Residue Integrals and Laurent Series with non-annular region

20170215

Based on

T.J. Cavicchi, Digital Signal Processing

Complex Analysis for Mathematics and Engineering
J. Mathews

Residue Theorem

D: simply connected domain

C: simple closed contour (CCW) in D

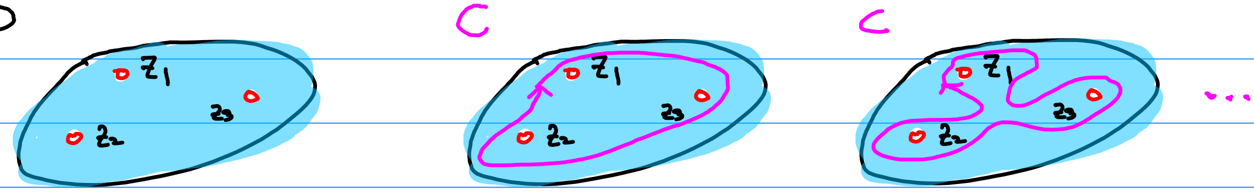
if $f(z)$ is analytic inside C and on C

except at the points $\boxed{z_1, z_2, \dots, z_k}$ in C

then $$\boxed{\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \operatorname{Res}\left(f(z), z_j\right)}$$

singular points of $f(z)$: $z_1, z_2, \dots, z_k$

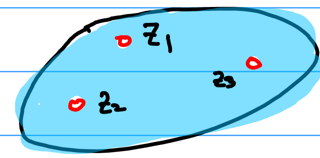
D C C



Integration of a function of a complex var.

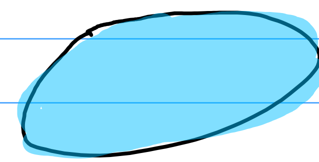
$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

finite number k of
Singular points z_k
residue theorem



$$\boxed{\oint_C f(z)\,dz} = 0 \qquad \text{if } f(z) \text{ is analytic within and on } C$$

no singularity



$$\boxed{\oint_C f(z)\,dz} = 0 \qquad \text{if } \boxed{f(z) = F'(z)} \text{ on } C$$

: $F(z)$ is an antiderivative of $f(z)$
fundamental theorem of calculus

Thomas J. Cavicchi
Digital Signal Processing, Wiley, 2000

$\boxed{\oint_C f(z)\,dz} = 0$ if $f(z)$ is continuous in D and

$\boxed{f(z) = F'(z)}$: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

Series Expansion

can expand $f(z)$ about **any point z_m**
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m
or at other points between z and z_m

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① **Laurent Series** Expansion of $f(z)$ at z_m
general n_1 — depend on $f(z)$ and z_m

② **z-transform** of $a_n^{\{m\}}$
general n_1 — depend on $f(z)$
$\boxed{z_m = 0}$

③ **Taylor Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ and z_m $(n_1 > 0)$

④ **MacLaurin Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ $(n_1 > 0)$
$\boxed{z_m = 0}$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$n_1 > 0$ pos powers

① Laurent Series	③ Taylor Series
② z-transform	④ MacLaurin Series

$\boxed{z_m = 0}$

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* Expansion of $f(z)$ about any point z_m over powers of $(z - z_m)$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$

for general $f(z)$

$$a_n^{\{m\}} = \sum_k \mathrm{Res}\left[\frac{f(z)}{(z - z_m)^{n+1}}, z_k\right]$$

for general $f(z)$

$$\boxed{a_n^{\{m\}} = \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0}$$

for analytic $f(z)$ within C

analytic $f(z)$ $\longrightarrow$ $\dfrac{f(z)}{(z - z_m)^{n+1}}$ has a pole at z_m order of $n+1$

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$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$\boxed{z_m}$: possible poles of $f(z)$
not necessarily poles

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$

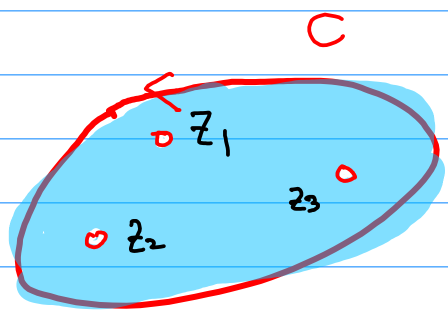
$\boxed{z_k}$: poles of $\frac{f(z)}{(z-z_m)^{n+1}}$
within C

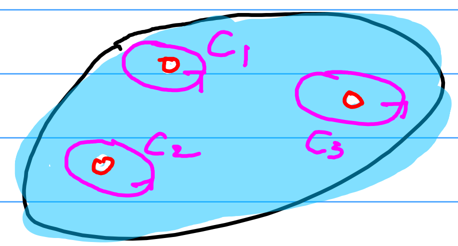
$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$

Residue Theorem and Laurent Series

assumed there are (K) singularities (poles) of $f(z)$ in a region

let $\boxed{C_k}$ is taken to enclose only one pole z

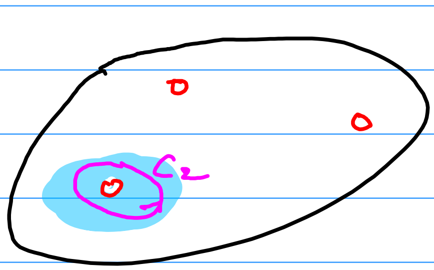




$a_n^{\{1\}}$ expanded at z_1

C_1 encloses z_1 only

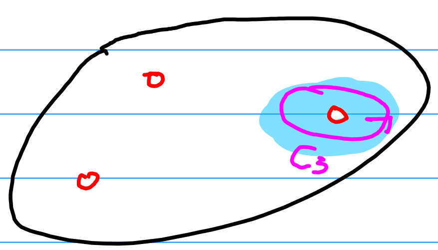
$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$



$a_n^{\{2\}}$ expanded at z_2

C_2 encloses z_2 only

$\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$



$a_n^{\{3\}}$ expanded at z_3

C_3 encloses z_3 only

$\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$

Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

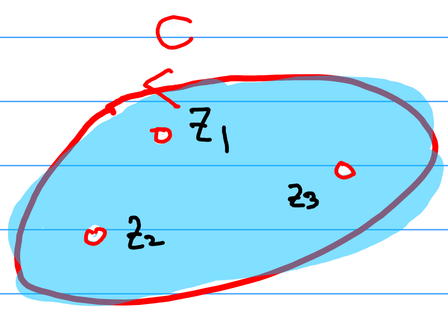
$z_1, z_2, \cdots, z_n$ within C

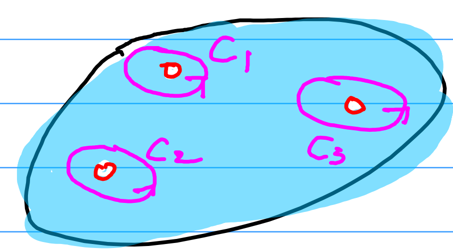
then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D





$\boxed{z_1}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k \qquad a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$

$\boxed{z_2}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k \qquad a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$

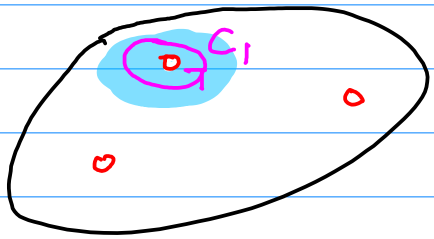
$\boxed{z_3}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k \qquad a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$

Laurent Series with Annular Region

expanded at each pole of $f(z)$

$\boxed{z_1}$ Laurent series expansion at z_1

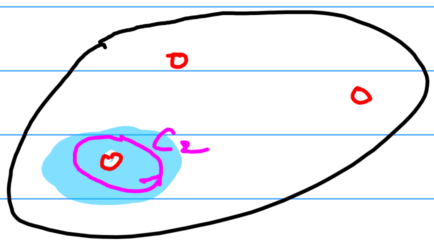
$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{1\}} (z - z_1)^n}$$



$$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1) = \frac{1}{2\pi i} \oint_{C_1} f(z)\,dz$$

$\boxed{z_2}$ Laurent series expansion at z_2

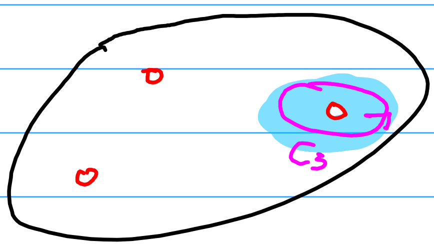
$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{2\}} (z - z_2)^n}$$



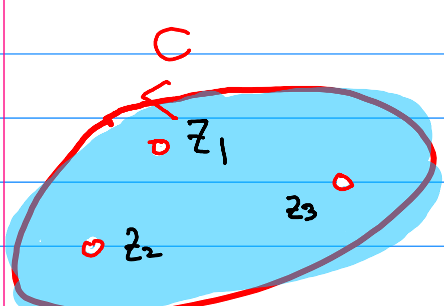
$$\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2) = \frac{1}{2\pi i} \oint_{C_2} f(z)\,dz$$

$\boxed{z_3}$ Laurent series expansion at z_3

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{3\}} (z - z_3)^n}$$

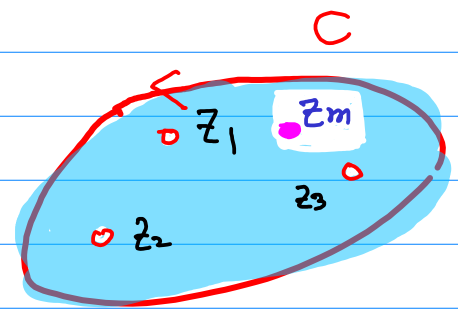


$$\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3) = \frac{1}{2\pi i} \oint_{C_3} f(z)\,dz$$



$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

Residue Theorem + Laurent Series

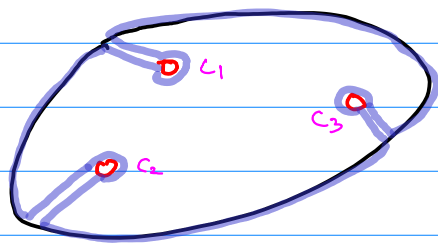


$*$ Whether z_m is singular or not

$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} \, dz$$

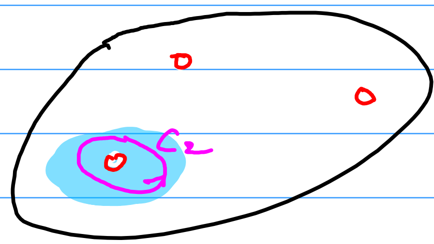
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

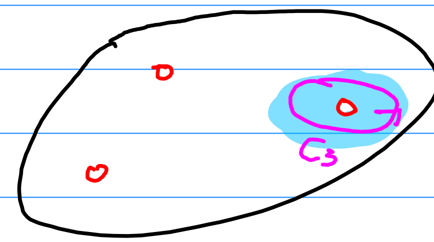


$n=-1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$





$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$ $\quad$ $\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$ $\quad$ $\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$

$$\boxed{a_{-1}^{\{m\}} = \tilde{a}_{-1}^{\{1\}} + \tilde{a}_{-1}^{\{2\}} + \tilde{a}_{-1}^{\{3\}}}$$

$$a_{-1}^{\{m\}} = \boxed{\begin{aligned} &\mathrm{Res}(f(z), z_1) \\ + &\mathrm{Res}(f(z), z_2) \\ + &\mathrm{Res}(f(z), z_3) \end{aligned}}$$

Laurent Series coefficient $a_{-1}^{\{i\}}$

- singular center z_i
- annular region

This can be a residue if

- singular center z_m
- annular region

Computing $a_n^{\{m\}}$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \qquad \boxed{n \leftarrow k}$$

$$f(z) = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^k$$

for a given n

$$\frac{f(z)}{(z-z_m)^{n+1}} = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1}$$

k : index variable
n : fixed value

$$\left[\begin{aligned} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz &= \oint_C \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1} dz \\ &= \sum_{k=n_1}^{\infty} \oint_C a_k^{\{m\}} (z - z_m)^{k-n-1} dz \end{aligned} \right] \qquad \boxed{k = n}$$

$$\oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz = \oint_C a_n^{\{m\}} \frac{1}{(z - z_m)} dz = 2\pi i \cdot a_n^{\{m\}}$$

$$\boxed{a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz}$$

$$\left\{ \begin{aligned} &\oint_C \underbrace{\boxed{\cdots (z - z_m)^{-3} + (z - z_m)^{-2}}}_{\rightarrow 0} + \boxed{\frac{1}{(z - z_m)}} + \underbrace{\boxed{1 + (z - z_m) + (z - z_m)^2 + \cdots}}_{\rightarrow 0} dz \\ &= \oint_C \boxed{\frac{1}{(z - z_m)}} dz = 2\pi i \end{aligned} \right.$$

Computing $a_n^{\{m\}}$ using Residues

expansion at z_m

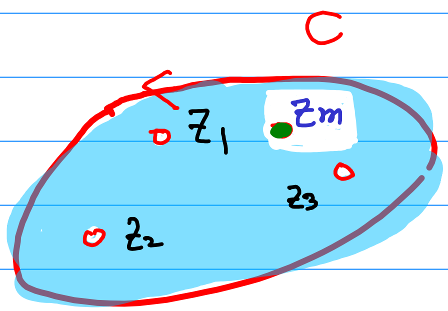
$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

$\boxed{n=-1}$ $\quad n+1=0 \quad (z-z_m)^{n+1}=1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} = \mathrm{Res}\left(f(z), z_m\right) = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z-z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

Residue → Laurent series → annular region

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}$$

$$\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i} \oint_C f(z)\, dz \\ &= \sum_k \mathrm{Res}\left(f(z), z_k \right) \end{aligned}$$

$$\vdots$$

$$a_{-3}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^2, z_k \right)$$

$$a_{-2}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^1, z_k \right)$$

$$a_{-1}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z), z_k \right)$$

$$a_{0}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^1}, z_k \right)$$

$$a_{1}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^2}, z_k \right)$$

$$a_{2}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^3}, z_k \right)$$

$$\vdots$$

Poles for Residue Computation

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

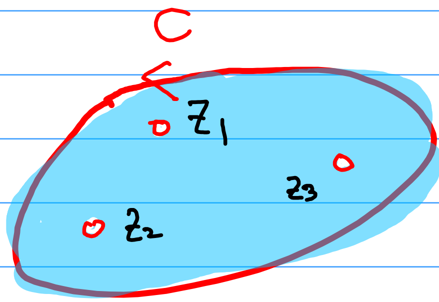
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$

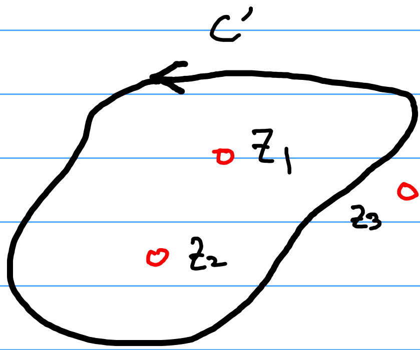
z_k within C : singularities of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$

$n \geq 0$ $\quad \{ \text{poles of } f(z) \} \cup \{ z = z_m \}$ $\quad n = 0, 1, 2 \cdots$

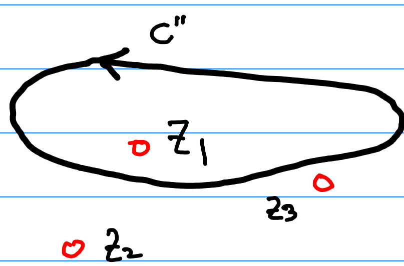
$n < 0$ $\quad \{ \text{poles of } f(z) \}$ $\quad n = -1, -2, \cdots$



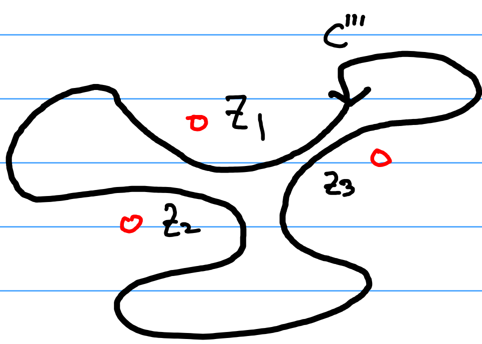
$$\int_C f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1) + 2\pi i \ \mathrm{Res}(f(z), z_2) + 2\pi i \ \mathrm{Res}(f(z), z_3)$$



$$\int_{C'} f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1) + 2\pi i \ \mathrm{Res}(f(z), z_2)$$

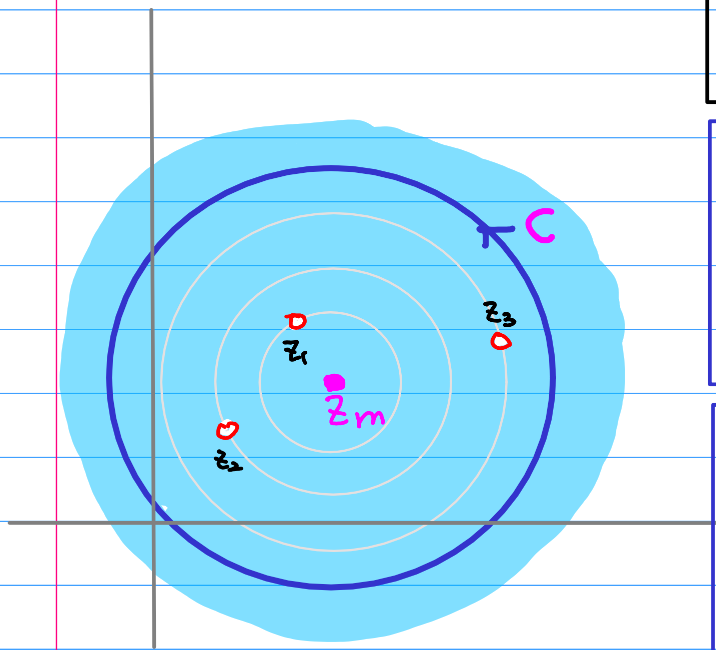


$$\int_{C''} f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1)$$



$$\int_{C''} f(z)\,dz = 0$$

Series Expansion at z_m



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n}$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$

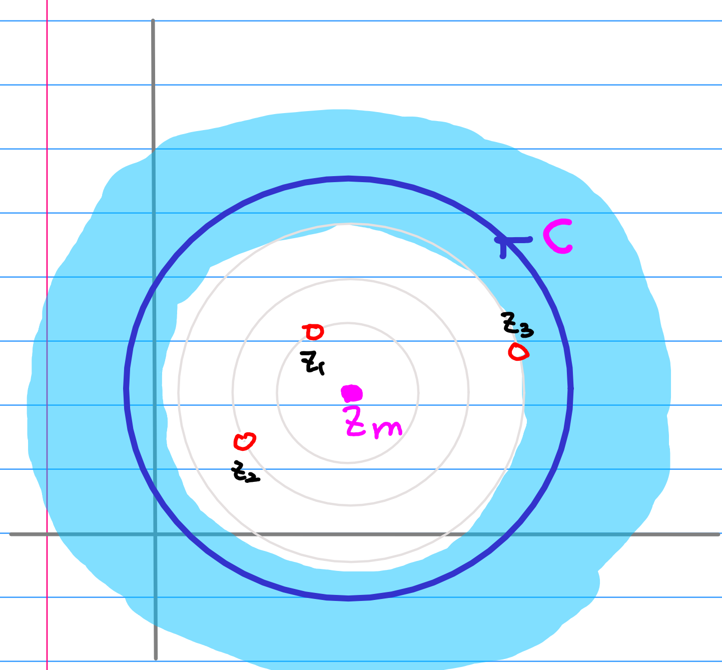
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

$n=-1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

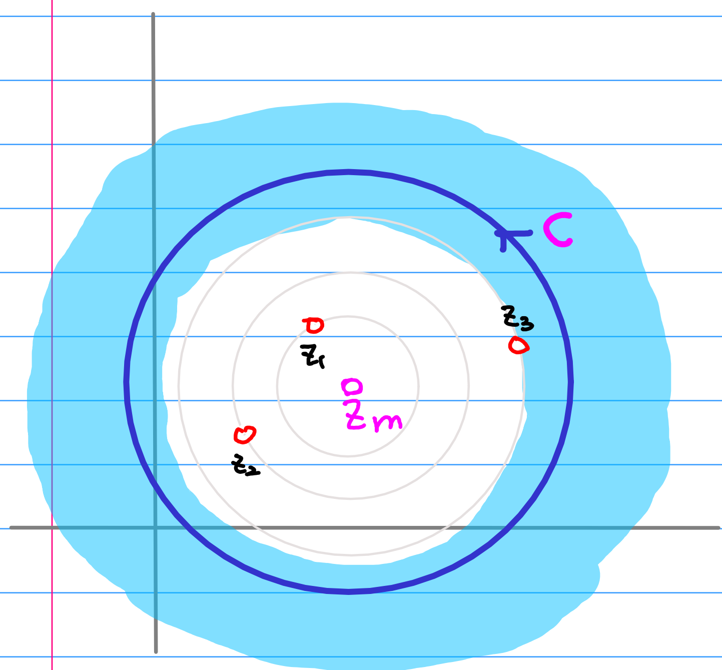
$$= \sum_k \mathrm{Res}\left(f(z), z_k \right)$$

[Annular Region]



z_m can be a pole of $\frac{f(z)}{(z - z_m)^{n+1}}$ only if $n \geq 0$

[Annular Region] & [z_m : isolated singularity]



$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} \, dz$$

$$= \sum_k \operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

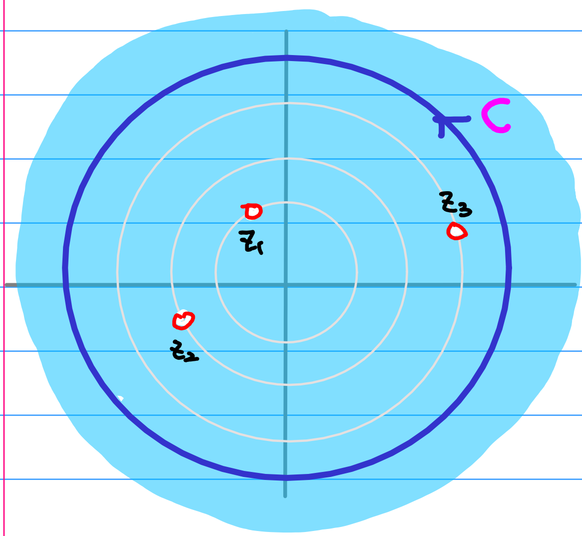
n=

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) \, dz$$

$$= \sum_k \operatorname{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} = \operatorname{Res}\left(f(z), z_m\right)$$

Series Expansion at $z=0$



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz$$

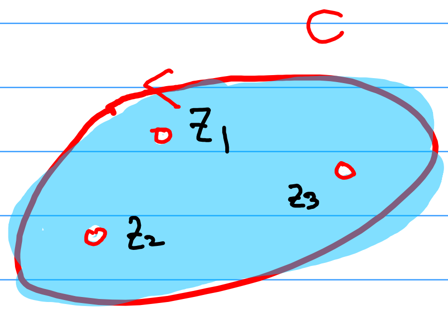
$$= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

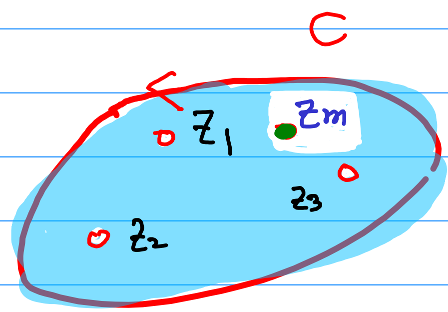
$n \geq 0$ $z_1, z_2, z_3, 0$

$n < 0$ z_1, z_2, z_3

Series Expansion at z_m no annular region



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$



$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$

$$= \sum_k \operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\ z_k\right)$$

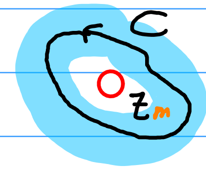
Let z_1, z_2, z_3 poles of $f(z)$

Then the poles of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$n \geq 0$	$z_1,\ z_2,\ z_3,$	z_m
$n < 0$	$z_1,\ z_2,\ z_3$	

if C encloses only one pole z_0,

and the expansion at that pole z_0 is assumed,



then

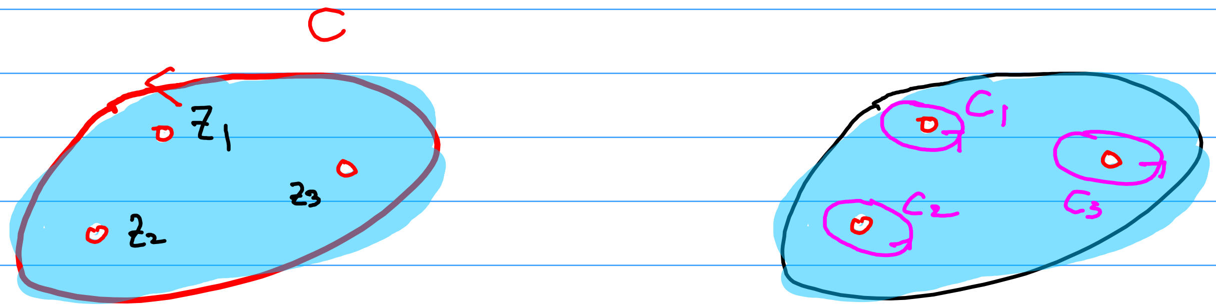
$$a_{-1}^{\{0\}} = \frac{1}{2\pi i} \oint_{C_0} f(z)\, dz = \mathrm{Res}(f(z), z_0)$$

Let $\boxed{\tilde{a}_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m)}$ notation $\tilde{}$

the residue of $f(z)$ at z_m

using C_m which is in the anulus ROC

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{m\}} (z - z_m)^n}$$



$$\boxed{\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{M} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{M} \mathrm{Res}(f(z), z_k)}$$

residue theorem

$$\boxed{a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)}$$

Laurent coefficient

C encloses k poles

C_k encloses only the k-th pole

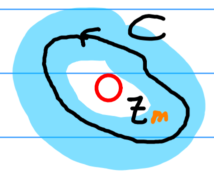
$\tilde{a}_{-1}^{\{k\}}$ the residue of the k-th pole enclosed by C, z_k

Non-anular region

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$



$\boxed{C}$ is in the same region of analyticity of $f(z)$

typically a circle centered on z_m non-annular ok

$\boxed{z_k}$ within C : singularities of $\boxed{\frac{f(z)}{(z-z_m)^{n+1}}}$

$\boxed{n_1 = n_{f,m}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$

$\boxed{a_n^{\{m\}}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$, $\boxed{\text{region of analyticity}}$

Whether $f(z)$ is singular at $z = z_m$ or not

or at other points between z and z_m

We can expand $f(z)$ about any point z_m

over powers of $(z - z_m)$.

Laurent's Theorem

f : analytic within the annular domain D

$$\boxed{r < |z - z_0| < R}$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k ,$$

valid for $\boxed{r < |z - z_0| < R}$

The coefficients a_k are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds , \qquad k = 0, \pm 1, \pm 2, \cdots$$

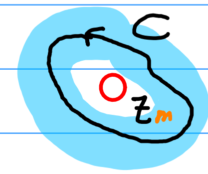
C : a simple closed curve
that lies entirely within D
that encloses z_0

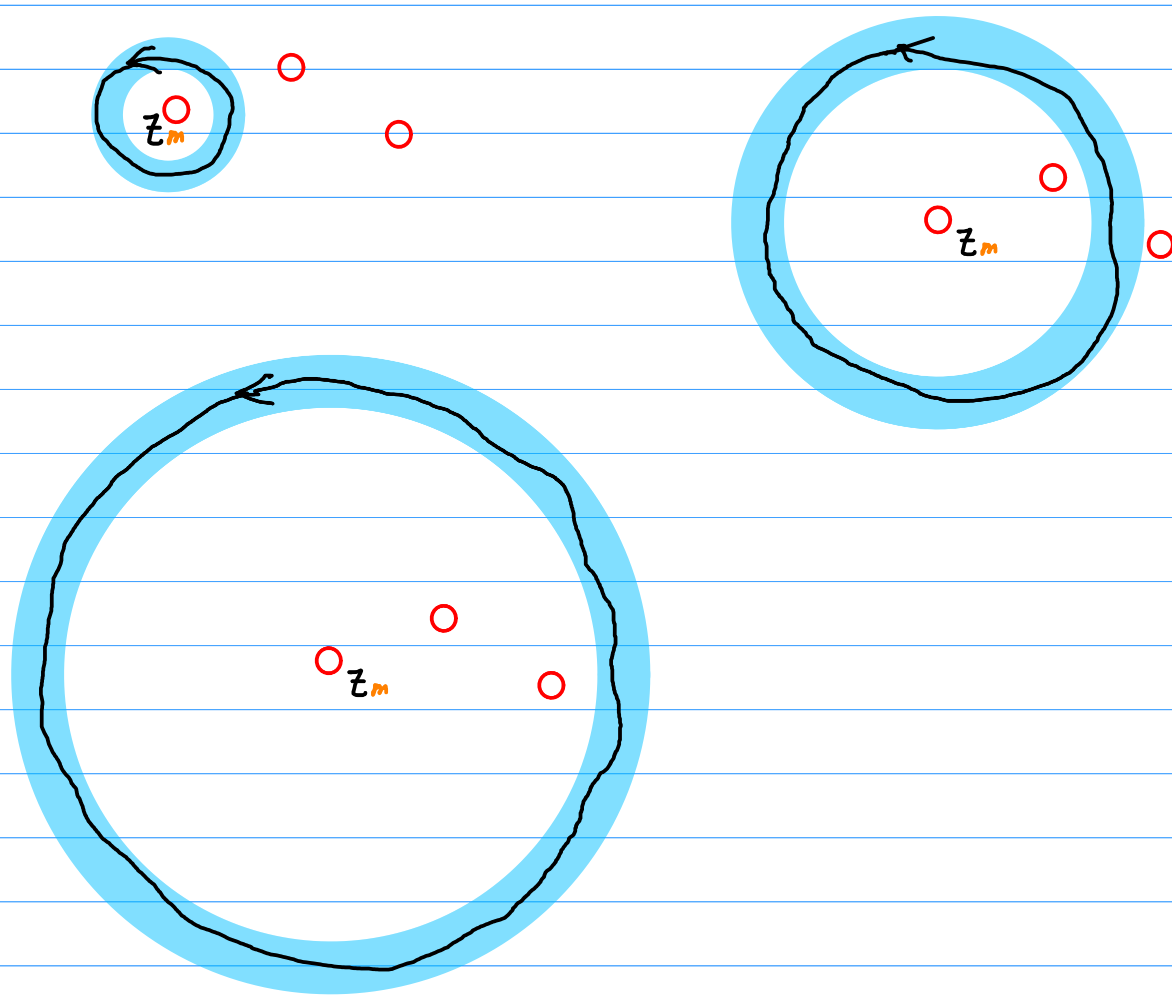
Curve C & Domain D of the Laurent Series

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

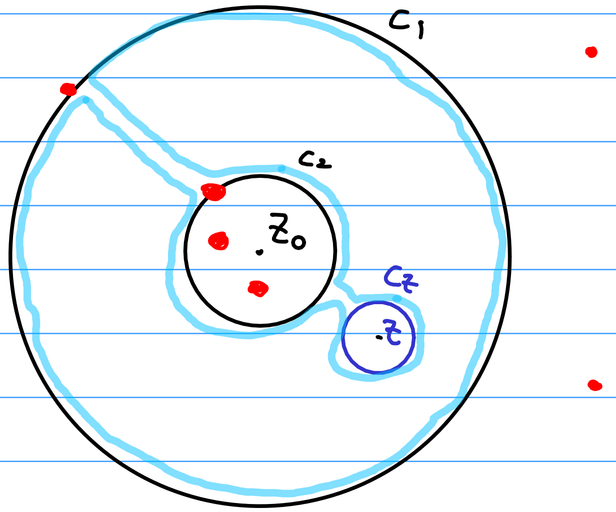
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}}\, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$





Expansion Points and Evaluation Points



z_0 : expansion point

z : evaluation point

which poles of $f(z)$ lie between the point of evaluation z and the point z_0 about which the expansion is formed

$$\frac{f(z')}{(z'-z_0)}$$ is analytic between C_1 & C_2

deformation theorem $C_1 - C_2$ coincide

common contour C

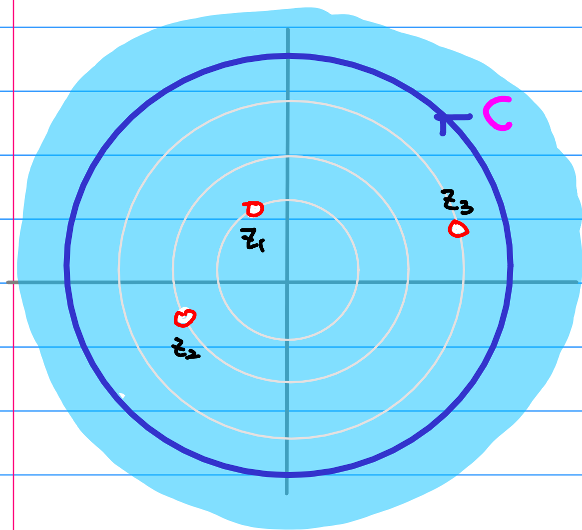
Residues

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds \Rightarrow \oint_C f(s)\, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

Series Expansion at $z=0$



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n$$

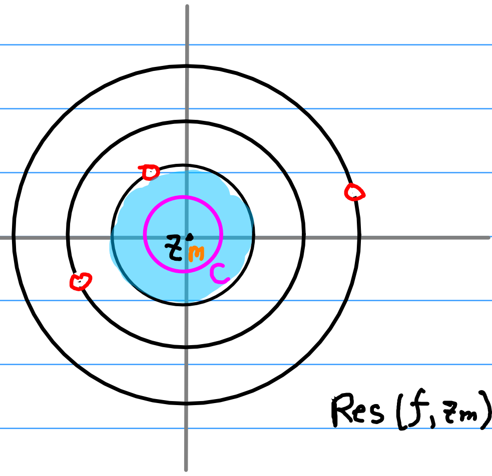
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

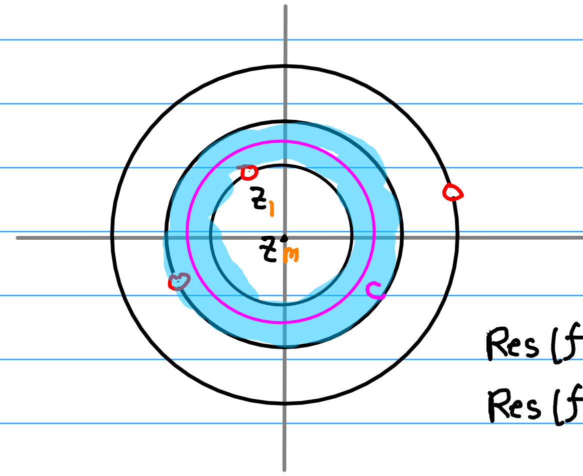
Poles z_k

$n \geq 0$ $\quad z_1, z_2, z_3, 0$

$n < 0$ $\quad z_1, z_2, z_3$

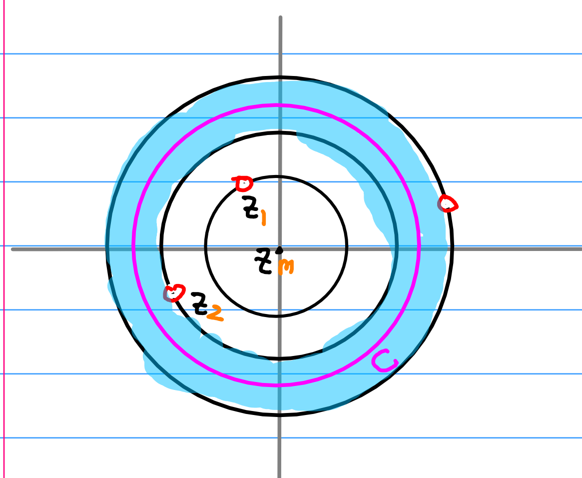


$\mathrm{Res}(f, z_m)$

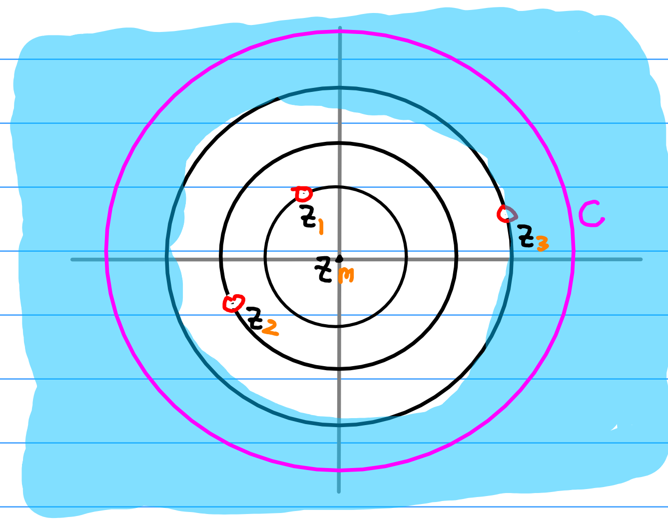


$\mathrm{Res}(f, z_1)$

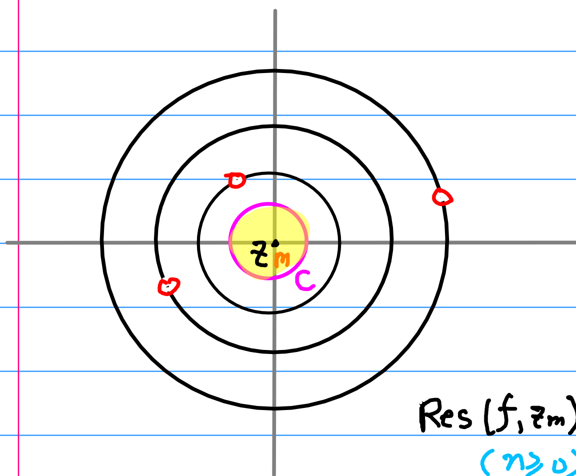
$\mathrm{Res}(f, z_m)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

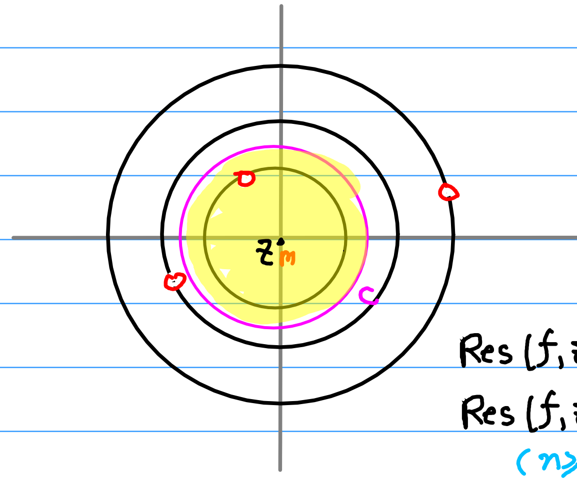


$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$
$+ \mathrm{Res}(f, z_m)$



$\mathrm{Res}(f, z_m)$

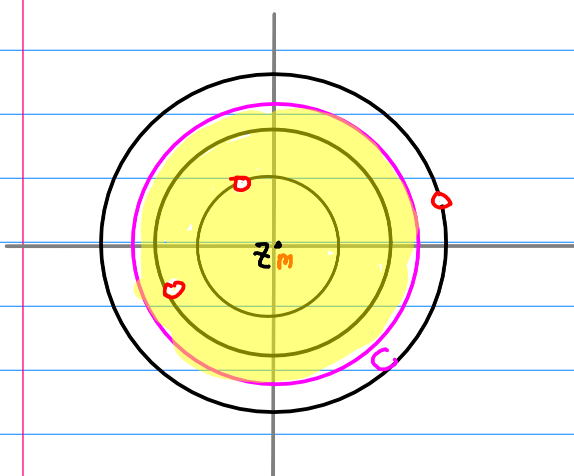
$(n \geq 0)$



$\mathrm{Res}(f, z_1)$

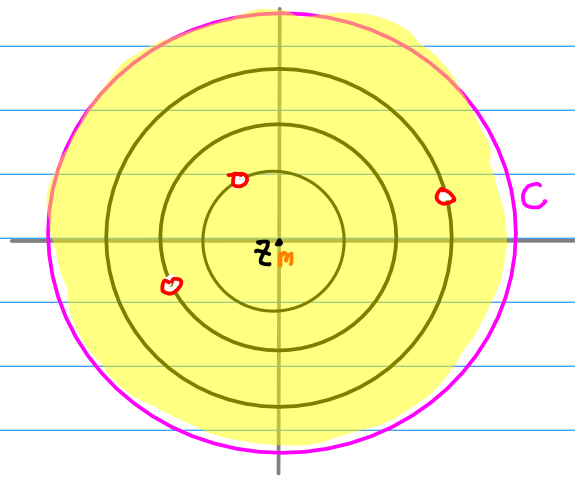
$\mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$

$+ \mathrm{Res}(f, z_m)$

$(n \geq 0)$

Inverse z-Transform $x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$

$$X(z) = \sum_{k=0}^{\infty} x_k\, z^{-k}$$

$$z^{n-1} X(z) = \left(\sum_{k=0}^{\infty} x_k\, z^{-k}\right) z^{n-1}$$

$$\boxed{\int z^{n-1}\, \text{LHS}\, dz = \int \text{RHS}\, z^{n-1}\, dz}$$

$$= \sum_{k=0}^{\infty} x_k\, z^{-k+n-1}$$

$$\boxed{[0, \infty) = [0, n-1] \cup [n] \cup [n+1, \infty)}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \sum_{k=n}^{n} x_k\, z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \frac{x_n}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z)\, z^{n-1}\, dz = \int_c \sum_{k=0}^{n-1} x_k\, z^{-k+n-1}\, dz + \int_c \frac{x_n}{z^1}\, dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1}\, dz + x_n \int_c \frac{1}{z^1}\, dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_n \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x[n] = \frac{1}{2\pi i} \int_c X(z)\, z^{n-1}\, dz$$

Z-transform

$\boxed{z_m = 0}$

$$x[n] = \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz = \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right)$$

$n > 0$ $\quad z_k$: poles of $f(z)$

$n = 0$ $\quad z_k$: poles of $f(z)$ + $z = 0$

$z^{n-1} = z^{-1} = \frac{1}{z}$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains z on its numerator

Also, think about modified partial fraction $\frac{X[z]}{z}$

Laurent Expansion

expansion at z_m

$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

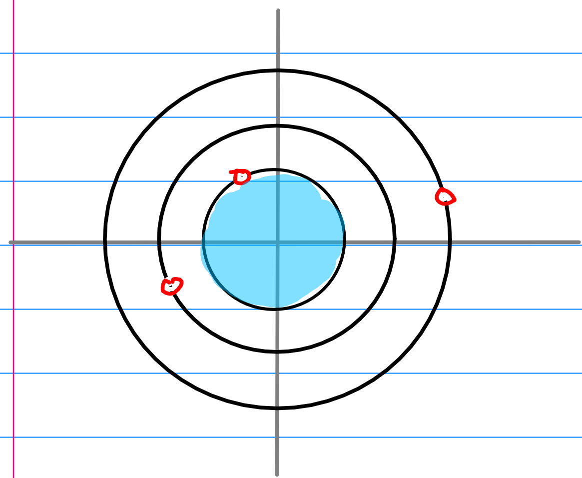
$\boxed{z_m = 0}$

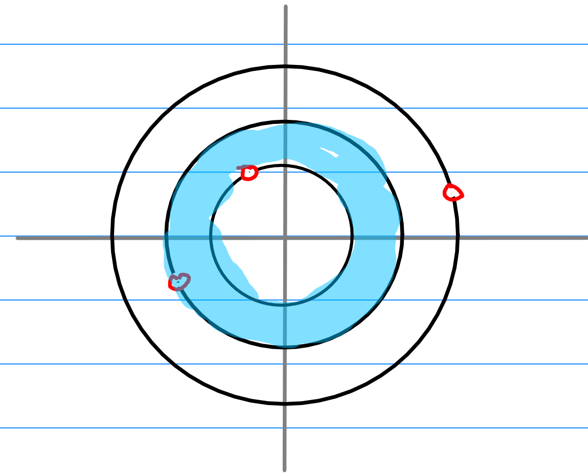
$$a_n^{\{0\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

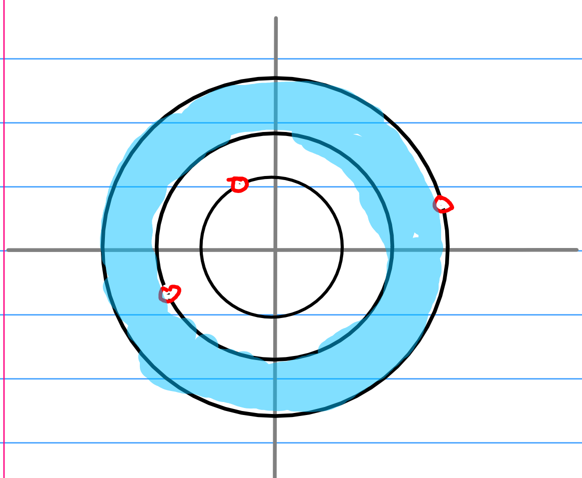
$$a_{-n}^{\{0\}} = \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz = \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right)$$

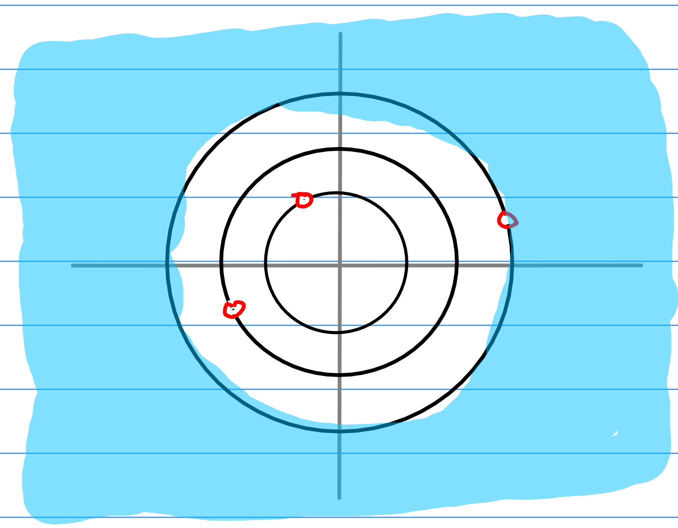
$$a_{-n}^{\{0\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{-n+1}}\, dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}}, z_k\right)$$

Different D, Different Laurent Series



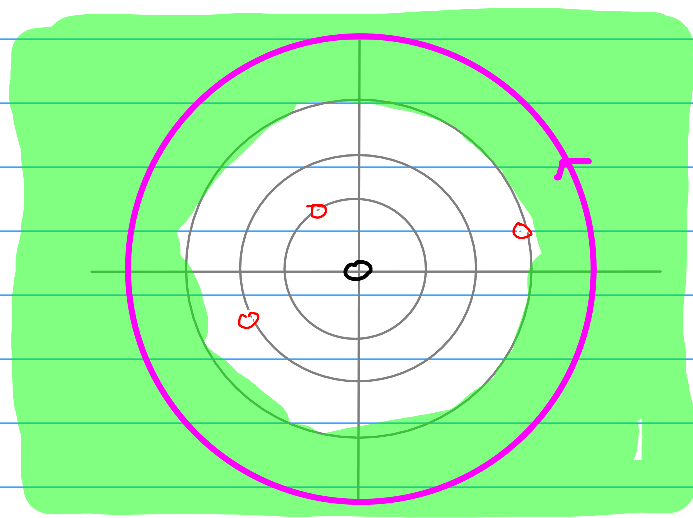






$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$



z-transform

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$D_1 : |z| < 1$

$D_2 : 1 < |z| < 2$

$D_3 : 2 < |z|$

① D_1 $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad |z|<1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

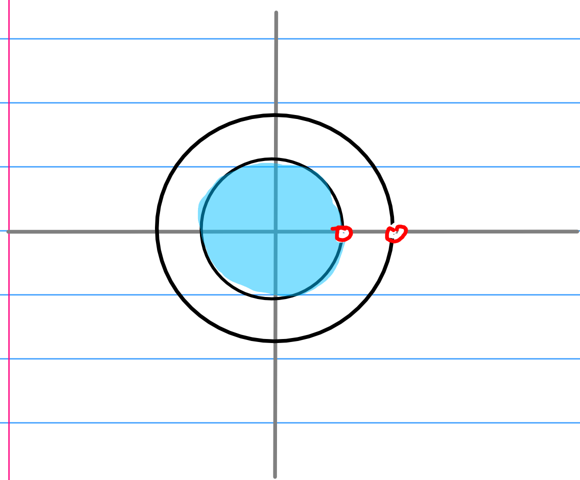
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $\quad |z|<1, \quad \left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

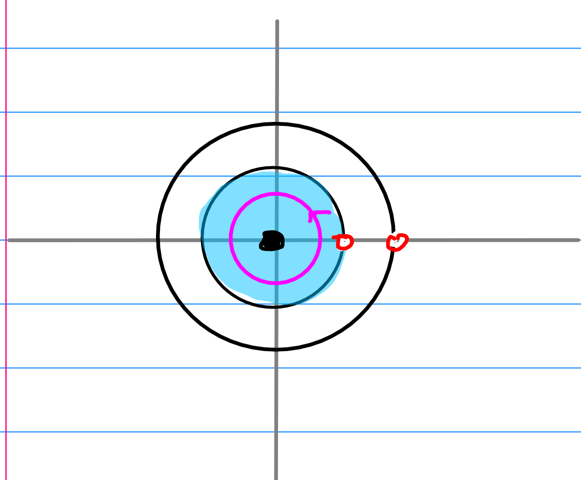


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1}-1\right)z^n \qquad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$
$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$
$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n!\left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

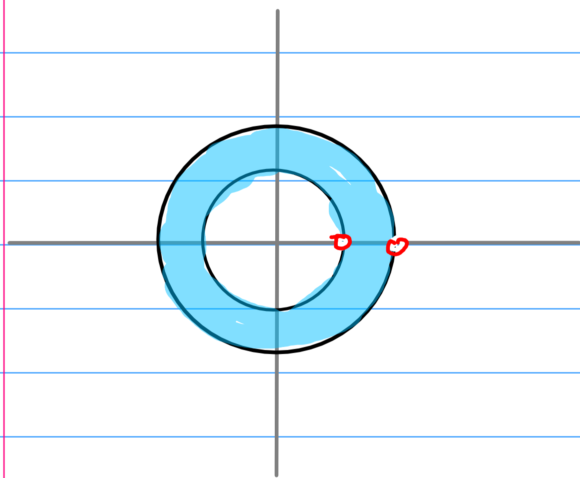
$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$
$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



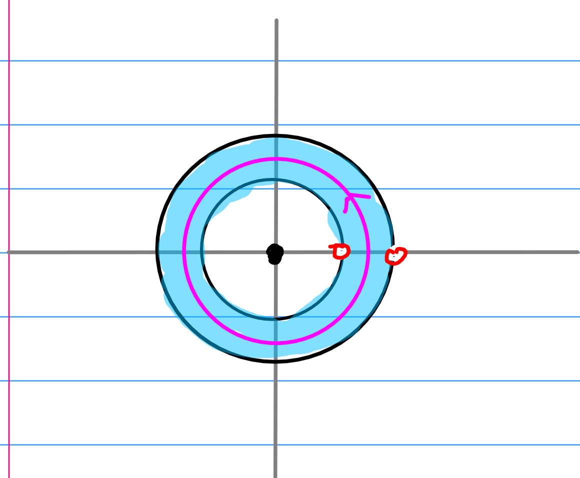
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$
$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$
$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$
$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

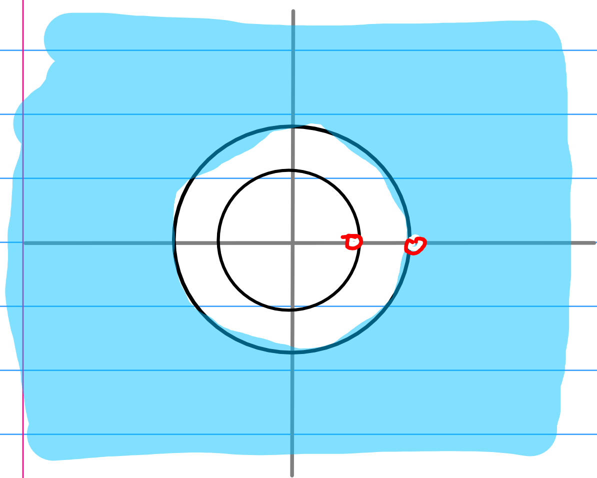
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
1	1	1	2^{-1}	2^{-2}	2^{-3}	

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

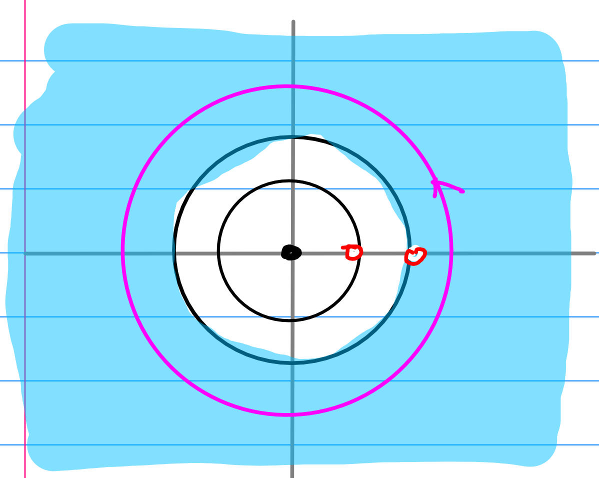
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$a_n = \sum_{k=1}^{M} \operatorname{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \operatorname{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$



$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z \to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

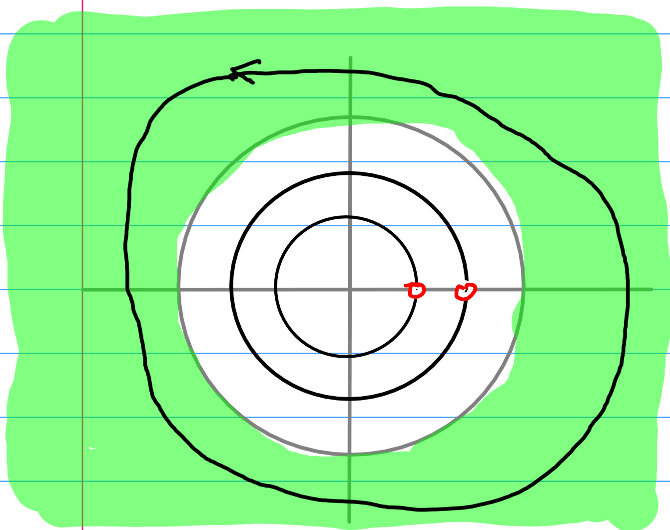
$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
0	0	0	$-1+2^{-1}$	$-1+2^{-2}$	$-1+2^{-3}$	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$
1	1	1	1	1	1	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$
-2^{2}	-2	-1	-2^{-1}	-2^{-2}	-2^{-3}	$\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$
$1-2^{2}$	$1-2$	0	0	0	0	

$$a_n = 1 - 2^{-n+1} \qquad n < 0$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1}) z^n = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$$x[n]$$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j \right)$$

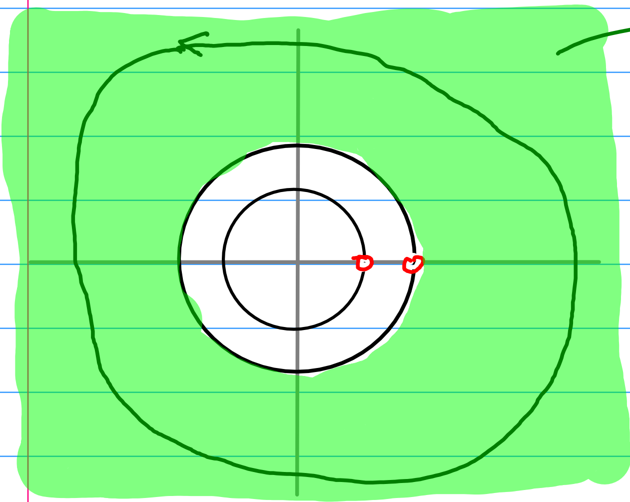
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1 \right) = (z-1) \frac{-1}{(z-1)(z-2)}\, z^{n-1} \Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2 \right) = (z-2) \frac{-1}{(z-1)(z-2)}\, z^{n-1} \Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow[\text{Converge}]{} \frac{1}{1-\frac{2}{z}}$$

$$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow[\text{Converge}]{} \frac{1}{1-\frac{1}{z}}$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots + \frac{1}{2}\left\{\left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots\right\} \xrightarrow[\text{Converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow[\text{Converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$