Monad P3 : Lambda Calculus (1F)

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7/3/22

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Lambda Calculus

CFG for Lambda Calculus (1)

The central concept in the **lambda calculus** is
an **expression** which we can think of as a program
that returns a result when evaluated
consisting of *another* **lambda calculus expression**.

Here is the grammar for lambda expressions:

expr → λ variable . expr | expr expr | variable | (expr) | constant

http://www.cs.columbia.edu/~aho/cs4115/Lectures/15-04-13.html

CFG for Lambda Calculus (2)

expr → **λ variable . expr | expr expr | variable | (expr) | constant**

A **variable** is an identifier.

A **constant** is a built-in function such as *addition* or *multiplication*,

or a constant such as an *integer* or *boolean*.

all **programming language constructs**

can be represented as **functions**

with the pure **lambda calculus**

so these **constants** are unnecessary.

However, some constants may be used for notational simplicity.

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Function Abstraction (1)

A **function abstraction**, often called a **lambda abstraction**,
is a **lambda expression** that defines a **function**.

A **function abstraction** consists of *four parts*:

a **lambda** followed by a **variable**, a **period**,
and then an **expression** as in **λx.expr**.

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Function Abstraction (2)

For example, the function abstraction **λx. + x 1**

defines a **function of x** that *adds* **x** to **1**.

Parentheses can be added to lambda expressions for clarity.

Thus, we could have written this function abstraction

as **λx.(+ x 1)** or even as **(λx. (+ x 1))**.

In C this function definition might be written as

```
int addOne (int x) {
  return (x + 1);     }
```

http://www.cs.columbia.edu/~aho/cs4115/Lectures/15-04-13.html

Function Abstraction (3)

Note that unlike C

the **lambda abstraction** does not give a **name** to the **function**.

The **lambda expression** itself is the **function**.

We say that **λx.expr** binds the **variable x** in **expr** and

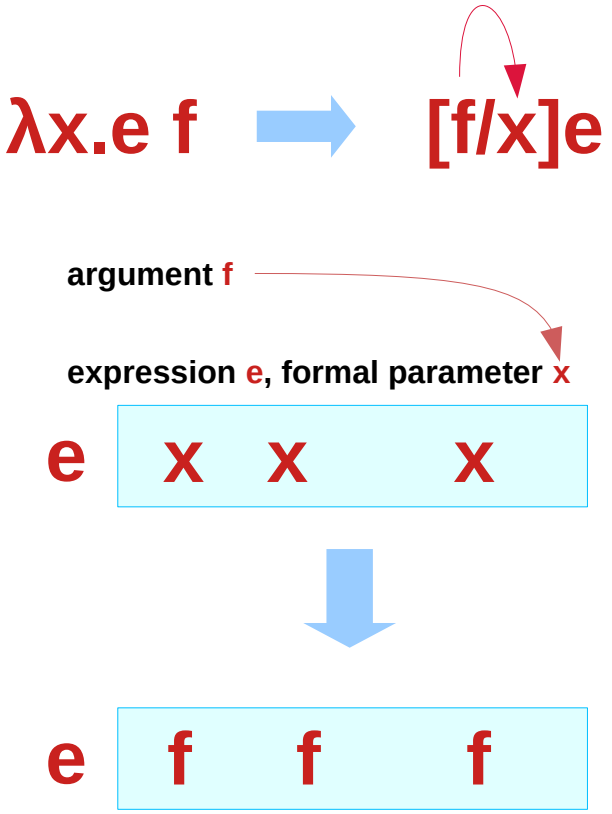
that **expr** is the **scope** of the **variable**.

http://www.cs.columbia.edu/~aho/cs4115/Lectures/15-04-13.html

Beta reduction

A **function application λx.e f** is evaluated
by substituting the **argument f**
for *all free occurrences* of the **formal parameter x**
in the body **e** of the **function definition**.

We will use the notation **[f/x]e** to indicate
that **f** is to be substituted for all free occurrences of **x**
in the expression **e**.



http://www.cs.columbia.edu/~aho/cs4115/Lectures/15-04-13.html

Function Application (1)

A **function application**, often called a **lambda application**,
consists of an **expression** followed by an **expression**:

expr expr.

The first **expression** is

a **function abstraction**

the second **expression** is

the **argument** to which the **function** is applied.

All **functions** in **lambda calculus** have exactly one **argument**.
Multiple-argument **functions** are represented by **currying**,

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Function Application (2)

the **lambda expression λx. (+ x 1) 2**
is an application of the **function λx. (+ x 1)** to the argument **2**.

This function application **λx. (+ x 1) 2** can be evaluated
by substituting the **argument 2** for the **formal parameter x**
in the **body (+ x 1)**.

Doing this we get (+ 2 1).

This substitution is called a **beta reduction**.

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Function Application (3)

Beta reductions are like macro substitutions in C.
To do beta reductions correctly
we may need to rename bound variables in lambda expressions
to avoid name clashes.

Function application associates left-to-right; thus, f g h = (f g)h.

Function application binds more tightly than λ; thus, λx. f g x = (λx. (f g)x).

Functions in the lambda calculus are first-class citizens;
that is to say, functions can be used as arguments to functions
and functions can return functions as results.

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Lambda calculus (4) – free and bound variables

In the function definition λx.x
the variable x in the body of the definition (the second x)
is bound because its first occurrence in the definition is λx.

A variable that is not bound in expr is said to be free in expr.

In the function (λx.xy), the variable x in the body of the function
is bound and the variable y is free.

Every variable in a lambda expression is either bound or free.
Bound and free variables have quite a different status in functions.

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Lambda calculus (5) – free and bound variables

In the expression (λx.x)(λy.yx):

- The variable x in the body of the leftmost expression is bound to the first lambda.
- The variable y in the body of the second expression is bound to the second lambda.
- The variable x in the body of the second expression is free.
- Note that x in second expression is independent of the x in the first expression.

In the expression (λx.xy)(λy.y):

- The variable y in the body of the leftmost expression is free.
- The variable y in the body of the second expression is bound to the second lambda.

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Lambda calculus (5) – free and bound variables

Given an expression e, the following rules define FV(e), the set of free variables in e:

If e is a variable x, then FV(e) = {x}.

If e is of the form λx.y, then FV(e) = FV(y) - {x}.

If e is of the form xy, then FV(e) = FV(x) ∪ FV(y).

An expression with no free variables is said to be closed.

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Lambda calculus (6) – beta reduction

Examples:

(λx.x)y → [y/x]x = y
in the express **x**,
substitute the parameter **x** with the argument **x**

(λx.xzx)y → [y/x]xzx = yzy
in the express **xzx**,
substitute the parameter **x** with the argument **y**

(λx.z)y → [y/x]z = z
in the express **z**,
substitute the parameter **x** with the argument **y**

since the formal parameter **x** does not appear in the body **z**.

This **substitution** in a **function application** is called
a **beta reduction** and we use a right arrow to indicate it.

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Lambda calculus (7) – beta reduction

If **expr1 → expr2**, we say **expr1** reduces to **expr2** in one step.

In general, **(λx.e)f → [f/x]e** means that

applying the **function (λx.e)** to the **argument expression f**

reduces to the **expression [f/x]e**

where the **argument expression f** is substituted

for the function's **formal parameter x** in the **function body e**.

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Lambda calculus (8) – beta reduction

A **lambda calculus expression** (aka a "**program**") is
"run" by *computing a final result*
by repeatly applying **beta reductions**.

We use →***** to denote the reflexive and transitive closure of →;
that is, zero or more applications of **beta reductions**.

http://www.cs.columbia.edu/~aho/cs4115/Lectures/15-04-13.html

Lambda calculus (9) – beta reduction

Examples:

(λx.x)y → y

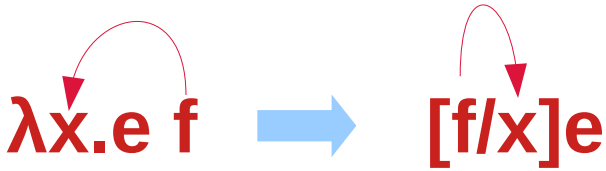
illustrating that **λx.x** is the **identity function**

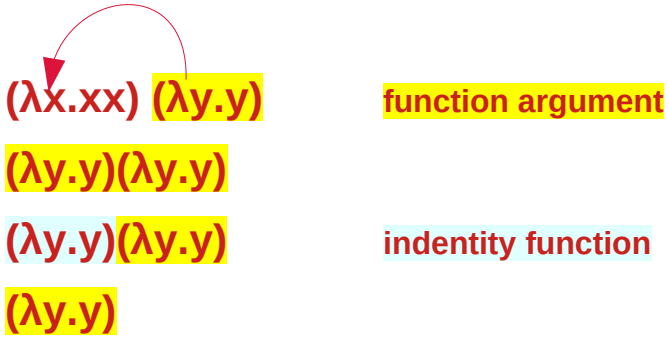
(λx.xx)(λy.y) → (λy.y)(λy.y) → (λy.y);

thus, we can write **(λx.xx)(λy.y) →* (λy.y)**.

we have applied a **function** to a **function**

as an argument and the **result** is a **function**.





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Lambda calculus (10) – beta reduction

Examples:

(λx.x)y → y

illustrating that **λx.x** is the **identity function**

(λx.xx)(λy.y) → (λy.y)(λy.y) → (λy.y);

thus, we can write **(λx.xx)(λy.y) →* (λy.y)**.

→* to denote the reflexive and transitive closure of **→**

that is, zero or more applications of beta reductions

Transitive relation

x R y and **y R z** then **x R z**

Reflexive relation

x R x

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Evaluation models of a function

Call-by-value:

arguments are evaluated before a function is entered

Call-by-name:

arguments are passed unevaluated

Call-by-need:

arguments are passed unevaluated

but an expression is only evaluated once

and shared upon subsequent references

http://dev.stephendiehl.com/fun/005_evaluation.html

Comparisons

Call by name is non-memoizing non-strict evaluation strategy
where the **value**(s) of the **argument**(s) need only be found
when actually used inside the **function's body**, each time anew:

Call by need is memoizing non-strict a.k.a. lazy evaluation strategy
where the **value**(s) of the **argument**(s) need only be found
when used inside the **function's body** for the first time,
and then are available for any further reference:

Call by value is strict evaluation strategy
where the **value**(s) of the **argument**(s) must be found
before entering the function's body:

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Comparisons

Call by name	non-memoizing	non-strict
Call by need	memoizing	non-strict
Call by value		strict

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Comparisons

Call by name the **value**(s) of the **argument**(s) need only be found when actually used inside the **function's body**, each time anew:	non-memoizing	non-strict
Call by need the **value**(s) of the **argument**(s) need only be found when used inside the **function's body** for the first time, and then are available for any further reference:	memoizing	non-strict
Call by value the **value**(s) of the **argument**(s) must be found before entering the function's body:		strict

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Memoization / Sharing

Memoization is a technique
for storing **values** of a **function**
instead of recomputing them
each time the **function** is called.

Sharing means that **temporary data** is physically stored,
if it is used multiple times.

https://wiki.haskell.org/Memoization

Strictness

Strict evaluation, or eager evaluation, is an evaluation strategy
where **expressions** are evaluated
as soon as they are bound to a **variable**.

> when **x = 3 * 7** is read, **3 * 7** is immediately computed
> and **21** is bound to **x**.

Conversely, with **lazy evaluation**
values are only computed when they are needed.

> In the example **x = 3 * 7**, **3 * 7** isn't evaluated until it's needed,
> like if you needed to output the value of **x**.

https://en.wikibooks.org/wiki/Haskell/Strictness https://wiki.haskell.org/Sharing

Laziness

Haskell is a **non**-**strict** language, and most implementations
use a strategy called **laziness** to run your program.
Basically **laziness == non-strictness + sharing**.

Laziness can be a useful tool for improving performance,
but more often than not it reduces performance
by adding a **constant overhead** to everything.

https://wiki.haskell.org/Performance/Strictness

Laziness

Because of **laziness**, the compiler can't
evaluate a function **argument**
and pass the **value** to the function,

it has to record the **expression**
in the **heap** in a **suspension** (or **thunk**)
in case it is evaluated later.

Storing and evaluating **suspensions** is costly, and unnecessary
if the **expression** was going to be evaluated anyway.

https://wiki.haskell.org/Performance/Strictness

Call by **name**

```
h x = x : (h x)
g xs = [head xs, head xs - 1]

g (h 2)  = let {xs = (h 2)} in [head xs, head xs - 1]
         = [let {xs = (h 2)} in head xs,      let {xs = (h 2)} in head xs - 1]
         = [head (h 2),                       let {xs = (h 2)} in head xs - 1]
         = [head (let {x = 2} in x : (h x)}), let {xs = (h 2)} in head xs - 1]
         = [let {x = 2} in x,                 let {xs = (h 2)} in head xs - 1]
         = [2,                                let {xs = (h 2)} in head xs - 1]
         = ....
```

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Call by **need**

```
h x = x : (h x)
g xs = [head xs, head xs - 1]

g (h 2)  = let {xs = (h 2)}          in [head xs, head xs - 1]
         = let {xs = (2 : (h 2))}    in [head xs, head xs - 1]
         = let {xs = (2 : (h 2))}    in [2,       head xs - 1]
      = ....
```

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Call by **value**

```
h x = x : (h x)
g xs = [head xs, head xs - 1]

g (h 2)  = let {xs = (h 2)}                  in [head xs, head xs - 1]
         = let {xs = (2 : (h 2))}            in [head xs, head xs - 1]
         = let {xs = (2 : (2 : (h 2)))}      in [head xs, head xs - 1]
         = let {xs = (2 : (2 : (2 : (h 2))))} in [head xs, head xs - 1]
         = ....
```

All the above assuming g (h 2) is entered at the GHCi prompt
and thus needs to be printed in full by it.

https://stackoverflow.com/questions/61601125/haskell-semantics-call-by-name-value

Reductions in the expression **f x**

Given an **expression f x**

Call-by-value:	Evaluate **x** to **v**
	Evaluate **f** to **λy.e**
	Evaluate **[y/v]e**
Call-by-name:	Evaluate **f** to **λy.e**
	Evaluate **[y/x]e**
Call-by-need:	Allocate a thunk v for **x**
	Evaluate **f** to **λy.e**
	Evaluate **[y/v]e**

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **value** (1)

Call by value is an extremely common evaluation model.
Many programming languages both imperative and functional
use this evaluation strategy.

The essence of **call-by-value** is that
there are two categories of expressions: **terms** and **values**.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **value** (2)

Values are **lambda expressions** and other **terms**
which are in **normal form** and cannot be reduced further.

All **arguments** to a **function** will be reduced to **normal form**
before they are bound inside the lambda and
reduction only proceeds once the **arguments** are reduced.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **value** (3)

For a simple arithmetic expression, the reduction proceeds as follows. Notice how the subexpression (2 + 2) is evaluated to normal form before being bound.

(\x. \y. y x) (2 + 2) (\x. x + 1)
=> (\x. \y. y x) 4 (\x. x + 1)
=> (\y. y 4) (\x. x + 1)
=> (\x. x + 1) 4
=> 4 + 1
=> 5

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **name** (1)

In **call-by-name** evaluation,
the **arguments** to lambda expressions are substituted as is,
evaluation simply proceeds from left to right
substituting the outermost lambda or reducing a value.

If a substituted expression is not used it is never evaluated.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **name** (2)

For example, the same expression we looked at for **call-by-value**
has the same normal form but arrives at it
by a different sequence of reductions:

(\x. \y. y x) (2 + 2) (\x. x + 1)
=> (\y. y (2 + 2)) (\x. x + 1)
=> (\x. x + 1) (2 + 2)
=> (2 + 2) + 1
=> 4 + 1
=> 5

Call-by-name is non-strict, although very few languages use this model.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **need** (1)

Call-by-need is a special type of non-strict evaluation
in which unevaluated **expressions** are represented
by **suspensions** or **thunks** which are passed
into a **function** unevaluated and
only evaluated when needed or forced.

When the **thunk** is forced
the **representation** of the **thunk** is updated
with the computed **value**
and is not recomputed upon further reference.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **need** (2)

The **thunks** for unevaluated lambda expressions
are allocated when evaluated,
and the resulting computed **value**
is placed in the same **reference**
so that subsequent **computations** share the result.

If the **argument** is never needed
it is never computed,
which results in a trade-off
between **space** and **time**.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by **need** (3)

Since the evaluation of subexpression
does not follow any pre-defined order,
any impure functions with side-effects
will be evaluated in an unspecified order.

As a result call-by-need can only effectively
be implemented in a purely functional setting.

http://dev.stephendiehl.com/fun/005_evaluation.html

Call by value (3)

For a simple arithmetic expression,
the reduction proceeds as follows.
Notice how the subexpression (2 + 2) is evaluated
to **normal form** before being bound.

(\x. \y. y x) (2 + 2) (\x. x + 1)
=> **(\x. \y. y x) 4 (\x. x + 1)**
=> **(\y. y 4) (\x. x + 1)**
=> **(\x. x + 1) 4**
=> **4 + 1**
=> *5*

http://dev.stephendiehl.com/fun/005_evaluation.html

References

[1] ftp://ftp.geoinfo.tuwien.ac.at/navratil/HaskellTutorial.pdf

[2] https://www.umiacs.umd.edu/~hal/docs/daume02yaht.pdf