Laurent Series and z-Transform
- Geometric Series
Time Shift A

20181002 Tue

Causal Signal $a(n) \Rightarrow f(z),\quad X(z)$

$(n \geq 0) \qquad (|z| < p) \qquad (|z| > p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \Rightarrow \frac{2}{2 - z} \quad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \Rightarrow \frac{z}{z - 0.5} \quad (|z| > 0.5)$$

Anti-Causal Signal $\quad a(n) \Rightarrow -f(z),\ -X(z)$

$(n<0) \qquad (|z|>p) \qquad (|z|<p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \qquad (n<0)$$

$$f_2(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$X_2(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1-(2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \qquad (n<0)$$

$$f_2(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \rightarrow \frac{2}{z-2} = -f(z) \qquad (|z|>2)$$

$$X_2(z) = \frac{(2z)}{1-(2z)} \rightarrow \frac{z}{0.5-z} = -X(z) \qquad (|z|<0.5)$$

$$a'_n = -\left(\frac{1}{2}\right)^n \qquad (n<0)$$

$$f(z) = \frac{2}{2-z} \rightarrow \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \qquad (|z|>2)$$

$$X(z) = \frac{z}{z-0.5} \rightarrow \frac{-(2z)}{1-(2z)} \qquad (|z|<0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $ROC(z) \leftarrow ROC(z^{-1})$

Causal $\quad z^{-1} \quad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \quad \longleftrightarrow \quad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \longleftrightarrow \quad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}| < 2) \longrightarrow f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \longrightarrow X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal) with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal) with the same formula as causal f(z)

Causal $\qquad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

Inverse z

$f(z^{-1})$, $ROC(z^{-1}) \Rightarrow a_{-n}$

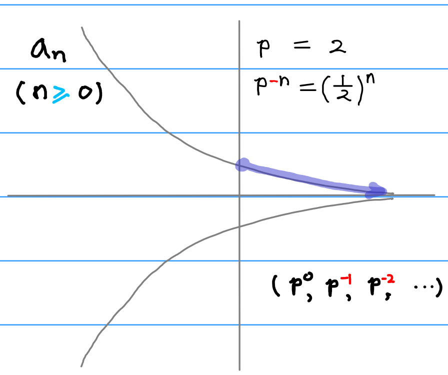
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z^{-1}) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$

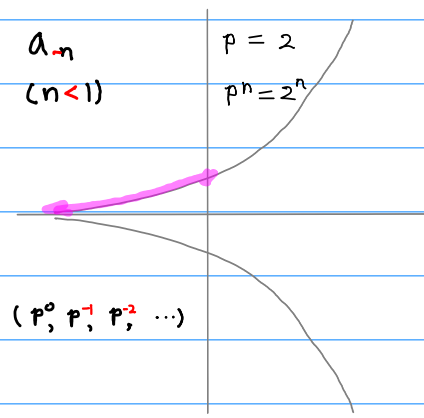
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, -1, -2, \cdots$$





Inverse z $\quad X(z^{-1}),\ ROC(z^{-1}) \Rightarrow a_{-n}$

Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z^{-1}) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$
$\boxed{X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)}$	$\boxed{X(z^{-1}) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)}$

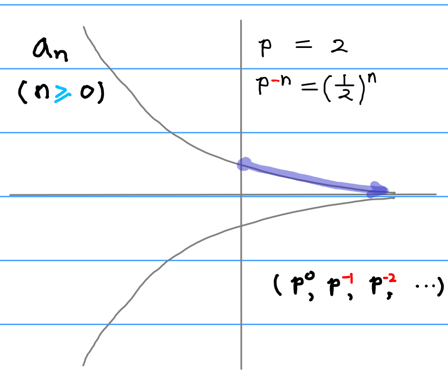
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

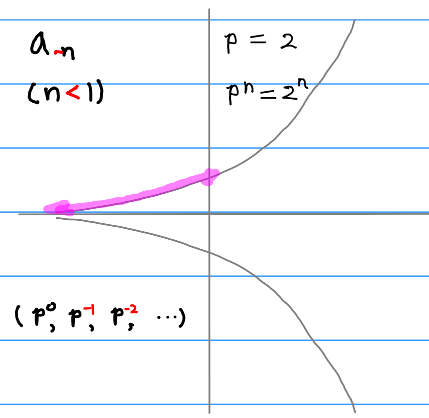
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$ROC(z) \leftarrow ROC(z^{-1})$

Causal → anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \quad \rightarrow \quad -f(z) = -\frac{2}{2-z} \quad (|z| > 2)$$

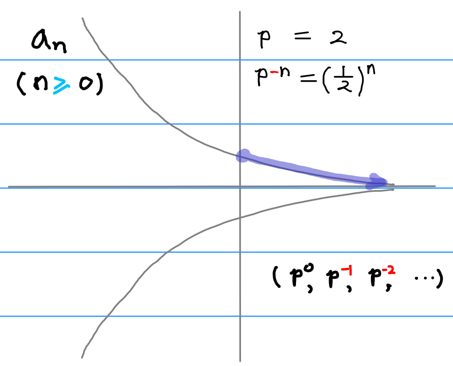
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \rightarrow \quad -X(z) = -\frac{z}{z-0.5} \quad (|z| < 0.5)$$

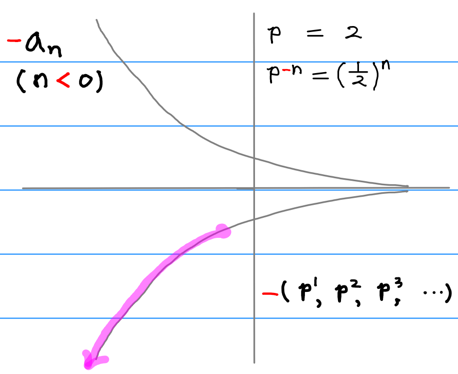
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$-f(z) = \frac{2}{z-2} = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots \qquad a_n = -\left(\frac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $f(z)$, $ROC(z^{-1}) \Rightarrow -a_n$

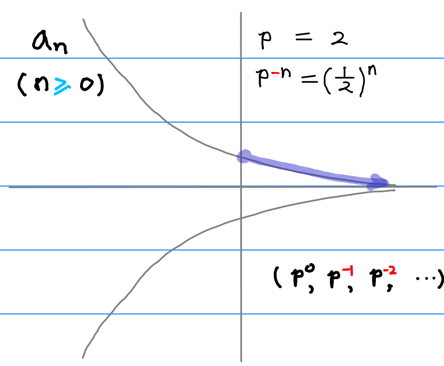
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

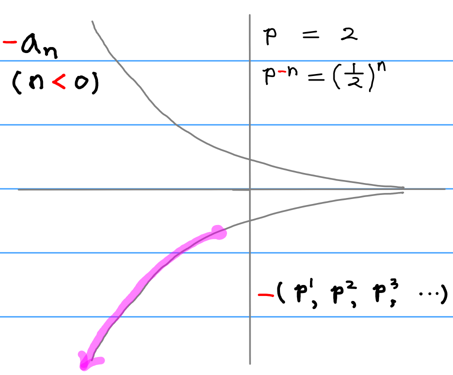
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad \therefore a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \qquad \therefore -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





Inverse ROC

$X(z)$, $ROC(z^{-1}) \Rightarrow -a_n$

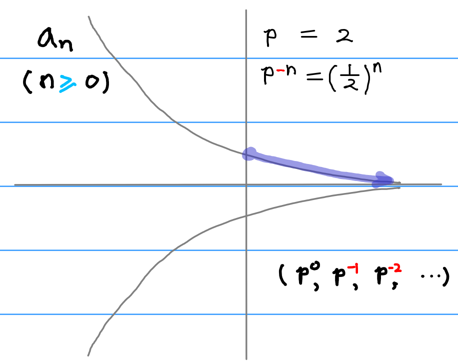
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

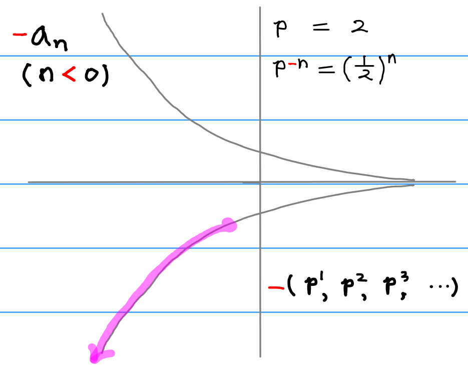
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \quad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





$$a_n \Rightarrow \pm f(z), \pm X(z)$$

causal & anti-causal	$\begin{pmatrix} \lvert z\rvert < p \\ \lvert z\rvert > p \end{pmatrix} \begin{pmatrix} \lvert z\rvert > p^{-1} \\ \lvert z\rvert < p^{-1} \end{pmatrix}$
$(n \geq 0)$ $(n < 0)$	

p=2

causal	anti-causal
$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$
$f(z) = \dfrac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$ $X(z) = \dfrac{1}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$	$f_a(z) = \dfrac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$ $X_a(z) = \dfrac{(2z)}{1-(2z)} \quad (\lvert z\rvert < 0.5)$
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$ $X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$f_a(z) = \dfrac{-2}{2-z} \quad (\lvert z\rvert > 2)$ $X_a(z) = \dfrac{-z}{z-0.5} \quad (\lvert z\rvert < 0.5)$

p=1/2

causal	anti-causal
$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
$f(z) = \dfrac{1}{1-(2z)} \quad (\lvert z\rvert < 0.5)$ $X(z) = \dfrac{1}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$	$f_a(z) = \dfrac{\left(\frac{1}{2z}\right)}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$ $X_a(z) = \dfrac{\left(\frac{z}{2}\right)}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$
$f(z) = \dfrac{0.5}{0.5-z} \quad (\lvert z\rvert < 0.5)$ $X(z) = \dfrac{z}{z-2} \quad (\lvert z\rvert > 2)$	$f_a(z) = \dfrac{-0.5}{0.5-z} \quad (\lvert z\rvert > 0.5)$ $X_a(z) = \dfrac{-z}{z-2} \quad (\lvert z\rvert < 2)$

1

$f(z)$	$f(z^{-1})$	a_n	$-b_n$
$g(z^{-1})$	$g(z)$	$-a_n$	b_n

$X(z^{-1})$	$X(z)$	$-y_n$	x_n
$Y(z)$	$Y(z^{-1})$	y_n	$-x_n$

Laurent Series $\quad a_n \leftrightarrow f(z) \quad b_n \leftrightarrow g(z)$

$n \geq 0$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n < 1$
causal a_n	$f(z)$	$\xleftrightarrow{z^{-1}}$	$f(z^{-1})$	anti-causal $-b_n$
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
$-a_n$ anti-causal	$g(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$g(z)$	b_n causal
$n < 0$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n \geq 1$

Z- Transform $\quad X(z) \leftrightarrow x_n \quad Y(z) \leftrightarrow y_n$

$n < 1$	$\lvert z\rvert < p$		$\lvert z\rvert > p^{-1}$	$n \geq 0$
anti- causal $-y_n$	$X(z^{-1})$	$\xleftrightarrow{z^{-1}}$	$X(z)$	causal x_n
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
causal y_n	$Y(z)$	$\xleftrightarrow{z^{-1}}$	$Y(z^{-1})$	anti- causal $-x_n$
$n \geq 1$	$\lvert z\rvert > p$		$\lvert z\rvert < p^{-1}$	$n < 0$

2

$f(z)$	$f(z^{-1})$	a_n	a_{-n}
$f(z)$	$f(z^{-1})$	$-a_n$	$-a_{-n}$

$X(z^{-1})$	$X(z)$	x_{-n}	x_n
$X(z^{-1})$	$X(z)$	$-x_{-n}$	$-x_n$

Laurent Series $\quad a_n \leftrightarrow f(z) \quad -a_{-n} = b_n \leftrightarrow g(z)$

causal a_n $\xleftrightarrow{z^{-1}}$ anti-causal a_{-n}

$-a_n$ anti-causal $\xleftrightarrow{z^{-1}}$ $-a_{-n}$ causal

Z- Transform $\quad X(z) \leftrightarrow x_n \quad Y(z) \leftrightarrow y_n = -x_{-n}$

anti-causal x_{-n} $\xleftrightarrow{z^{-1}}$ causal x_n

causal $-x_{-n}$ $\xleftrightarrow{z^{-1}}$ anti-causal $-x_n$

$\frac{z}{2-z}$	$\frac{z}{z-0.5}$
$\frac{z}{2-z}$	$\frac{z}{z-0.5}$

0.5^{n}	0.5^{-n}
-0.5^{n}	-0.5^{-n}

$\frac{z}{2-z}$	$\frac{z}{z-0.5}$
$\frac{z}{2-z}$	$\frac{z}{z-0.5}$

0.5^{-n}	0.5^{n}
-0.5^{-n}	-0.5^{n}

f(z) Laurent Series

$n \in [0, \infty)$ $|z| < 2$

causal $a_n = 0.5^n$ $\boxed{\frac{z}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $|z| > 0.5$ $\boxed{\frac{z}{z-0.5}}$

$n \in (-\infty, 0]$ anti-causal $a_{-n} = 0.5^{-n}$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\cdot \frac{p}{p} \frac{z}{z}$

$-a_n = -0.5^n$ anti-causal $\boxed{\frac{z}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty, -1]$ $|z| > 2$ $|z| < 0.5$ $n \in [1, \infty)$

X(z) z-Transform

$n \in (-\infty, 0]$ $|z| < p$

anti-causal X(z) $a_{-n} = 0.5^{-n}$ $\boxed{\frac{z}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $|z| > p^{-1}$ $\boxed{\frac{z}{z-0.5}}$

$n \in [0, \infty)$ causal X(z) $a_n = 0.5^n$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\cdot \frac{p}{p} \frac{z}{z}$

$-a_{-n} = -0.5^{-n}$ causal X(z) $\boxed{\frac{z}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $-a_n = -0.5^n$ anti-causal X(z)

$n \in [1, \infty)$ $|z| > p$ $|z| < p^{-1}$ $n \in (-\infty, -1]$

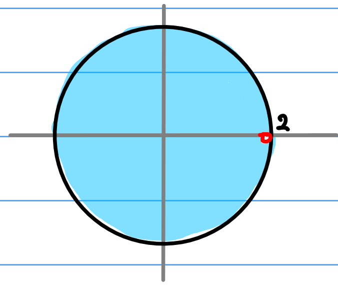
$f(z) = \boxed{\begin{matrix} \frac{2}{2-z} & \frac{z}{z-0.5} \\ \frac{2}{2-z} & \frac{z}{z-0.5} \end{matrix}}$ $\quad$ $a(n) = \boxed{\begin{matrix} 0.5^{n} & 0.5^{-n} \\ -0.5^{n} & -0.5^{-n} \end{matrix}}$

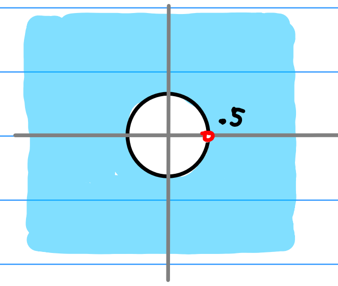
$n \in [0, \infty)$ $\quad$ $|z| < 2$

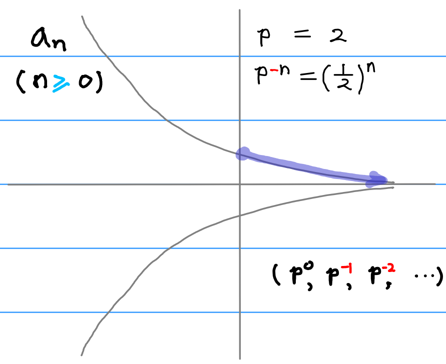
$\boxed{\begin{matrix} \text{causal} \\ a_n = 0.5^n \end{matrix}}$ $\quad$ $\boxed{\frac{2}{2-z}}$

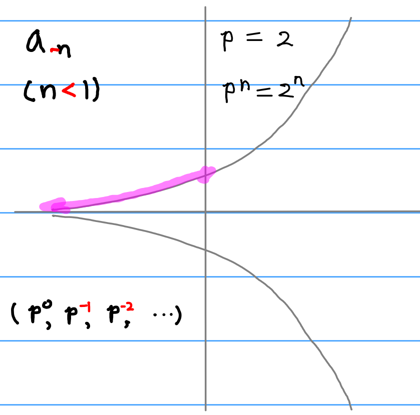
$|z| > 0.5$ $\quad$ $n \in (-\infty, 0]$

$\boxed{\frac{z}{z-0.5}}$ $\quad$ $\boxed{\begin{matrix} \text{anti-causal} \\ a_{-n} = 0.5^{-n} \end{matrix}}$







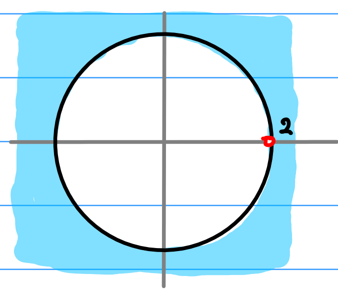


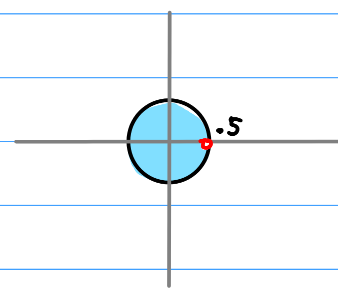
$\boxed{\begin{matrix} -a_n = -0.5^n \\ \text{anti-causal} \end{matrix}}$ $\quad$ $\boxed{\frac{2}{2-z}}$

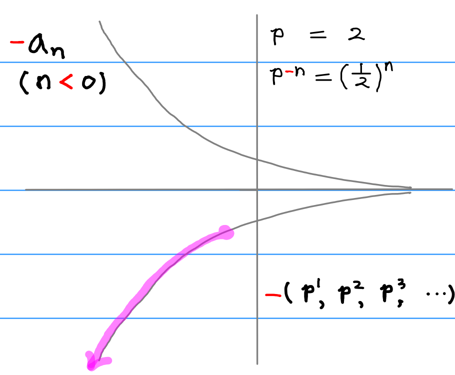
$n \in (-\infty, -1]$ $\quad$ $|z| > 2$

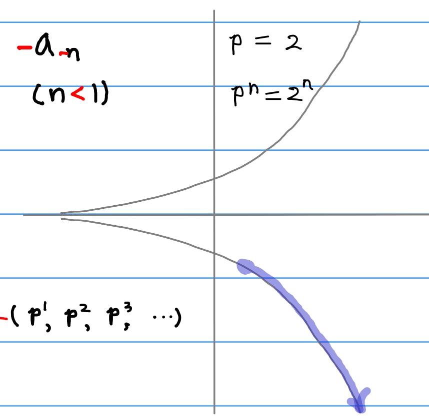
$\boxed{\frac{z}{z-0.5}}$ $\quad$ $\boxed{\begin{matrix} -a_{-n} = -0.5^{-n} \\ \text{causal} \end{matrix}}$

$|z| < 0.5$ $\quad$ $n \in [1, \infty)$









$$X(z) = \boxed{\begin{matrix} \frac{2}{2-z} & \frac{z}{z-0.5} \\ \frac{2}{2-z} & \frac{z}{z-0.5} \end{matrix}} \qquad a(n) = \boxed{\begin{matrix} 0.5^{-n} & 0.5^{n} \\ -0.5^{-n} & -0.5^{n} \end{matrix}}$$

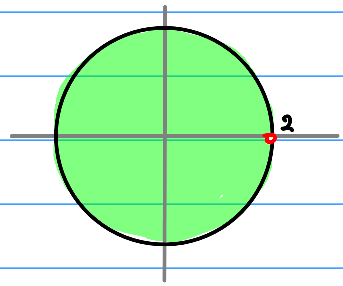
$n < 1$ $\quad$ $|z| < 2$

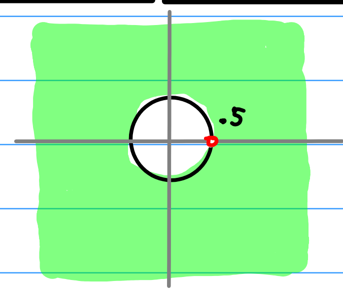
x_{-n} $\quad$ $\boxed{\begin{matrix}\text{anti-causal} \\ x_{-n} = 0.5^{-n}\end{matrix}}$ $\quad$ $\boxed{-\frac{0.5}{1-0.5z}}$

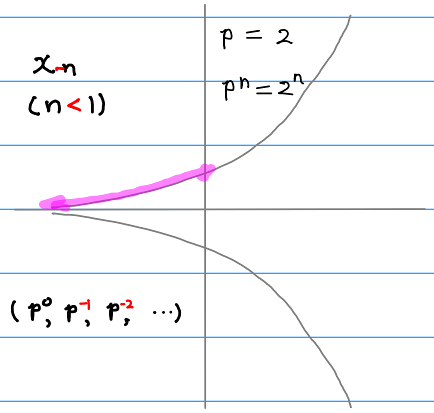
$\xleftrightarrow{z^{-1}}$

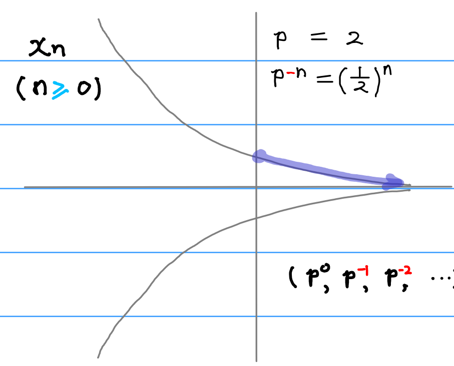
$|z| > 0.5$ $\quad$ $n \geq 0$

$\boxed{-\frac{0.5}{1-0.5z^{-1}}}$ $\quad$ $\boxed{\begin{matrix}\text{causal} \\ x_n = 0.5^{n}\end{matrix}}$ $\quad$ x_n









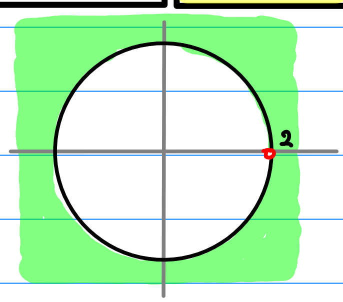
$n \geq 1$ $\quad$ $|z| > 2$

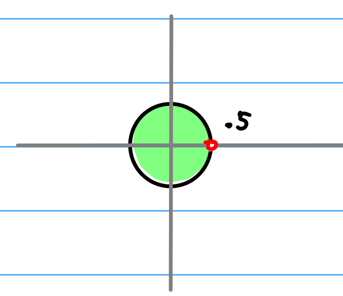
y_n $\quad$ $\boxed{\begin{matrix}-x_{-n} = -0.5^{-n} \\ \text{causal}\end{matrix}}$ $\quad$ $\boxed{\frac{z^{-1}}{1-2z^{-1}}}$

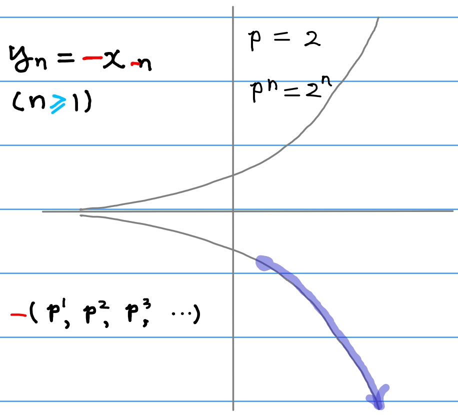
$\xleftrightarrow{z^{-1}}$

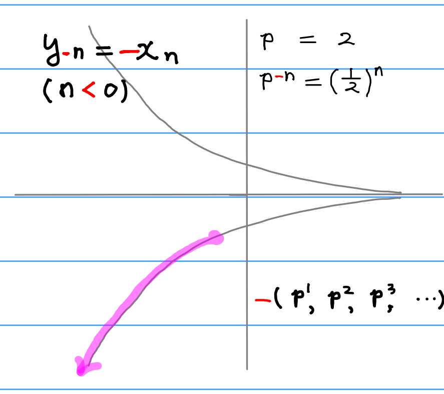
$|z| < 0.5$ $\quad$ $n < 0$

$\boxed{\frac{z}{1-2z}}$ $\quad$ $\boxed{\begin{matrix}-x_n = -0.5^{n} \\ \text{anti-causal}\end{matrix}}$ $\quad$ y_{-n}





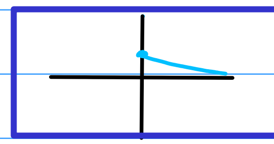
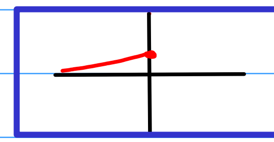
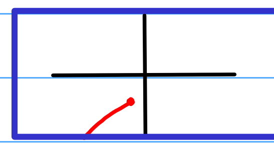
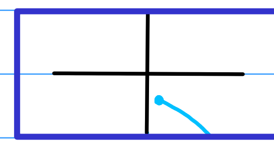




0.5^n

a(n), Range => f(z), ROC

$n \in [0,\infty)$: 0.5^n	$n \in (-\infty,0]$: 0.5^{-n}	→	$\lvert z\rvert < 2$: $\frac{1}{1-0.5z}$	$\lvert z\rvert \geq 0.5$: $\frac{1}{1-0.5z^{-1}}$
$n \in (-\infty,-1]$: -0.5^n	$n \in [1,\infty)$: -0.5^{-n}	→	$\lvert z\rvert \geq 2$: $\frac{-2z^{-1}}{1-2z^{-1}}$	$\lvert z\rvert < 0.5$: $\frac{-2z}{1-2z}$

$n \in [0,\infty)$: 0.5^n	$n \in (-\infty,0]$: 0.5^{-n}	$n \in [0,\infty)$: 	$n \in (-\infty,0]$: 
$n \in (-\infty,-1]$: -2^{-n}	$n \in [1,\infty)$: -2^{n}	$n \in (-\infty,-1]$: 	$n \in [1,\infty)$: 

$\frac{2}{2-z}$ → $\frac{1}{1-0.5z}$ causal 0.5^n $(n \geq 0)$

$\frac{2}{2-z}$ → $\frac{-2z^{-1}}{1-2z^{-1}}$ anti -0.5^n $(n<0)$

$\frac{z}{z-0.5}$ → $\frac{1}{1-0.5z^{-1}}$ anti 0.5^{-n} $(n<1)$

$\frac{z}{z-0.5}$ → $\frac{-2z}{1-2z}$ causal -0.5^{-n} $(n \geq 1)$

$\boxed{0.5^{n}}$ $\boxed{0.5^{-n}}$ $\rightarrow$ $\boxed{\frac{1}{1-0.5z}}$ $\boxed{\frac{1}{1-0.5z^{-1}}}$

$\boxed{-0.5^{n}}$ $\boxed{-0.5^{-n}}$ $\rightarrow$ $\boxed{\frac{-2z^{-1}}{1-2z^{-1}}}$ $\boxed{\frac{-2z}{1-2z}}$

$n \in [0, \infty)$

$\boxed{0.5^{n}}$ $(0.5)^0, (0.5)^1, (0.5)^2$

$$\frac{1}{1-0.5z}$$

$$\frac{2}{2-z}$$

$$\boxed{\frac{1}{1-0.5z}}$$

$n \in (-\infty, 0]$

$\boxed{0.5^{-n}}$ $(0.5)^0, (0.5)^1, (0.5)^2, \cdots$

$$\frac{1}{1-0.5z^{-1}}$$

$$\frac{z}{z-0.5}$$

$$\boxed{\frac{1}{1-0.5z^{-1}}}$$

$\boxed{-0.5^{n}}$ $(0.5)^{-1}, (0.5)^{-2}, (0.5)^{-3}$

$n \in (-\infty, -1]$

$$\frac{-2z^{-1}}{1-2z^{-1}}$$

$$\frac{2}{2-z}$$

$$\boxed{\frac{-2z^{-1}}{1-2z^{-1}}}$$

$\boxed{-0.5^{-n}}$ $(0.5)^{-1}, (0.5)^{-2}, (0.5)^{-3}$

$n \in [1, \infty)$

$$\frac{-2z}{1-2z}$$

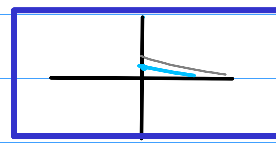
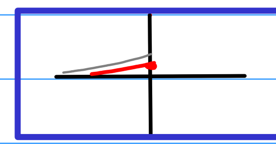
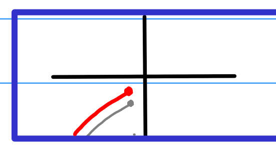
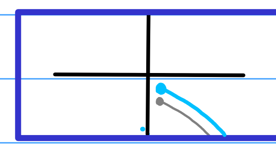
$$\frac{z}{z-0.5}$$

$$\boxed{\frac{-2z}{1-2z}}$$

0.5^{n+1}

a(n), Range => f(z), ROC

$n \in [0,\infty)$	$n \in (-\infty,0]$		$\lvert z\rvert < 2$	$\lvert z\rvert \geq 0.5$
0.5^{n+1}	0.5^{-n+1}	$\rightarrow$	$\frac{0.5}{1-0.5z}$	$\frac{0.5}{1-0.5z^{-1}}$
-0.5^{n+1}	-0.5^{-n+1}	$\rightarrow$	$\frac{-z^{-1}}{1-2z^{-1}}$	$\frac{-z}{1-2z}$
$n \in (-\infty,-1]$	$n \in [1,\infty)$		$\lvert z\rvert \geq 2$	$\lvert z\rvert < 0.5$

$n \in [0,\infty)$	$n \in (-\infty,0]$	$n \in [0,\infty)$	$n \in (-\infty,0]$
0.5^{n+1}	0.5^{-n+1}		
-2^{-n-1}	-2^{-n+1}		
$n \in (-\infty,-1]$	$n \in [1,\infty)$	$n \in (-\infty,-1]$	$n \in [1,\infty)$

$\frac{1}{2-z}$

- $\frac{0.5}{1-0.5z}$ causal 0.5^{n+1} $(n \geq 0)$
- $\frac{-z^{-1}}{1-2z^{-1}}$ anti -0.5^{n+1} $(n<0)$

$\frac{0.5z}{z-0.5}$

- $\frac{0.5}{1-0.5z^{-1}}$ anti 0.5^{-n+1} $(n<1)$
- $\frac{-z}{1-2z}$ causal -0.5^{-n+1} $(n \geq 1)$

$\boxed{0.5^{n+1}}$ $\boxed{0.5^{-n+1}}$ $\rightarrow$ $\boxed{\frac{0.5}{1-0.5z}}$ $\boxed{\frac{0.5}{1-0.5z^{-1}}}$

$\boxed{-0.5^{n+1}}$ $\boxed{-0.5^{-n+1}}$ $\rightarrow$ $\boxed{\frac{-z^{-1}}{1-2z^{-1}}}$ $\boxed{\frac{-z}{1-2z}}$

$n \in [0, \infty)$

$\boxed{0.5^{n+1}}$ $(0.5)^1, (0.5)^2, (0.5)^3$

$$\frac{0.5}{1-0.5z}$$

$$\frac{1}{2-z}$$

$$\boxed{\frac{0.5}{1-0.5z}}$$

$n \in (-\infty, 0]$

$\boxed{0.5^{-n+1}}$ $(0.5)^1, (0.5)^2, (0.5)^3$

$$\frac{0.5}{1-0.5z^{-1}}$$

$$\frac{0.5z}{z-0.5}$$

$$\boxed{\frac{0.5}{1-0.5z^{-1}}}$$

$\boxed{-0.5^{n+1}}$ $(0.5)^0, (0.5)^{-1}, (0.5)^{-2}$

$n \in (-\infty, -1]$

$$\frac{-z^{-1}}{1-2z^{-1}}$$

$$\frac{1}{2-z}$$

$$\boxed{\frac{-z^{-1}}{1-2z^{-1}}}$$

$\boxed{-0.5^{-n+1}}$ $(0.5)^0, (0.5)^{-1}, (0.5)^{-2}$

$n \in [1, \infty)$

$$\frac{-z}{1-2z}$$

$$\frac{0.5z}{z-0.5}$$

$$\boxed{\frac{-z}{1-2z}}$$

(Causal / Anti-Causal) forms of $\left(\frac{1}{z-p}\ ,\ \frac{1}{z^{-1}-p}\right) * p$

2 formulas (A)

$$\boxed{\frac{1}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$\boxed{\frac{1}{z-p}}$

- Causal: $\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$
- Anti-Causal: $\boxed{\frac{z^{-1}}{1-pz^{-1}}}$

$\boxed{\frac{1}{z^{-1}-p}}$

- Anti-Causal: $\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$
- Causal: $\boxed{\frac{z}{1-pz}}$

2 formulas (B)

$$\boxed{\frac{p}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$\boxed{\frac{p}{z-p}}$

- Causal: $\boxed{-\frac{1}{1-p^{-1}z}}$
- Anti-Causal: $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$

$\boxed{\frac{p}{z^{-1}-p}}$

- Anti-Causal: $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$
- Causal: $\boxed{\frac{pz}{1-pz}}$

$$\xrightarrow{p}$$

$$\xleftarrow[1/p]{}$$

$$\text{Ⓐ} \left(\frac{1}{z-p}, \frac{1}{z^{-1}-p} \right) f(z) \longrightarrow a_n$$

2 formulas Ⓐ

$$\boxed{\frac{1}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{1}{z-p}}$$

causal

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z}} \triangleq f(z) \qquad -(p^{-1})^{n+1} \quad (n \geq 0)$$

$$\|$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}} \triangleq g(z^{-1}) \qquad (p^{-1})^{n+1} \quad (n < 0)$$

anti- causal

anti- causal

$$\boxed{\frac{1}{z^{-1}-p}}$$

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq f(z^{-1}) \qquad -(p^{-1})^{-n+1} \quad (n < 1)$$

$$\|$$

$$\boxed{\frac{z}{1-pz}} \triangleq g(z) \qquad (p^{-1})^{-n+1} \quad (n \geq 1)$$

causal

$$\boxed{\frac{1}{2-z}}$$

$$\boxed{\frac{0.5}{1-0.5z}}$$ causal 0.5^{n+1} $(n \geq 0)$

$$\boxed{\frac{-z^{-1}}{1-2z^{-1}}}$$ anti -0.5^{n+1} $(n < 0)$

$$\boxed{\frac{0.5z}{z-0.5}}$$

$$\boxed{\frac{0.5}{1-0.5z^{-1}}}$$ anti 0.5^{-n+1} $(n < 1)$

$$\boxed{\frac{-z}{1-2z}}$$ causal -0.5^{-n+1} $(n \geq 1)$

$$\text{Ⓐ}\ \left(\frac{1}{z-p},\ \frac{1}{z^{-1}-p}\right)\ f(z) \rightarrow a_n \quad \text{method } 1$$

causal

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{n+1} \rightarrow -(p^{-1})^{n+1}$

$n=0,1,2,\cdots$

$-(p^{-1}z^{0}+p^{-2}z^{1}+p^{-3}z^{2}+\cdots)$

anti- causal

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $(p) \rightarrow (p)^{-n-1} \rightarrow (p^{-1})^{n+1}$

$n=-1,-2,-3,\cdots$

$(p^{0}z^{-1}+p^{1}z^{-2}+p^{2}z^{-3}+\cdots)$

anti- causal

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{-n+1} \rightarrow -(p^{-1})^{-n+1}$

$n=0,-1,-2,\cdots$

$-(p^{-1}z^{0}+p^{-2}z^{-1}+p^{-3}z^{-2}+\cdots)$

causal

p $\boxed{\dfrac{z}{1-pz}}$ $(p) \rightarrow (p)^{n-1} \rightarrow (p^{-1})^{-n+1}$

$n=1,2,3,\cdots$

$(p^{0}z^{1}+p^{1}z^{2}+p^{2}z^{3}+\cdots)$

Ⓐ $\left(\frac{1}{z-p}, \frac{1}{z^{-1}-p}\right) f(z) \longrightarrow a_n$ method 2

causal

p^{-1} $\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1})$ z → $n=0,1,2,\cdots$ → $-(p^{-1})^{n+1} = -(p)^{-n-1}$

-1

anti- causal

p $\boxed{\frac{z^{-1}}{1-pz^{-1}}}$ (p) z^{-1}/z^{-1} → $n=-1,-2,-3,\cdots$ → $(p^{-1})^{n+1} = (p)^{-n-1}$

$-n$

anti- causal

p $\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ z^{-1} → $n=0,-1,-2,\cdots$ → $-(p^{-1})^{-n+1} = -(p)^{n-1}$

-1

causal

p^{-1} $\boxed{\frac{z}{1-pz}}$ (p) z/z → $n=1,2,3,\cdots$ → $(p^{-1})^{-n+1} = (p)^{n-1}$

$$\text{(B)} \quad \left(\frac{p}{z-p} \,,\, \frac{p}{z^{-1}-p} \right) \quad f(z) \longrightarrow a_n$$

2 formulas $\quad f(z) \longrightarrow a_n$

$$\boxed{\frac{p}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}} \quad \begin{cases} -\boxed{\dfrac{1}{1-p^{-1}z}} \triangleq f'(z) \quad \text{causal} & -(p^{-1})^n \quad (n \geq 0) \\ \qquad \qquad \| \\ \boxed{\dfrac{pz^{-1}}{1-pz^{-1}}} \triangleq g'(z^{-1}) \quad \text{anti- causal} & (p^{-1})^n \quad (n < 0) \end{cases}$$

$$\boxed{\frac{p}{z^{-1}-p}} \quad \begin{cases} -\boxed{\dfrac{1}{1-p^{-1}z^{-1}}} \triangleq f'(z^{-1}) \quad \text{anti- causal} & -(p^{-1})^{-n} \quad (n < 1) \\ \qquad \qquad \| \\ \boxed{\dfrac{pz}{1-pz}} \triangleq g'(z) \quad \text{causal} & (p^{-1})^{-n} \quad (n \geq 1) \end{cases}$$

$$\boxed{\frac{2}{2-z}} \quad \begin{cases} \boxed{\dfrac{1}{1-0.5z}} \quad \text{causal} \quad 0.5^n \ (n \geq 0) \\ \boxed{\dfrac{-2z^{-1}}{1-2z^{-1}}} \quad \text{anti} \quad -0.5^n \ (n < 0) \end{cases}$$

$$\boxed{\frac{z}{z-0.5}} \quad \begin{cases} \boxed{\dfrac{1}{1-0.5z^{-1}}} \quad \text{anti} \quad 0.5^{-n} \ (n < 1) \\ \boxed{\dfrac{-2z}{1-2z}} \quad \text{causal} \quad -0.5^{-n} \ (n \geq 1) \end{cases}$$

Ⓑ $\left(\frac{p}{z-p}, \frac{p}{z^{-1}-p}\right) f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{n} \rightarrow -(p^{-1})^{n}$

$n = 0, 1, 2, \cdots$

$-(p^{0}z^{0} + p^{-1}z^{1} + p^{-2}z^{2} + \cdots)$

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ $(p) \rightarrow (p)^{-n} \rightarrow (p^{-1})^{n}$

$n = -1, -2, -3, \cdots$

$(p^{1}z^{-1} + p^{2}z^{-2} + p^{3}z^{-3} + \cdots)$

p $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{-n} \rightarrow -(p^{-1})^{-n}$

$n = 0, -1, -2, \cdots$

$-(p^{0}z^{0} + p^{-1}z^{-1} + p^{-2}z^{-2} + \cdots)$

p^{-1} $\boxed{\frac{pz}{1-pz}}$ $(p) \rightarrow (p)^{n} \rightarrow (p^{-1})^{-n}$

$n = 1, 2, 3, \cdots$

$(p^{1}z^{1} + p^{2}z^{2} + p^{3}z^{3} + \cdots)$

$$\text{(B)}\ \left(\frac{p}{z-p},\ \frac{p}{z^{-1}-p}\right)\ f(z) \longrightarrow a_n \qquad \text{method } 2$$

causal

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1})$ z → $n=0,1,2,\dots$ → $-(p^{-1})^n = -(p)^{-n}$

-1

anti- causal

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ (p) z^{-1}/z^{-1} → $n=-1,-2,-3,\dots$ → $(p^{-1})^n = (p)^{-n}$

$-n$

anti- causal

p $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ z^{-1} → $n=0,-1,-2,\dots$ → $-(p^{-1})^{-n} = -(p)^n$

-1

causal

p^{-1} $\boxed{\frac{pz}{1-pz}}$ (p) z/z → $n=1,2,3,\dots$ → $(p^{-1})^{-n} = (p)^n$

2 formulas

$$\boxed{\frac{p}{z-p}} \xleftrightarrow{z^{-1}} \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

causal anti- causal

$$\boxed{\frac{p}{z-p}} \to \boxed{-\frac{1}{1-p^{-1}z}} \triangleq f'(z) = X'(z^{-1})$$

$$\boxed{\frac{p}{z-p}} \to \boxed{\frac{pz^{-1}}{1-pz^{-1}}} \triangleq Y'(z) = g'(z^{-1})$$

causal anti- causal

causal anti- causal

$$\boxed{\frac{p}{z^{-1}-p}} \to \boxed{-\frac{1}{1-p^{-1}z^{-1}}} \triangleq X'(z) = f'(z^{-1})$$

$$\boxed{\frac{p}{z^{-1}-p}} \to \boxed{\frac{pz}{1-pz}} \triangleq g'(z) = Y'(z^{-1})$$

causal anti- causal

$$\left(\frac{p}{p-z}\,,\ \frac{p}{p-z^{-1}}\right) = \left(\frac{p}{p-z}\,,\ \frac{z}{z-p^{-1}}\right)$$

$$\boxed{\frac{2}{2-z}} \xleftrightarrow{z^{-1}} \boxed{\frac{2}{2-z^{-1}}} = \boxed{\frac{z}{z-0.5}}$$

$$\Big\downarrow -\tfrac{1}{2} \qquad\qquad \Big\downarrow -\tfrac{1}{2} \qquad\qquad \Big\downarrow -\tfrac{1}{2}$$

$$\boxed{-\frac{1}{2-z}} \xleftrightarrow{z^{-1}} \boxed{-\frac{1}{2-z^{-1}}} = \boxed{-\frac{0.5z}{z-0.5}}$$

$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

$n \in [0,\infty)$ $\quad |z| < 2$ $\qquad\qquad$ $|z| > 0.5$ $\quad n \in (-\infty, 0]$

$$\boxed{\text{causal} \quad a_n = 0.5^n}\ \boxed{\frac{2}{2-z}} \xleftrightarrow{z^{-1}} \boxed{\frac{z}{z-0.5}}\ \boxed{\text{anti-causal} \quad a_{-n} = 0.5^{-n}}$$

$$\cdot\ \frac{p}{p}\,\frac{z^{-1}}{z^{-1}} \qquad\qquad \cdot\ \frac{p}{p}\,\frac{z}{z}$$

$$\boxed{-a_n = -0.5^n \quad \text{anti-causal}}\ \boxed{\frac{2}{2-z}} \xleftrightarrow{z^{-1}} \boxed{\frac{z}{z-0.5}}\ \boxed{-a_{-n} = -0.5^{-n} \quad \text{causal}}$$

$n \in (-\infty, -1]$ $\quad |z| > 2$ $\qquad\qquad$ $|z| < 0.5$ $\quad n \in [1,\infty)$

$$\frac{1}{p}\left(\frac{p}{p-z},\ \frac{z}{z-p^{-1}}\right) = \left(\frac{1}{p-z},\ \frac{p^{-1}z}{z-p^{-1}}\right)$$

$\frac{2}{2-z}$

- $\frac{1}{1-0.5z}$ causal $0.5^{n}\ (n \geq 0)$
- $\frac{-2z^{-1}}{1-2z^{-1}}$ anti $-0.5^{n}\ (n<0)$

$\frac{z}{z-0.5}$

- $\frac{1}{1-0.5z^{-1}}$ anti $0.5^{-n}\ (n<1)$
- $\frac{-2z}{1-2z}$ causal $-0.5^{-n}\ (n \geq 1)$

$\frac{1}{2-z}$

- $\frac{0.5}{1-0.5z}$ causal $0.5^{n+1}\ (n \geq 0)$
- $\frac{-z^{-1}}{1-2z^{-1}}$ anti $-0.5^{n+1}\ (n<0)$

$\frac{0.5z}{z-0.5}$

- $\frac{0.5}{1-0.5z^{-1}}$ anti $0.5^{-n+1}\ (n<1)$
- $\frac{-z}{1-2z}$ causal $-0.5^{-n+1}\ (n \geq 1)$

$$\frac{1}{2}\left(\frac{z}{2-z}, \frac{z}{z-0.5}\right) = \left(\frac{0.5}{1-0.5z}, \frac{z}{1-2z}\right)$$

$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
causal $a_n = 0.5^n$	$\frac{z}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p}\frac{z}{z}$	
$-a_n = -0.5^n$ anti-causal	$\frac{z}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_{-n} = -0.5^{-n}$ causal
$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1,\infty)$

	$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal 0.5^{n+1}	$\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$\frac{0.5}{1-0.5z^{-1}}$	anti-causal 0.5^{-n+1} 2^{+n-1}
b	-0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	2^{+n-1} -0.5^{-n+1} causal
	$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1,\infty)$

$$\left(\frac{p}{p-z},\ \frac{z}{z-p^{-1}}\right)$$

$\frac{2}{2-z}$ → $\frac{1}{1-0.5z}$ causal $0.5^{n}\ (n\geq 0)$

$\frac{2}{2-z}$ → $\frac{-2z^{-1}}{1-2z^{-1}}$ anti $-0.5^{n}\ (n<0)$

$\frac{z}{z-0.5}$ → $\frac{1}{1-0.5z^{-1}}$ anti $0.5^{-n}\ (n<1)$

$\frac{z}{z-0.5}$ → $\frac{-2z}{1-2z}$ causal $-0.5^{-n}\ (n\geq 1)$

$n\in[0,\infty)$ $|z|<2$

causal $a_n = 0.5^{n}$ $\boxed{\frac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\frac{z}{z-0.5}$ anti-causal $a_{-n} = 0.5^{-n}$

$|z|>0.5$ $n\in(-\infty,0]$

$\cdot\frac{p}{p}\ \frac{z^{-1}}{z^{-1}}$ $\cdot\frac{p}{p}\ \frac{z}{z}$

$-a_n = -0.5^{n}$ anti-causal $\frac{2}{2-z}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $-a_{-n} = -0.5^{-n}$ causal

$n\in(-\infty,-1]$ $|z|>2$ $|z|<0.5$ $n\in[1,\infty)$

$$\left(\frac{1}{p - z}\ ,\ \frac{p^{-1} z}{z - p^{-1}}\right)$$

$\boxed{\frac{1}{2 - z}}$

- $\boxed{\frac{0.5}{1 - 0.5z}}$ causal 0.5^{n+1} $(n \geq 0)$
- $\boxed{\frac{-z^{-1}}{1 - 2z^{-1}}}$ anti -0.5^{n+1} $(n < 0)$

$\boxed{\frac{0.5z}{z - 0.5}}$

- $\boxed{\frac{0.5}{1 - 0.5z^{-1}}}$ anti 0.5^{-n+1} $(n < 1)$
- $\boxed{\frac{-z}{1 - 2z}}$ causal -0.5^{-n+1} $(n \geq 1)$

	$n \in [0, \infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal 0.5^{n+1}	$\frac{0.5}{1 - 0.5z}$	$\xleftrightarrow{z^{-1}}$	$\frac{0.5}{1 - 0.5z^{-1}}$	anti-causal 0.5^{-n+1}
					2^{+n-1}

					-2^{+n-1}
b_n	-0.5^{n+1} anti-causal	$\frac{z^{-1}}{1 - 2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1 - 2z}$	-0.5^{-n+1} causal
	$n \in (-\infty, -1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1, \infty)$

Time Shift

$p = 2$

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5 - z}$ $X(z) = \frac{z}{z-2}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5 - z}$ $X(z) = -\frac{z}{z-2}$

6 $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5 - z}$ $X(z) = \frac{1}{z-2}$

8 $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5 - z}$ $X(z) = -\frac{1}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5 - z)z}$ $X(z) = \frac{z^2}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5 - z)z}$ $X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \leftrightarrow \frac{1}{2}$

1	$(n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
2	$(n \geq 0)$	$a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
3	$(n < 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
4	$(n < 0)$	$a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
5	$(n \geq 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
6	$(n \geq 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$
7	$(n < 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
8	$(n < 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$
9	$(n \geq -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = \frac{2}{(2-z)z}$	$X(z) = \frac{z^2}{z-0.5}$
10	$(n \geq -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = \frac{0.5}{(0.5-z)z}$	$X(z) = \frac{z^2}{z-2}$
11	$(n < -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = -\frac{2}{(2-z)z}$	$X(z) = -\frac{z^2}{z-0.5}$
12	$(n < -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = -\frac{0.5}{(0.5-z)z}$	$X(z) = -\frac{z^2}{z-2}$

Shift to the right

shift to the right →	$\times z$	$\times z^{-1}$
delete a_0		

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

6 $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5-z}$ $X(z) = \frac{1}{z-2}$

shift to the right →	$\times z$	$\times z^{-1}$
insert a_0		

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

8 $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5-z}$ $X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z}{z-0.5}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z}{z-2}$

Shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z}{z-0.5}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$

b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$

b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$

b^{n-1} $n=1,2,3,\cdots$

$$1 \longleftrightarrow \frac{1}{1}$$

1. $(n \geq 0) \quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

2. $(n \geq 0) \quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

3. $(n < 0) \quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

4. $(n < 0) \quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

5. $(n \geq 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

6. $(n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

7. $(n < 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

8. $(n < 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

9. $(n \geq -1) \quad a_{n+1} = (1)^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

10. $(n \geq -1) \quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

11. $(n < -1) \quad a_{n+1} = (1)^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

12. $(n < -1) \quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

1 $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the right → $\times z$ $\times z^{-1}$

delete a_0

5 $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

6 $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the right → $\times z$ $\times z^{-1}$

insert a_0

7 $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

8 $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

1 $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

delete a_0

9 $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

10 $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

insert a_0

11 $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

12 $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

Causality

$f(z)\ (|z|<p) \leftrightarrow a_n\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1})\ (|z|<p) \leftrightarrow x_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z^{-1})\ (|z|>p^{-1}) \leftrightarrow a_{-n}\ (n < 1)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z)\ (|z|>p^{-1}) \leftrightarrow x_n\ (n \geq 0)\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z)\ (|z|>p) \leftrightarrow -a_n\ (n < 0)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$X(z^{-1})\ (|z|>p) \leftrightarrow -x_{-n}\ (n \geq 1)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$f(z^{-1})\ (|z|<p^{-1}) \leftrightarrow -a_{-n}\ (n \geq 1)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$X(z)\ (|z|<p^{-1}) \leftrightarrow -x_n\ (n < 0)\quad (p^{0}, p^{1}, p^{2}, \cdots)$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ g(z^{-1}) & g(z) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ Y(z) & Y(z^{-1}) \end{array} \qquad \begin{array}{cc} a_n & a_{-n} \\ b_{-n} & b_n \end{array} \qquad \begin{array}{cc} x_{-n} & x_n \\ y_n & y_{-n} \end{array}$$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ f(z) & f(z^{-1}) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ X(z^{-1}) & X(z) \end{array}$$

$$\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array} \qquad \begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array}$$

$$\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array} \qquad \begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}$$

$f(z)$	$g(z)$
$f(z)$	$g(z)$

$Y(z)$	$X(z)$
$Y(z)$	$X(z)$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$[0, \infty)$	$(-\infty, 0]$
$(-\infty, -1]$	$[1, \infty)$

$(-\infty, 0]$	$[0, \infty)$
$[1, \infty)$	$(-\infty, -1]$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^0, p^1, p^2, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^0, p^1, p^2, \cdots)$

$$\boxed{\begin{array}{ll} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \qquad \boxed{\begin{array}{ll} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad a_n = -2^{-n}$$

$$\boxed{\begin{array}{ll} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}} \qquad \boxed{\begin{array}{ll} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad x_n = -2^{n}$$

$$\boxed{\begin{array}{ll} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array}} \qquad \boxed{\begin{array}{ll} -(-2^{-1}, -2^{-2}, -2^{-3}, \dots) & -(-2^{-1}, -2^{-2}, -2^{-3}, \dots) \\ (-2^{0}, -2^{1}, -2^{2}, \dots) & (-2^{0}, -2^{1}, -2^{2}, \dots) \end{array}}$$

$$\boxed{\begin{array}{ll} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \qquad \boxed{\begin{array}{ll} \dfrac{2^{-1}}{1-2^{-1}z} & \dfrac{2^{-1}}{1-2^{-1}z^{-1}} \\ -\dfrac{z^{-1}}{1-2z^{-1}} & -\dfrac{z}{1-2z} \end{array}} \qquad \begin{array}{ll} \dfrac{(\frac{1}{2})}{1-(\frac{z}{2})} & \dfrac{(\frac{1}{2})}{1-(\frac{1}{2z})} \\ -\dfrac{(\frac{1}{z})}{1-(\frac{2}{z})} & -\dfrac{z}{1-2z} \end{array}$$

$$\boxed{\begin{array}{ll} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \qquad \boxed{\begin{array}{ll} |z| < 2 & |z| > 2^{-1} \\ |z| > 2 & |z| < 2^{-1} \end{array}}$$

$$\boxed{\begin{array}{ll} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}} \qquad \boxed{\begin{array}{ll} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}}$$