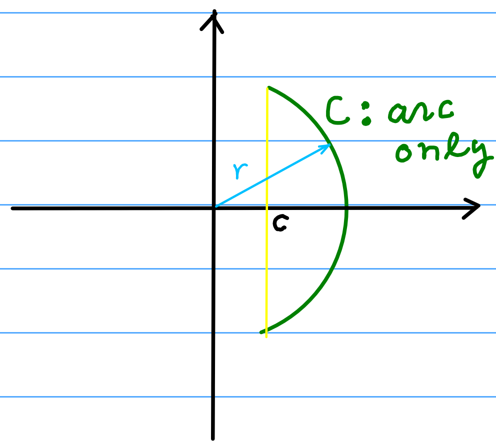
as $z \to \infty$ $\quad f(z) \to 0$

as $r \to \infty$ $\quad \int_C e^{tz} f(z)\, dz \to 0$

$t < 0$

$$\lim_{z \to \infty} f(z) = 0$$

$$\lim_{r \to \infty} \int_C e^{tz} f(z)\, dz = 0$$



$$t<0 \quad \lim_{z \to \infty} f(z) = 0 \;\Rightarrow\; \lim_{r \to \infty} \int_C e^{tz} f(z)\, dz = 0$$

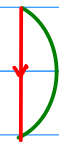
$$\lim_{r \to \infty} \int_{D} e^{tz} f(z)\, dz = \lim_{r \to \infty} \left[\int_{|} e^{tz} f(z)\, dz + \underbrace{\int_{)} e^{tz} f(z)\, dz}_{0} \right]$$

Bromwich contour

arc only

$$\lim_{r \to \infty} \int e^{tz} f(z)\, dz = -2\pi j \sum_k \mathrm{Res}_k$$

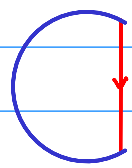
$\boxed{t < 0}$



arc to the right

$$\boxed{\lim_{r\to\infty} \int_{\downarrow} e^{tz} f(z)\, dz = -2\pi j \sum_k Res_k}$$

$\boxed{t > 0}$



arc to the left

$$\boxed{\lim_{r\to\infty} \int_{\uparrow} e^{tz} f(z)\, dz = +2\pi j \sum_k Res_k}$$

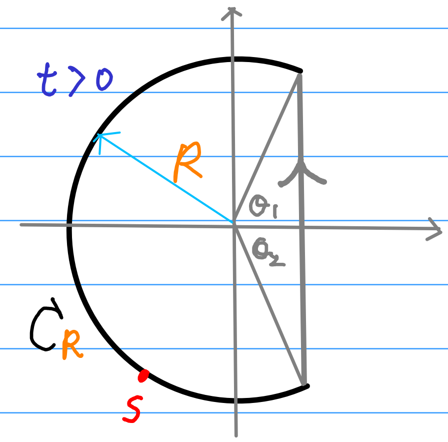
Proof II $\boxed{t > 0}$ $\boxed{c > 0}$

for s on C_R

$$\boxed{s = Re^{i\theta}} \quad \theta_1 \leq \theta \leq \theta_2$$

$$ds = iRe^{i\theta}d\theta$$

$$|ds| = R\,d\theta$$



For s on C_R,

suppose that $F(s)$ satisfies

$$\boxed{|F(s)| \leq \frac{M}{|s|^p} \quad \text{some } p > 0, \text{ all } R > R_0}$$

$$\boxed{\lim_{R\to\infty} \int e^{ts} F(s)\,ds = 0 \qquad (t > 0)}$$

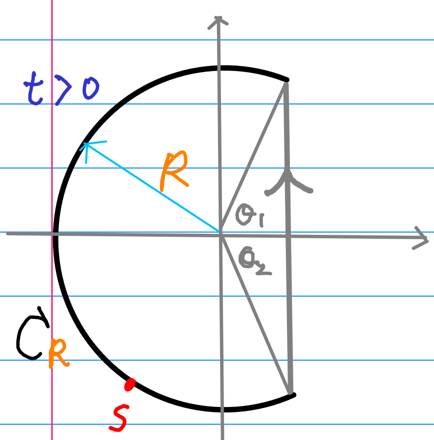
$$\boxed{|F(s)| \leqslant \frac{M}{|s|^p}}$$

$$\boxed{t > 0} \qquad \boxed{c > 0}$$

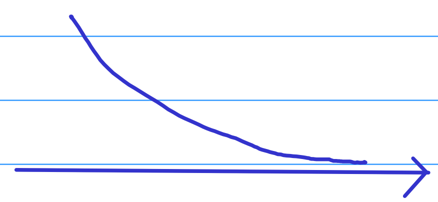
$$\boxed{\lim_{s \to \infty} F(s) = 0} \qquad \Rightarrow \qquad \boxed{\lim_{R \to \infty} \int_{C_R} e^{ts} f(s)\, ds = 0}$$

$$\boxed{\lim_{s \to \infty} e^{ts} \neq 0}$$

$$|e^{ts}| = e^{tR\cos\theta}$$



$$|F(s)| \leqslant \frac{M}{|s|^p} = M \cdot |s|^{-p}$$



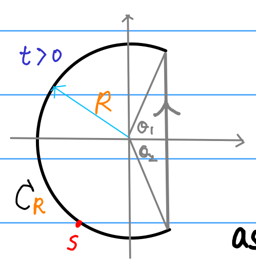
for sufficiently large R

all poles of $F(s)$ are included in the arc C_R

$\Rightarrow$ $F(s)$ continuous on C_R for all large R

$$\left|\int_{C_R} e^{st} F(s)\, ds\right| \leq \int_{C_R} |e^{st}||F(s)||ds|$$

$$\leq \int_{C_R} e^{tR\cos\theta} \frac{M}{|s|^p} |ds|$$



$\theta_1 \to \frac{\pi}{2}$

$\theta_2 \to \frac{3\pi}{2}$

as $R \to \infty$

$\boxed{s = Re^{i\theta}}$ $\theta_1 \leq \theta \leq \theta_2$

$ds = iRe^{i\theta} d\theta$

$|ds| = R\, d\theta$ $|s| = R$

$$\leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\theta} \frac{M}{R^p} R\, d\theta$$

$$\leq \frac{M}{R^{p-1}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\theta}\, d\theta$$

$\theta = \varphi + \frac{\pi}{2}$ $\varphi = \theta - \frac{\pi}{2}$

$\cos(\theta) = \cos(\varphi + \frac{\pi}{2}) = -\sin\varphi$

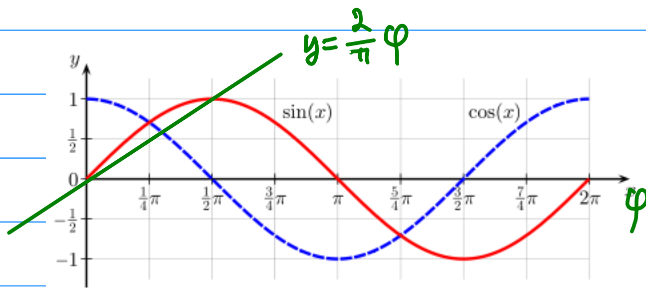
$$\leq \frac{M}{R^{p-1}} \int_0^{\pi} e^{-tR\sin\varphi}\, d\varphi$$

$$\leq \frac{2M}{R^{p-1}} \int_0^{\frac{\pi}{2}} e^{-tR\sin\varphi}\, d\varphi$$

$$\left|\int_{C_R} e^{st} F(s)\, ds\right| \leq \int_{C_R} |e^{st}|\,|F(s)|\,|ds|$$

$$\cdots$$

$$\leq \frac{2M}{R^{p-1}} \int_0^{\frac{\pi}{2}} e^{-tR\sin\varphi}\, d\varphi$$



$$\sin\varphi \geq \frac{2}{\pi}\varphi \qquad 0 \leq \varphi \leq \frac{\pi}{2}$$

$$\left|\int_{C_R} e^{st} F(s)\, ds\right| \leq \frac{2M}{R^{p-1}} \int_0^{\frac{\pi}{2}} e^{-tR\frac{2}{\pi}\varphi}\, d\varphi$$

$$\leq \frac{2M}{R^{p-1}} \frac{-1}{tR\frac{2}{\pi}} \left[e^{-tR\frac{2}{\pi}\varphi}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{M\pi}{R^{p}} \left(\frac{-1}{t}\right)\left(e^{-tR} - 1\right)$$

$$= \frac{M\pi}{R^{p}t} \left(1 - e^{-tR}\right)$$

$$\to 0 \qquad \text{as } R \to \infty$$

$$\left|\int_{C_R} e^{st} F(s)\, ds\right| \leq \int_{C_R} |e^{st}|\,|F(s)|\,|ds|$$

$$\leq \int_{C_R} e^{tR\cos\theta} \frac{M}{|s|^p} |ds|$$

$$\leq \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\theta} \frac{M}{R^p} R\, d\theta$$

$$\leq \frac{M}{R^{p-1}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\theta}\, d\theta$$

$$\leq \frac{M}{R^{p-1}} \int_0^{\pi} e^{-tR\sin\varphi}\, d\varphi$$

$$\leq \frac{2M}{R^{p-1}} \int_0^{\frac{\pi}{2}} e^{-tR\sin\varphi}\, d\varphi$$

$$\leq \frac{2M}{R^{p-1}} \frac{-1}{tR\frac{2}{\pi}} \left[e^{-tR\frac{2}{\pi}\varphi} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{M\pi}{R^p} \left(\frac{-1}{t}\right) \left(e^{-tR} - 1\right)$$

$$= \frac{M\pi}{R^p t} \left(1 - e^{-tR}\right)$$

$$\rightarrow 0 \qquad \text{as } R \to \infty$$

Melin I

θ = real & positive

$\phi(z)$ analytic in the strip $\alpha < x < \beta$ α, β : real

$\int_{x-j\infty}^{x+j\infty} |\phi(z)|\, dz = \int_{-\infty}^{+\infty} |\phi(x+iy)|\, dy$ converges

$\phi(z) \to 0$ uniformly as $|y| \to \infty$ in the strip $\alpha < x < \beta$

$$f(\theta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \theta^{-z} \phi(z)\, dz$$

$$\rightarrow \quad \phi(z) = \int_0^{\infty} \theta^{z-1} f(\theta)\, d\theta$$

Melin II

θ = real & positive

$\alpha < Re\{z\} < \beta$

$f(\theta)$: continuous / piecewise continuous

$$\phi(z) = \int_0^\infty \theta^{z-1} f(\theta)\, d\theta$$: absolutely convergent

$$\rightarrow \quad f(\theta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \theta^{-z} \phi(z)\, dz$$

$$\boxed{f(\theta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \theta^{-z} \phi(z)\, dz}$$

$$\boxed{\phi(z) = \int_0^{\infty} \theta^{z-1} f(\theta)\, d\theta}$$

Melin III

$$f(\theta) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \theta^{-z} \phi(z)\, dz$$

$$\phi(z) = \int_0^\infty \theta^{z-1} f(\theta)\, d\theta$$

$\theta = e^{-t}$, t : real

$z \to p$

$f(e^{-t}) \to g(t)$

$$g(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zt} \phi(z)\, dz$$

$$\phi(p) = \int_{-\infty}^{\infty} \theta^{-pt} g(t)\, dt$$

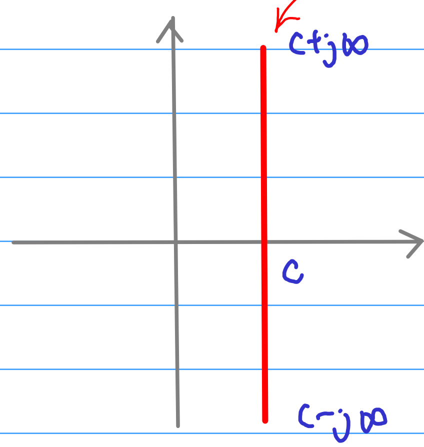
The Bromwich Contour

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zt} F(z)\, dz$$

$F(z)$ singularities are on the left of the path

$t > 0$ time

Bromwich Contour Br



Finite Number of Poles

$$F(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

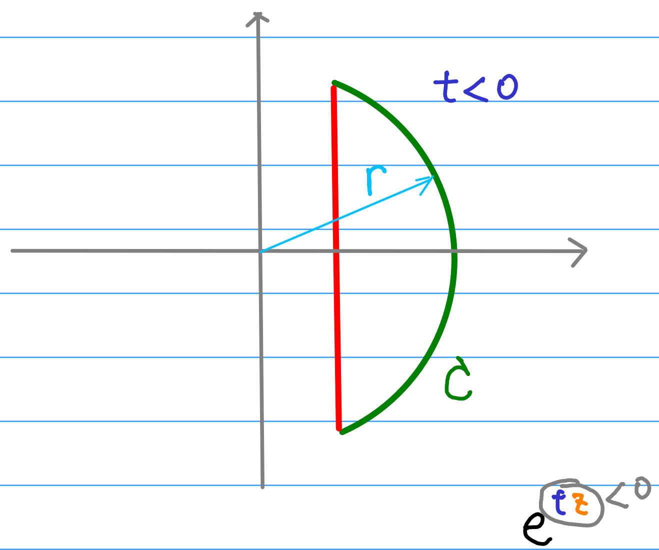
Assume

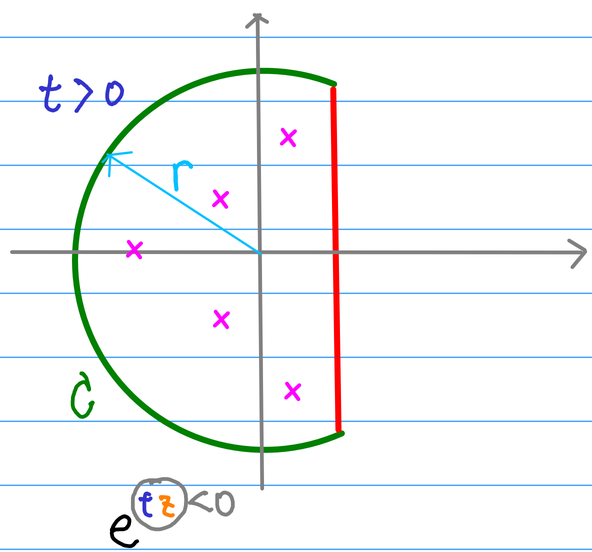
- $F(z)$ has n poles : $p_1, p_2, \cdots, p_n$

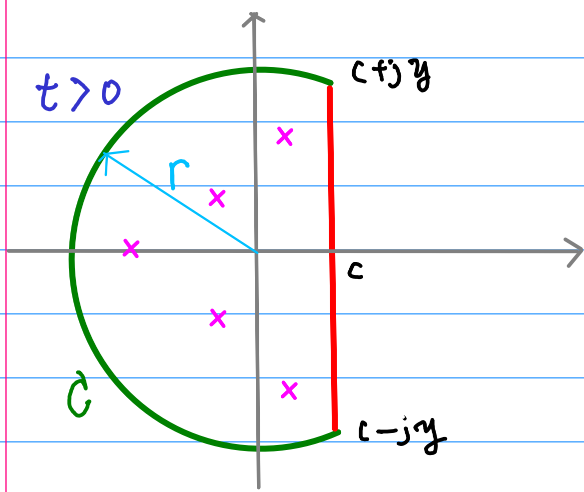
  Special case : rational function $\frac{N(z)}{D(z)}$

  $N(z)$, $D(z)$: polynomial of z

- all these n poles lies on the left of the Br contour

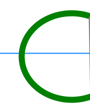




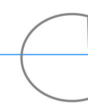


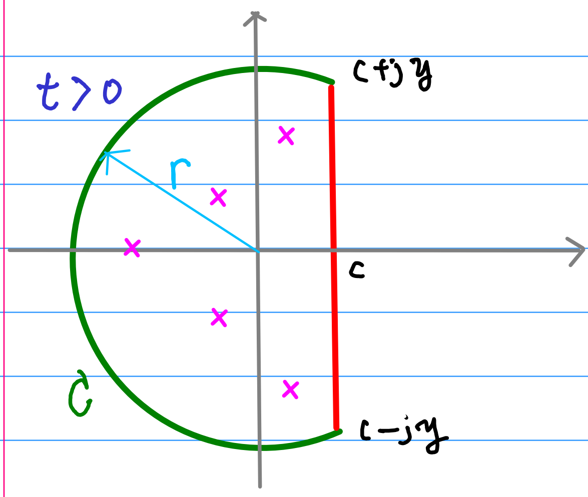
From Jordan's lemma

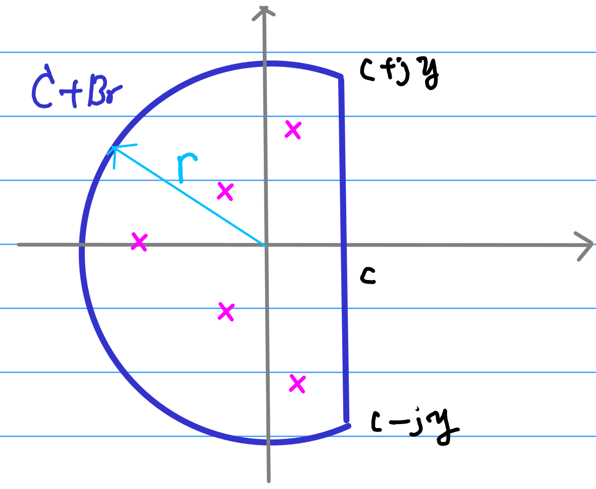
$$\boxed{\lim_{z \to 0} f(z) = 0} \Rightarrow \boxed{\lim_{r \to \infty} \int_C e^{tz} f(z)\, dz = 0}$$



$$\boxed{\lim_{y \to \infty} \int_{c-jy}^{c+jy} e^{tz} F(z)\, dz} = \boxed{\int_{Br} e^{tz} F(z)\, dz}$$





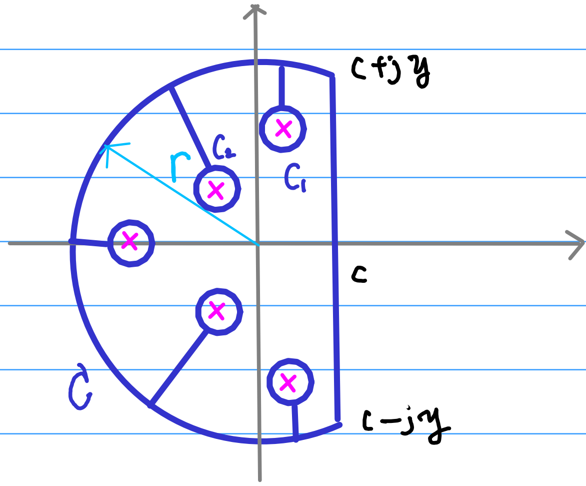


$$\nearrow 0$$

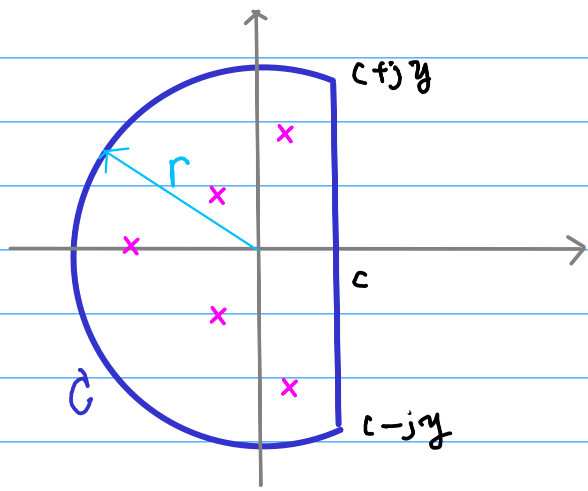
$$\boxed{\lim_{r\to\infty}\int_C e^{tz} f(z)\,dz = 0} + \boxed{\int_{Br} e^{tz} F(z)\,dz}$$

$$= \boxed{\lim_{r\to\infty}\int_{C+Br} e^{tz} f(z)\,dz}$$

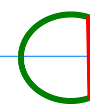
$$\boxed{\int_{Br} e^{tz} F(z)\,dz} = \boxed{\lim_{r\to\infty}\int_{C+Br} e^{tz} f(z)\,dz}$$

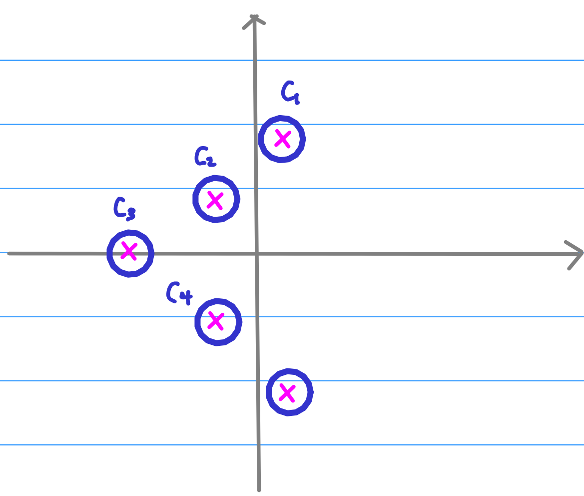


Using Cauchy's Theorem

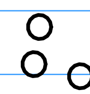


$$\boxed{\lim_{r\to\infty} \int_{c+\beta r} e^{tz} f(z)\, dz}$$





$$\boxed{\sum_{k=1}^{n} \int_{C_k} e^{tz} f(z)\, dz}$$



$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} e^{zt} F(z)\, dz$$

$$= \sum_{k=1}^{n} \int_{C_k} e^{tz} f(z)\, dz$$

$$= \sum_{k=1}^{n} \mathrm{Res}\left(e^{tz} f(z),\ z_k \right)$$ Residue Integral

Line Integration

$$\boxed{\int_C f(z)\, dz} = 2\pi j \sum_k \mathrm{Res}\left(f(z), p_k \right)$$

if $f(z)$ has a simple pole at $z = p$

$$\mathrm{Res}\left(f(z), p \right) = \lim_{z \to p} \boxed{(z-p)\, f(z)} \qquad \frac{K}{(z-p)},$$

if $f(z)$ has a pole of order k at $z = q$

$$\mathrm{Res}\left(f(z), q \right) = \lim_{z \to q} \boxed{\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \boxed{(z-q)^k f(z)}} \qquad \frac{A_{k-1}}{(z-q)},$$

$$f(t) = \frac{1}{2\pi j} \int_C e^{zt} F(z)\, dz$$

① for simple poles

$$\mathrm{Res}\left(e^{tz} f(z),\ z_k\right) = \left[e^{zt} F(z)(z - z_k)\right]\Big|_{z=z_k}$$

$$= \left[e^{zt} F_k(z)\right]\Big|_{z=z_k}$$

$$\boxed{f(t) = \sum_{k=1}^{n} \left[e^{zt}\ F_k(z)\right]\Big|_{z=z_k}} \qquad t > 0$$

$$\boxed{F_k(z) = F(z)(z - z_k)}$$

② for a multiple pole of $m+1$ multiplicity

$$\mathrm{Res}\left(e^{tz} f(z),\ z_k\right) = \left[\frac{1}{m!} \frac{d^m}{dz^m} \left(e^{zt} F(z)(z - z_k)^{m+1}\right)\right]\Big|_{z=z_k}$$

$$= \left[\frac{1}{m!} \frac{d^m}{dz^m} \left(e^{zt} F_k(z)\right)\right]\Big|_{z=z_k}$$

$$\boxed{f(t) = \sum_{k=1}^{n} \left[\frac{1}{m!} \frac{d^m}{dz^m} \left(e^{zt} F_k(z)\right)\right]\Big|_{z=z_k}} \qquad t > 0$$

$$\boxed{F_k(z) = F(z)(z - z_k)^{m+1}}$$

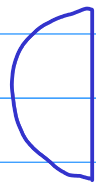
③ Infinitely many poles

if we can find circular arcs with radii $\to \infty$, such that

$$\lim_{z \to \infty} F(z) = 0 \quad \text{on} \quad C_n$$

$$\lim_{n \to \infty} \int_{C_n} F(z) e^{zt} \, dz = 0 \qquad t > 0 \qquad \text{(Jordan's Lemma)}$$

$$C_n' = C_n + \text{vertical line } \operatorname{Re} z = c \qquad \text{closed}$$



$$f(t) = \lim_{n \to \infty} \frac{1}{2\pi j} \int_{C_n'} e^{zt} F(z) \, dz \qquad t > 0$$

for simple poles $z_1, z_2, \cdots, z_n$ of $F(z)$

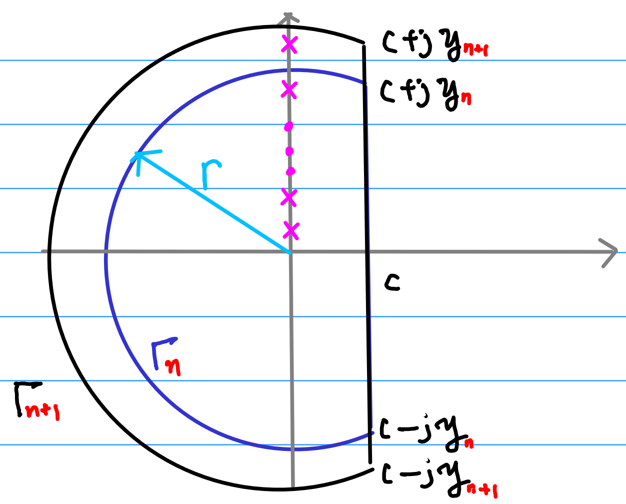
$$f(t) = \sum_{k=1}^{\infty} F_k(z_k) e^{z_k t}$$

$F(z)$ infinitely many poles $z_1, z_2, z_3 \cdots$
all to the left of the line $Re(z) = c > 0$

Without loss of generality
$|z_1| \leqslant |z_2| \leqslant |z_3| \cdots$

$|z_k| \to \infty$ as $k \to \infty$

Γ_1	arc enclosing	z_1
Γ_2	arc enclosing	z_1, z_2
Γ_3	arc enclosing	z_1, z_2, z_3
	. . .	
Γ_n	arc enclosing	$z_1, z_2, \ldots, z_n$
	. . .	

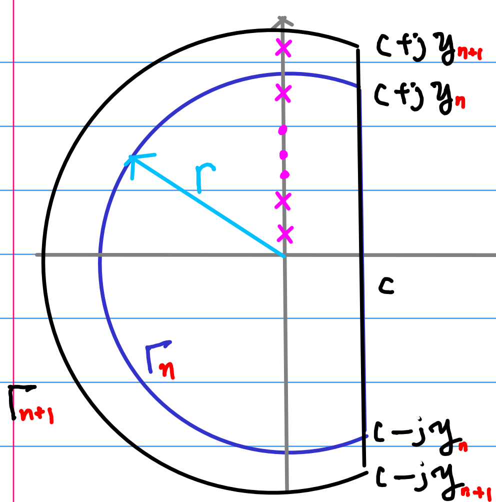


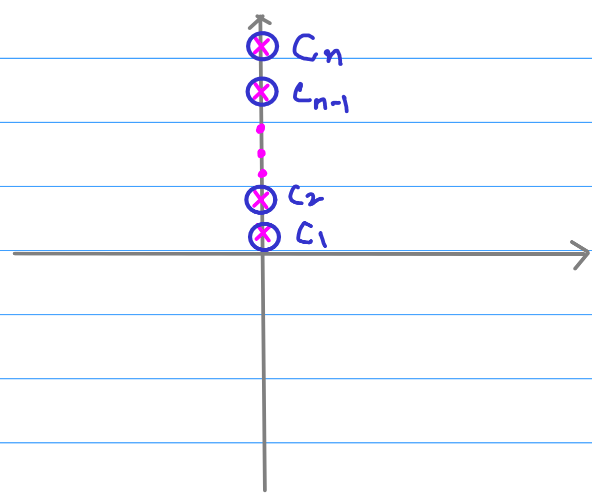
Γ_n arc enclosing $z_1, z_2, \cdots, z_n$

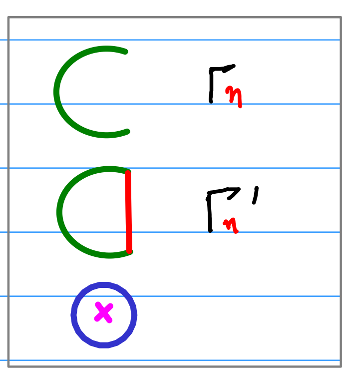
$[c-jy_n, c+jy_n]$

$\Gamma_n' = \Gamma_n + [c-jy_n, c+jy_n]$

C_k : circle enclosing z_k among n poles



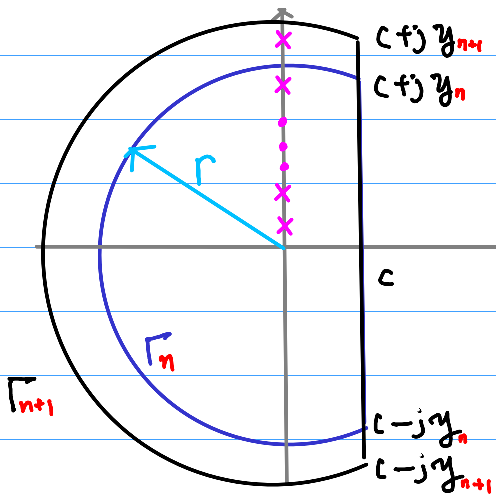


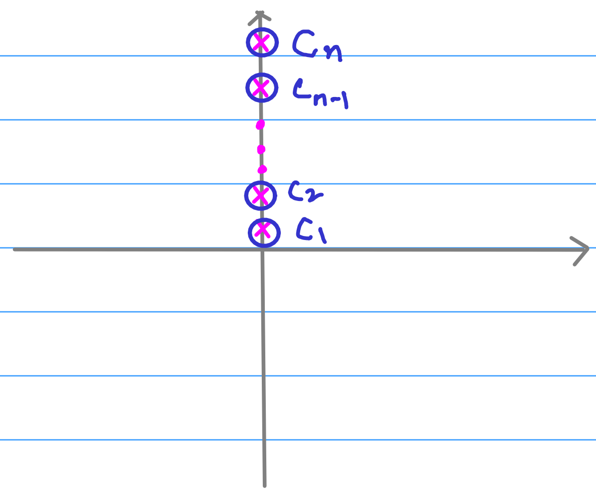


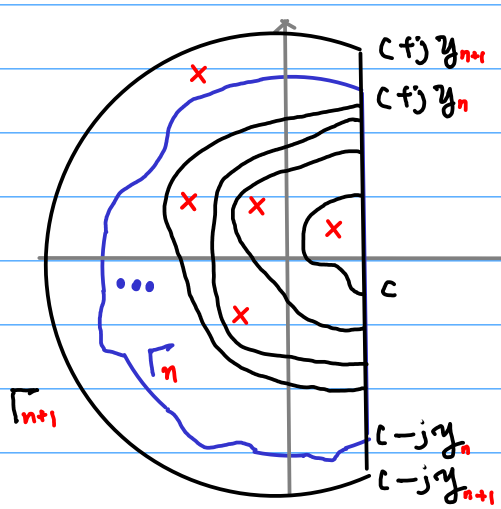
$$\frac{1}{2\pi j}\int_{\Gamma_n'} e^{zt} F(z)\, dz$$

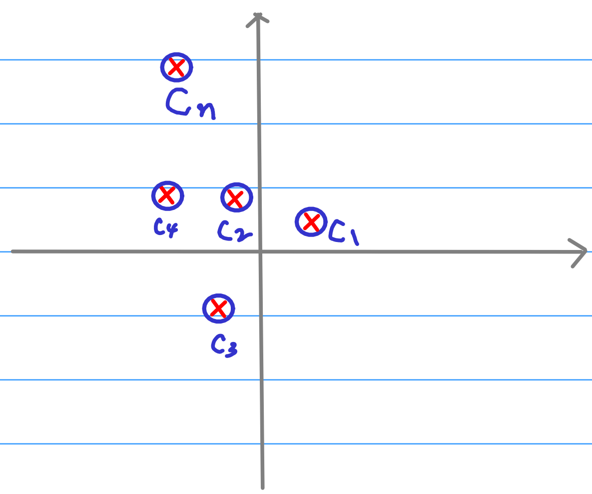
$$= \sum_{k=1}^{n} \int_{C_k} e^{tz} f(z)\, dz$$

$$= \sum_{k=1}^{n} \mathrm{Res}\left(e^{tz} f(z), z_k\right)$$


$c+jy_{n+1}$
$c+jy_n$
r
c
Γ_n
Γ_{n+1}
$c-jy_n$
$c-jy_{n+1}$


C_n
C_{n-1}
C_2
C_1


$c+jy_{n+1}$
$c+jy_n$
c
Γ_n
Γ_{n+1}
$c-jy_n$
$c-jy_{n+1}$


C_n
C_4
C_2
C_1
C_3

$$\boxed{\frac{1}{2\pi j}\int_{\Gamma_n'} e^{zt} F(z)\, dz = \sum_{k=1}^{n} \mathrm{Res}\left(e^{tz} f(z),\ z_k\right) = \frac{1}{2\pi j}\boxed{\int_{\Gamma_n} e^{zt} F(z)\, dz} + \frac{1}{2\pi j}\int_{c-jg_n}^{c+jg_n} e^{zt} F(z)\, dz}$$

if $\displaystyle \lim_{n\to\infty} \boxed{\int_{\Gamma_n} e^{zt} F(z)\, dz} = 0$

$$\lim_{n\to\infty} \frac{1}{2\pi j}\int_{c-jg_n}^{c+jg_n} e^{zt} F(z)\, dz = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Res}\left(e^{tz} f(z),\ z_k\right)$$

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} e^{zt} F(z)\, dz = \sum_{k=1}^{\infty} \mathrm{Res}\left(e^{tz} f(z),\ z_k\right)$$

$$f(t) = \lim_{n\to\infty} \frac{1}{2\pi j}\int_{\Gamma_n'} e^{zt} F(z)\, dz \qquad t > 0$$

for simple poles $z_1, z_2, \ldots, z_n$ of $F(z)$

$$f(t) = \sum_{k=1}^{\infty} F_k(z_k)\, e^{z_k t}$$

$$F_k(z) = F(z)(z - z_k)$$

$$F(s) = \frac{1}{s^2 + w^2}$$

$$f(t) = \frac{1}{2\pi j} \oint \frac{e^{st}}{(s+jw)(s-jw)} \, dt$$

$$\mathrm{Res}\left[(s-jw) \frac{e^{st}}{s^2+w^2} \right]_{s=+jw} = \left. \frac{e^{st}}{s+jw} \right|_{s=+jw} = \frac{e^{+jwt}}{2jw}$$

$$\mathrm{Res}\left[(s+jw) \frac{e^{st}}{s^2+w^2} \right]_{s=-jw} = \left. \frac{e^{st}}{s+jw} \right|_{s=-jw} = \frac{e^{-jwt}}{-2jw}$$

$$f(t) = \sum \mathrm{Res} = \frac{e^{+jwt} - e^{-jwt}}{2jw} = \frac{\sin wt}{w}$$

Deduction of the $\mathcal{L}^{-1}\{F(s)\}$ formula

$$\boxed{f(t) \Longleftrightarrow F(s)}$$

$F(s) = \mathcal{L}\{f(t)\}$
$f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$\boxed{\oint \frac{F(z)}{s-z}\, dz = j2\pi F(s)}$$

Cauchy 2nd Int formula

Bromwich Contour + Inverse Laplace Transform

$$\lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} F(z) \boxed{\mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\}} dz = j2\pi\, \mathcal{L}^{-1}\{F(s)\}$$

we know $\boxed{e^{zt} \Longleftrightarrow \frac{1}{s-z}}$ Laplace Transform pair

$$f(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} \boxed{e^{zt}} F(z)\, dz$$

$$= \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \boxed{e^{zt}} F(z)\, dz$$

$$f(t) = \frac{1}{2\pi j} \lim_{R \to \infty} \oint_{\Gamma_1} F(s) e^{st}\, ds$$

$f(z)$: **analytic** on and inside simple close curve C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw$$

the value of $f(z)$ at a point $z = a$ inside C

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw\right] \qquad f'(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{(w-z)^2} dw$$

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw\right] \qquad f''(z) = \frac{2}{2\pi i} \oint \frac{f(w)}{(w-z)^3} dw$$

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left[\frac{1}{2\pi i} \oint \frac{f(w)}{w-z} dw\right] \qquad f^{(3)}(z) = \frac{3!}{2\pi i} \oint \frac{f(w)}{(w-z)^4} dw$$

• • •

적분은 ⓦ에 대해서

미분은 ⓩ에 대해서

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$f(z)$ is *infinitely differentiable* in that neighborhood

$$\oint \frac{F(z)}{s-z} dz = j2\pi F(s)$$

$$\mathcal{L}^{-1}\left\{\oint \frac{F(z)}{s-z} dz\right\} = \mathcal{L}^{-1}\{j2\pi F(s)\}$$

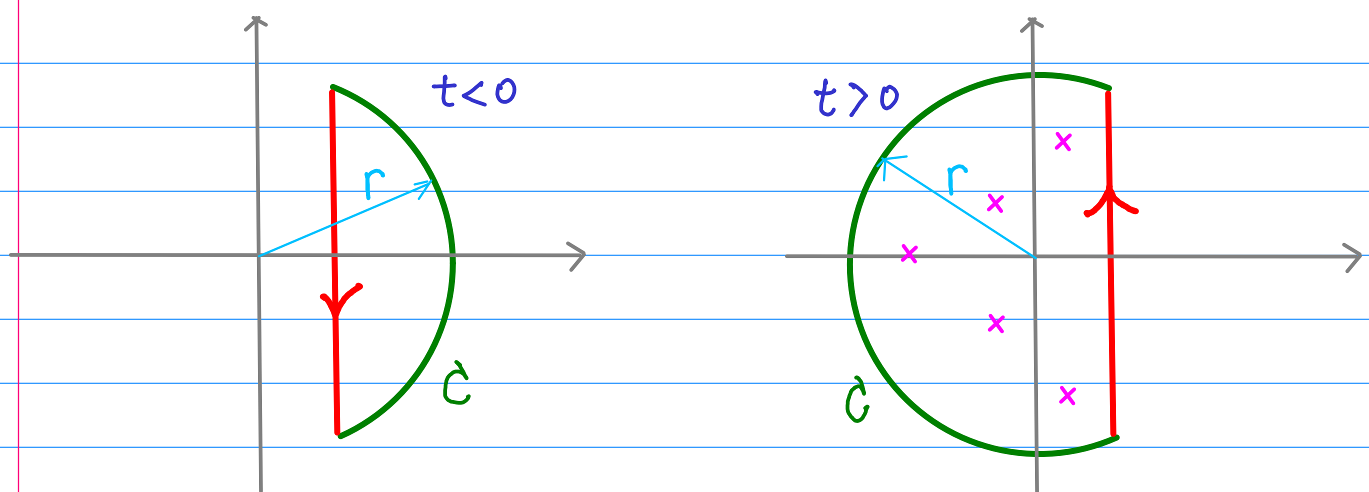
$$\oint F(z) \mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\} dz = j2\pi \mathcal{L}^{-1}\{F(s)\}$$

$$\lim_{\omega\to\infty} \int_{\sigma-j\omega}^{\sigma+j\omega} F(z) \mathcal{L}^{-1}\left\{\frac{1}{s-z}\right\} dz = j2\pi \mathcal{L}^{-1}\{F(s)\}$$

$$f(t) = \frac{1}{2\pi j} \lim_{R \to \infty} \oint_{\Gamma'} F(s) e^{st} \, ds$$

$$\boxed{\Gamma'} = \sum \text{ residues of } F(s)e^{st} \text{ at the poles to the left of } C \quad (t > 0)$$

$$\boxed{\Gamma'} = -\sum \text{ residues of } F(s)e^{st} \text{ at the poles to the right of } C \quad (t < 0)$$



$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$h(n) = \frac{1}{2\pi j} \oint H(z) z^{n-1} \, dz$$

$$= \sum_{k} \mathrm{Res}\{ H(z) z^{n-1}, z_k \}$$

One-Sided z-transform

$$F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$f(nT) = \frac{1}{2\pi j} \oint_c F(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\{ F(z)\, z^{n-1}, z_k \}$$

① Simple poles

$$F(z) = \frac{H(z)}{G(z)}$$

$$\lim_{z \to a} (z-a)\, F(z)\, z^{n-1} = \lim_{z \to a} \left[(z-a) \frac{H(z)}{G(z)}\, z^{n-1} \right]$$

② Multiple poles

$$\frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m\, F(z)\, z^{n-1} \right]$$

Two-sided z-transform

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT)\, z^{-n}$$

$$f(nT) = \frac{1}{2\pi j} \oint_c F(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\{F(z)\, z^{n-1}, z_k\}$$

① Simple poles

$$F(z) = \frac{H(z)}{G(z)}$$

$$\lim_{z \to a} (z-a) F(z)\, z^{n-1} = \lim_{z \to a} \left[(z-a) \frac{H(z)}{G(z)}\, z^{n-1} \right]$$

② Multiple poles

$$\frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m F(z)\, z^{n-1} \right]$$

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

$$= C_0 + C_1 z^{-1} + \cdots + C_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D(z)}$$

A proper function $(M < N)$

$$F(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \qquad a_N \neq 0 \qquad M < N$$

$$= \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M-1}}{z^N + a_1 z^{N-1} + \cdots + a_N}$$