Cross Power Density Spectrum

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

$$W(t) = X(t) + Y(t)$$

$$R_{WW}(t, t+\tau) = E\left[W(t)W(t+\tau)\right]$$

$$= E\left[\{X(t) + Y(t)\}\{X(t+\tau) + Y(t+\tau)\}\right]$$

$$= R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau)$$

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$$S_{WW}(\omega) = S_{WW}(\omega) + S_{YY}(\omega)$$

$$+\mathscr{F}\left\{A[R_{XY}(t, t+\tau)]\right\} + \mathscr{F}\left\{A[R_{YX}(t, t+\tau)]\right\}$$

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$$= R_{XX}(t, t+\tau) + R_{YY}(t, t+\tau) + R_{XY}(t, t+\tau) + R_{YX}(t, t+\tau)$$

Definition

$$x_T(t) = \begin{cases} x(t) & -T < t < +T \\ 0 & elsewhere \end{cases}$$

$$y_T(t) = \begin{cases} y(t) & -T < t < +T \\ 0 & elsewhere \end{cases}$$

$$x_T(t) \Longleftrightarrow X_T(\omega)$$

$$y_T(t) \Longleftrightarrow Y_T(\omega)$$

Definition

$$P_{XY}(T) = \frac{1}{2T}\int_{-T}^{+T} x_T(t) y_T(t) dt$$
$$= \frac{1}{2T}\int_{-T}^{+T} x(t) y(t) dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2T} X_T^*(\omega) Y_T(\omega) d\omega$$

Definition

$$\overline{P}_{XY}(T) = \frac{1}{2T}\int_{-T}^{+T} R_{XX}(t,t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2T} E\left[X_T^*(\omega)Y_T(\omega)\right] d\omega$$

$$P_{XY} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} R_{XX}(t,t)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega)Y_T(\omega)\right] d\omega$$

Definition

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[X_T^*(\omega) Y_T(\omega)\right]$$

$$P_{XY} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) d\omega$$

$$S_{YX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[Y_T^*(\omega) X_T(\omega)\right]$$

$$P_{YX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{YX}(\omega) d\omega$$

1. $S_{XY}(\omega) = S_{YX}(-\omega) = S^*_{YX}(\omega)$
2. $\Re[S_{XY}(\omega)]$and $\Re[S_{YX}(\omega)]$are even function of ω
3. $\Im[S_{XY}(\omega)]$and $\Im[S_{YX}(\omega)]$are odd function of ω
4. $S_{XY}(\omega) = 0$and $S_{YX}(\omega) = 0$ if $X(t)$ and $Y(t)$ are orthogonal
5. if $X(t)$ and $Y(t)$ are uncorrelated and have constant mean $\overline{X}$ and $\overline{Y}$, then $S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \overline{X}\,\overline{Y}\delta(\omega)$
6. $A[R_{XY}(t, t+\tau)] \Longleftrightarrow S_{XY}(\omega)$
7. $A[R_{YX}(t, t+\tau)] \Longleftrightarrow S_{YX}(\omega)$

Definition

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{YX}(\omega) = \int_{-\infty}^{+\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) e^{+j\omega\tau} d\omega$$

$$R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{YX}(\omega) e^{+j\omega\tau} d\omega$$