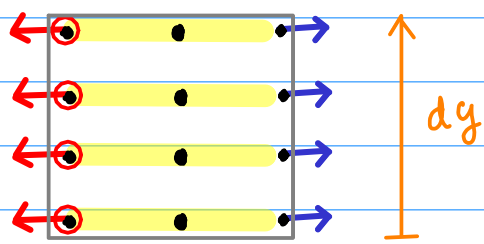
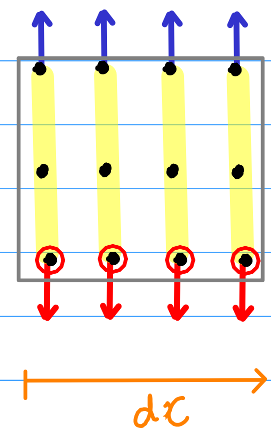
Vector Calculus (H.1) Green's Theorem

20160202

$$\boxed{\frac{\partial P}{\partial x} dA = \frac{\partial P}{\partial x} dx\, dy} \qquad \boxed{\frac{\partial Q}{\partial y} dA = \frac{\partial Q}{\partial y} dy\, dx}$$

$$dP\, dy\, \vec{i} \qquad dQ\, dx\, \vec{j}$$





Increasing / Decreasing P & Q $\quad \frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$

⇒ Outbound, Inbound P & Q

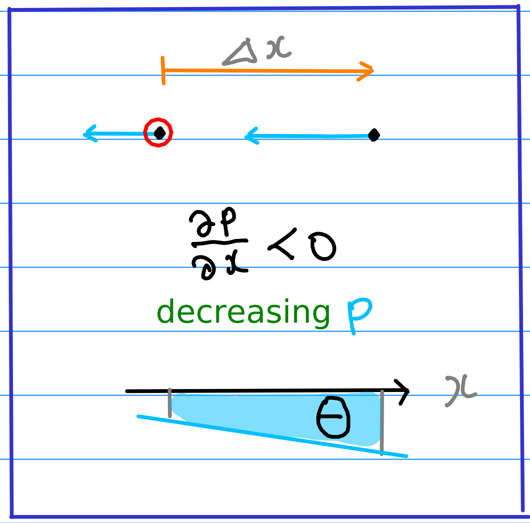
Differentials of P & Q $\quad dP, dQ$

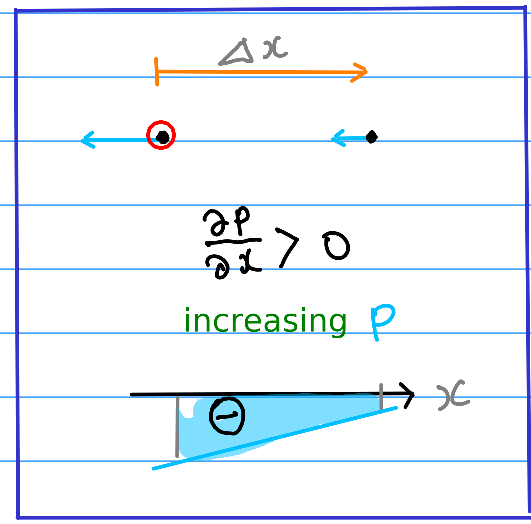
Outbound / Inbound Region

Increasing / Decreasing P & Q $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$

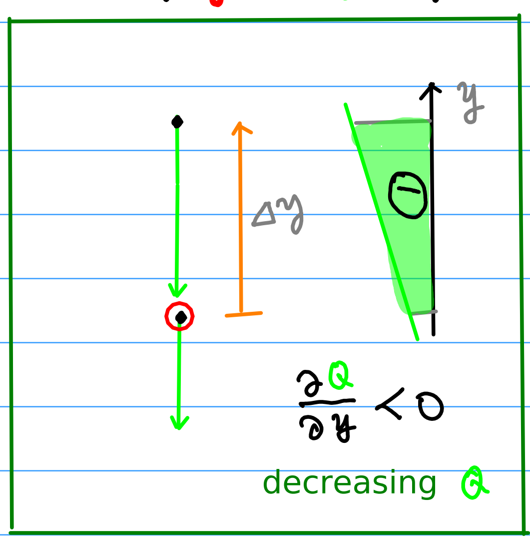
$P(x,y)\,\vec{i}$

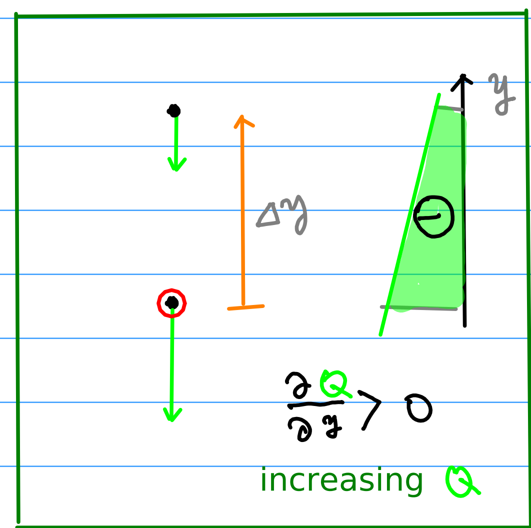
$P(x,y)$





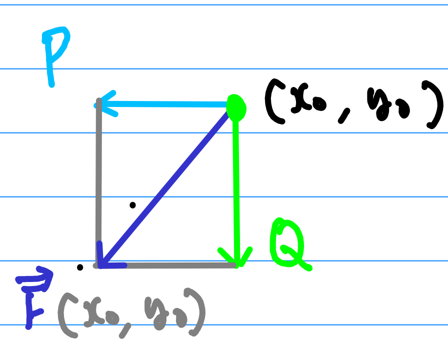
$Q(x,y)\,\vec{j}$ $Q(x,y)$



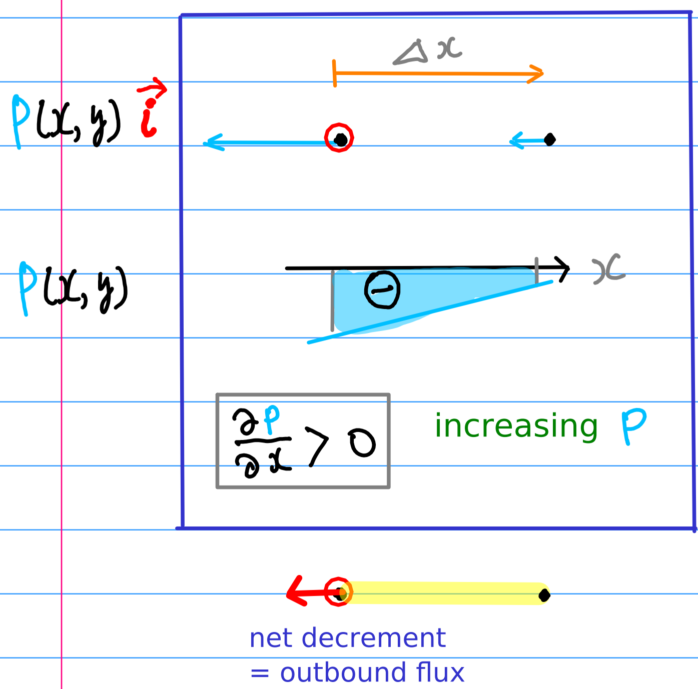


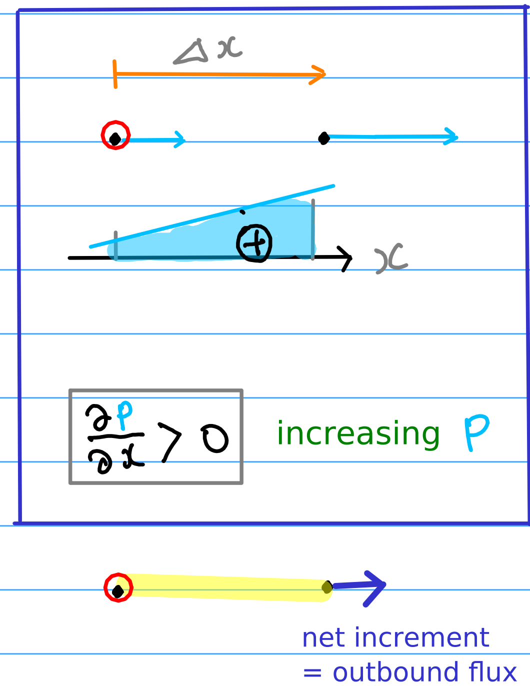
P : 1st component of a vector $\vec{F}$

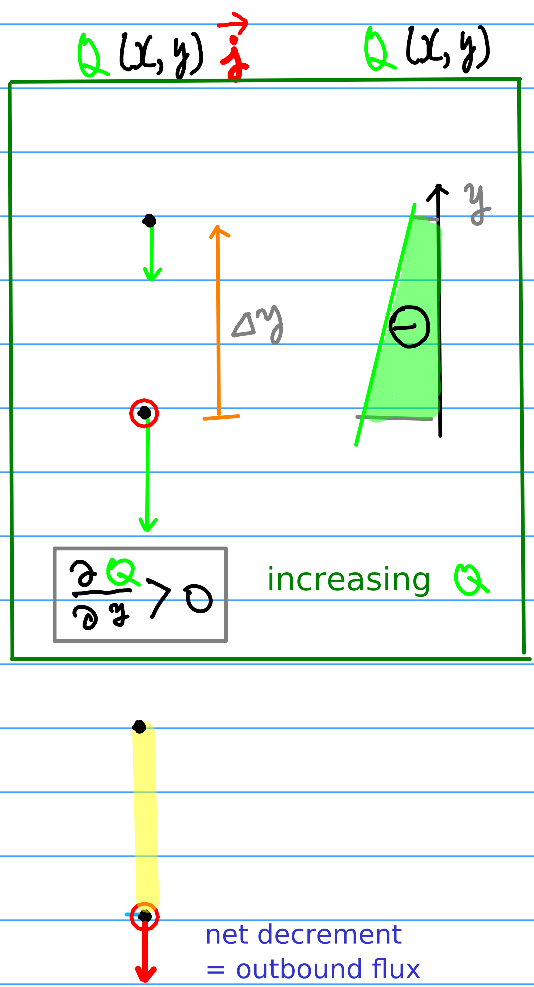
Q : 2nd component of a vector $\vec{F}$

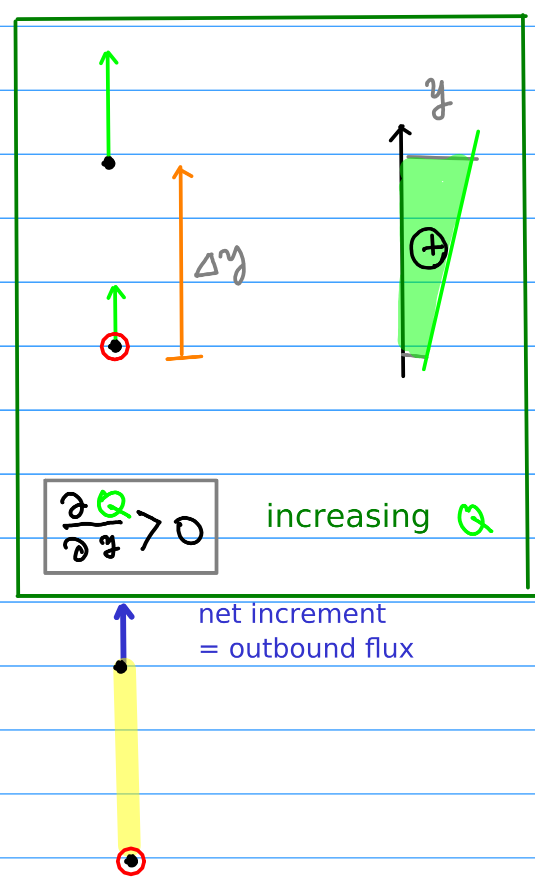


Increasing, Outbound P & Q $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$





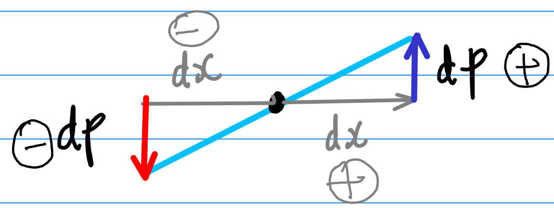




Differentials of P & Q dP, dQ

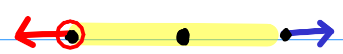
$$\boxed{\text{increasing } P \qquad \frac{\partial P}{\partial x} > 0}$$

$P(x, y)$



$$\boxed{dP = \frac{\partial P}{\partial x}\, dx}$$

$dP\, \vec{i}$

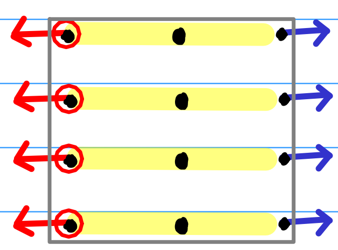


net decrement = outward flux

net increment = outward flux

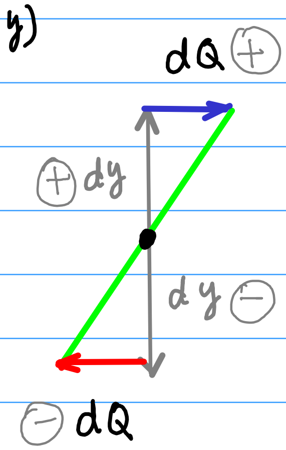
$$\boxed{\frac{\partial P}{\partial x}\, dA = \frac{\partial P}{\partial x}\, dx\, dy}$$

$dP\, dy\, \vec{i}$



$$\boxed{\text{increasing } Q \qquad \frac{\partial Q}{\partial y} > 0}$$

$Q(x, y)$



$$\boxed{dQ = \frac{\partial Q}{\partial y}\, dy}$$

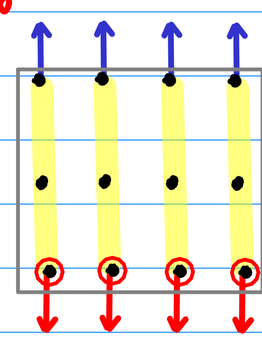
$dQ\, \vec{j}$

net increment = outward flux

net decrement = outward flux

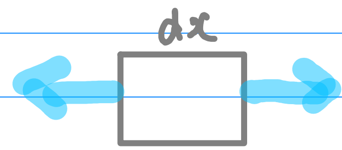
$$\boxed{\frac{\partial Q}{\partial y}\, dA = \frac{\partial Q}{\partial y}\, dy\, dx}$$

$dQ\, dx\, \vec{j}$

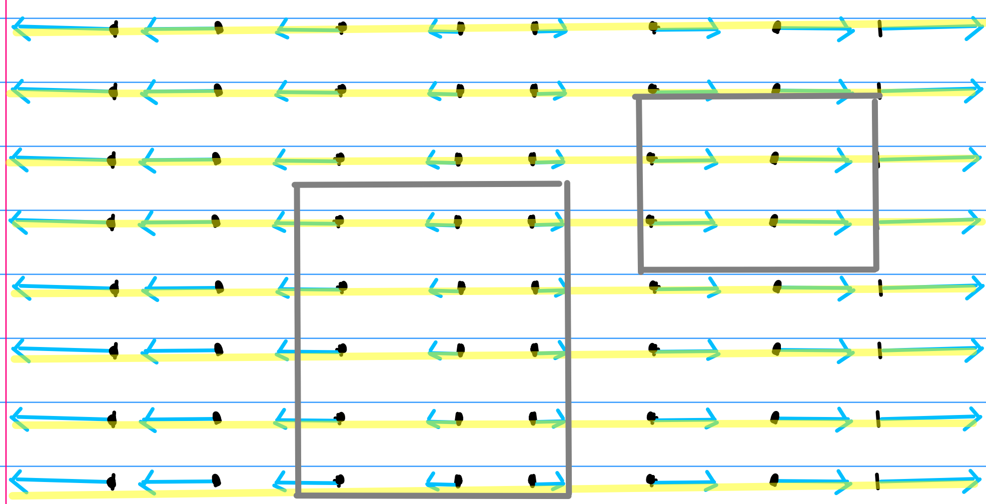


Outbound / Inbound Region

$\frac{\partial P}{\partial x} > 0$



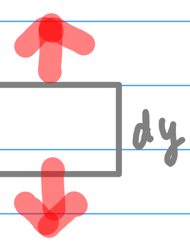
increasing = outbound



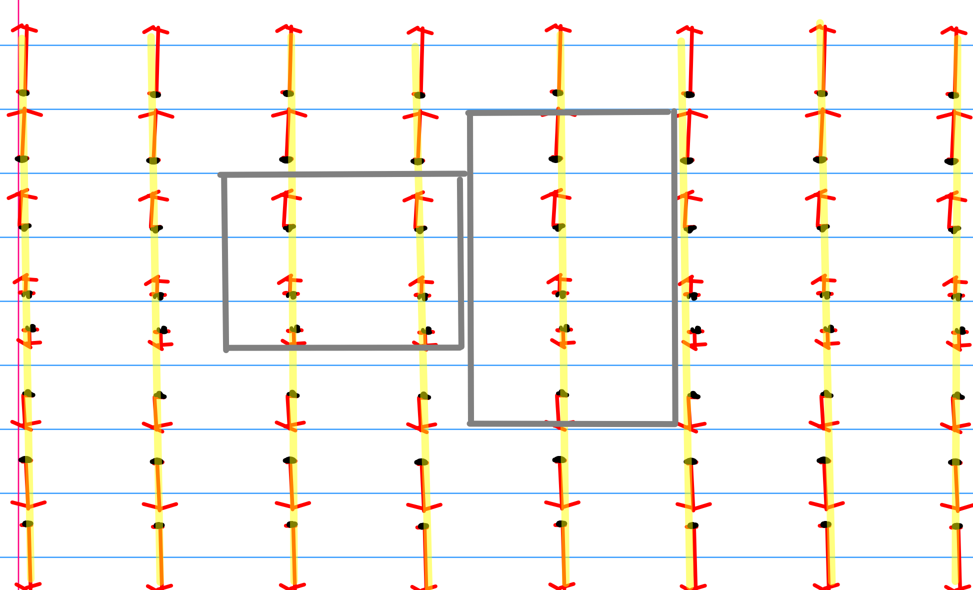
net result of $\frac{\partial P}{\partial x}$ over a region

> 0 : out bound

$\frac{\partial Q}{\partial y} > 0$



increasing = outbound



net result of $\frac{\partial Q}{\partial y}$ over a region

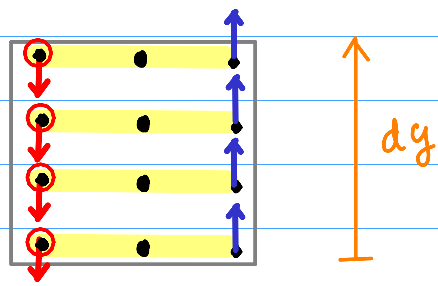
> 0 : out bound

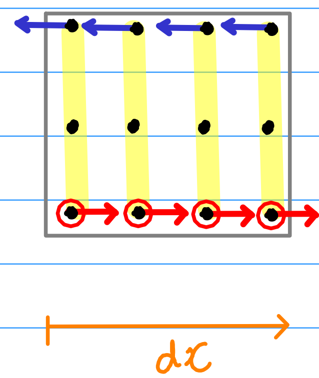
$$\boxed{\frac{\partial Q}{\partial x}\,dA = \frac{\partial Q}{\partial x}\,dx\,dy}$$

$$\boxed{-\frac{\partial P}{\partial y}\,dA = -\frac{\partial P}{\partial y}\,dy\,dx}$$

$dQ\,dy\,\vec{i}$

$dP\,dx\,\vec{j}$





Increasing / Decreasing P & Q — $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$

⇒ CCW / CW Circulation P & Q

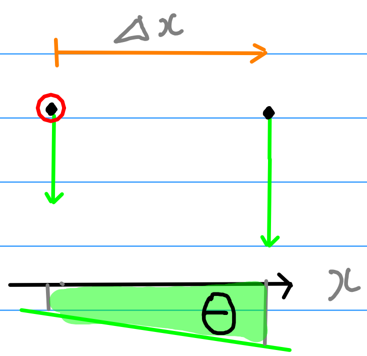
Differentials of P & Q — dQ dP

CCW / CW Rotating Region

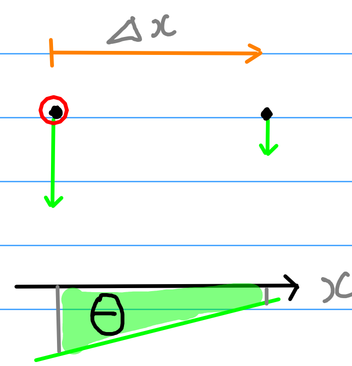
Increasing / Decreasing P & Q $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$

$Q(x,y)\,\vec{j}$

$Q(x,y)$

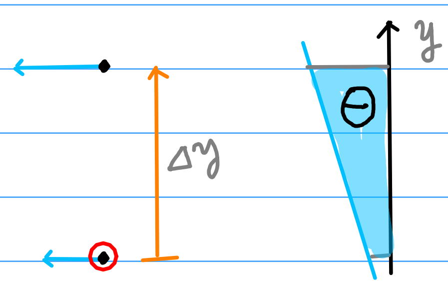


$\frac{\partial Q}{\partial x} < 0$ decreasing Q

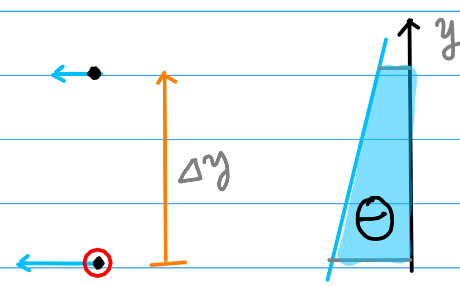


$\frac{\partial Q}{\partial x} > 0$ increasing Q

$P(x,y)\,\vec{i}$ $P(x,y)$



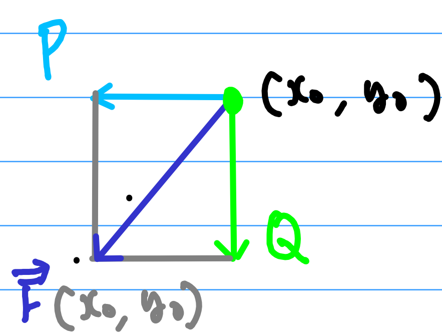
$\frac{\partial P}{\partial y} < 0$ decreasing P



$\frac{\partial P}{\partial y} > 0$ increasing P

P : 1st component of a vector $\vec{F}$

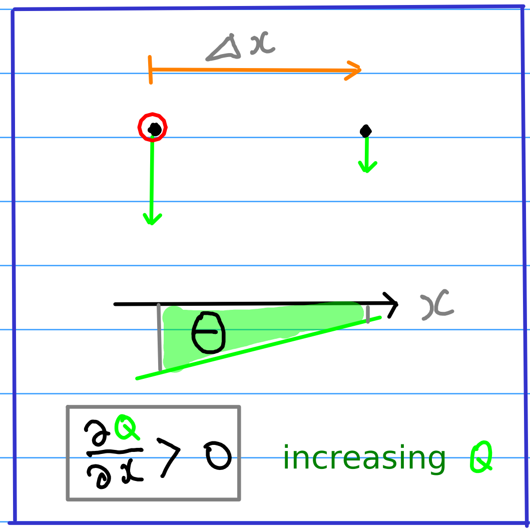
Q : 2nd component of a vector $\vec{F}$

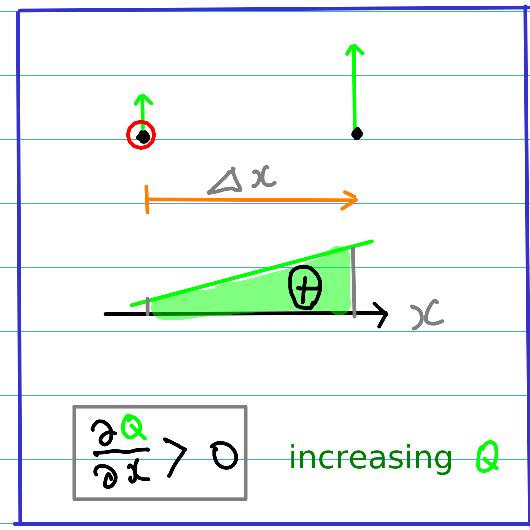


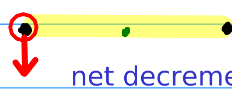
Increasing, Circulation P & Q $\frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial y}$

$Q(x,y)\vec{j}$

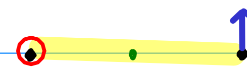
$Q(x,y)$





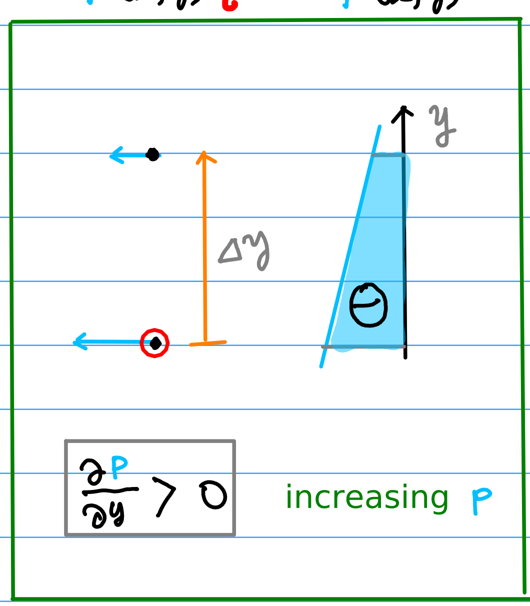


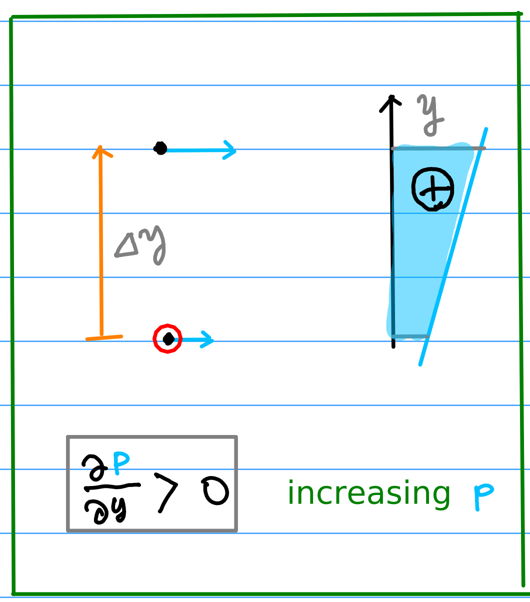
net decrement
= ccw circulation

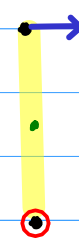


net increment
= ccw circulation

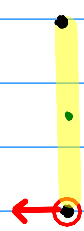
$P(x,y)\vec{i}$ $P(x,y)$







net increment
= cw circulation
= ccw



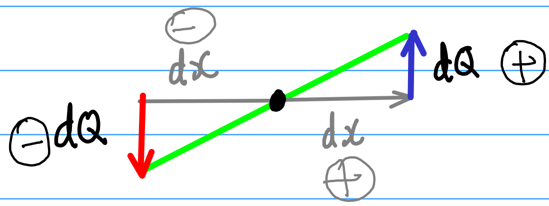
net decrement
= cw circulation
− ccw

Differentials of P & Q

$dQ \quad dP$

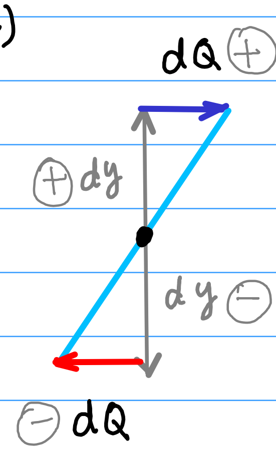
increasing Q $\quad \frac{\partial Q}{\partial x} > 0$

$Q(x,y)$



increasing P $\quad \frac{\partial P}{\partial y} > 0$

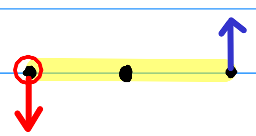
$P(x,y)$



$$dQ = \frac{\partial Q}{\partial x}\, dx$$

$dQ\ \vec{j}$

net increment
= ccw circulation

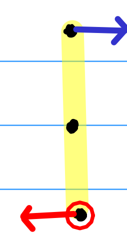


net decrement
= ccw circulation

$$dP = \frac{\partial P}{\partial y}\, dy$$

$dP\ \vec{i}$

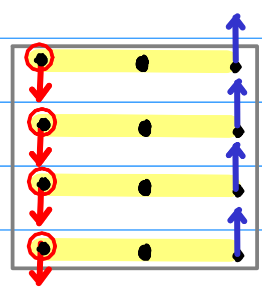
net increment
= cw circulation
– ccw



net decrement
= cw circulation
– ccw

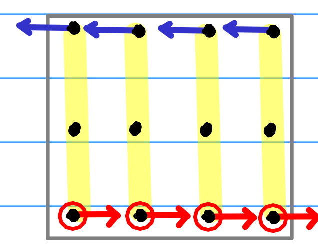
$$\frac{\partial Q}{\partial x}\, dA = \frac{\partial Q}{\partial x}\, dx\, dy$$

$dQ\ dy\ \vec{i}$



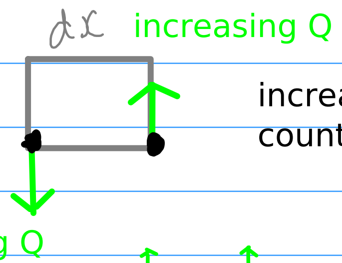
$$-\frac{\partial P}{\partial y}\, dA = -\frac{\partial P}{\partial y}\, dy\, dx$$

$dP\ dx\ \vec{j}$

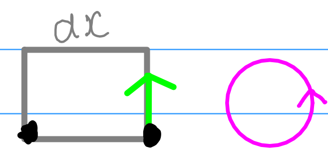


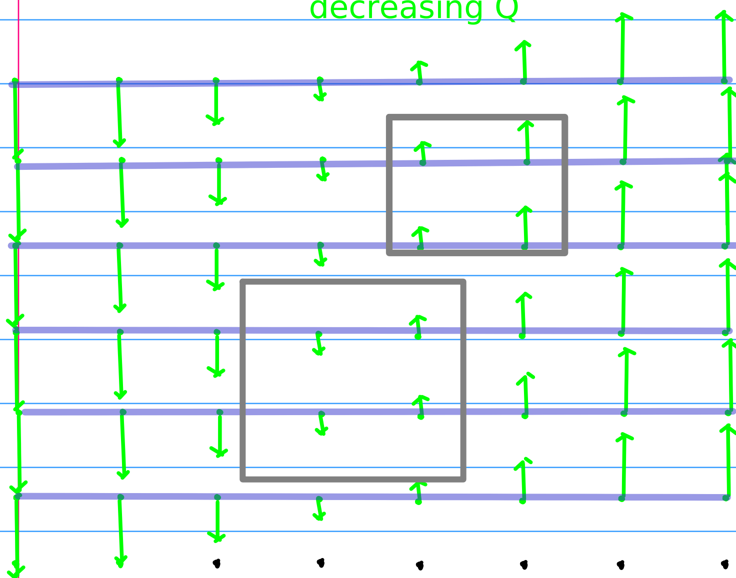
CCW / CW Rotating Region

$$\frac{\partial Q}{\partial x} > 0$$



increasing =
counter-clock-wise

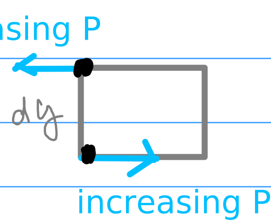




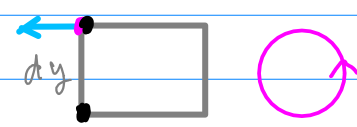
net result of $\frac{\partial Q}{\partial x}$
over a region

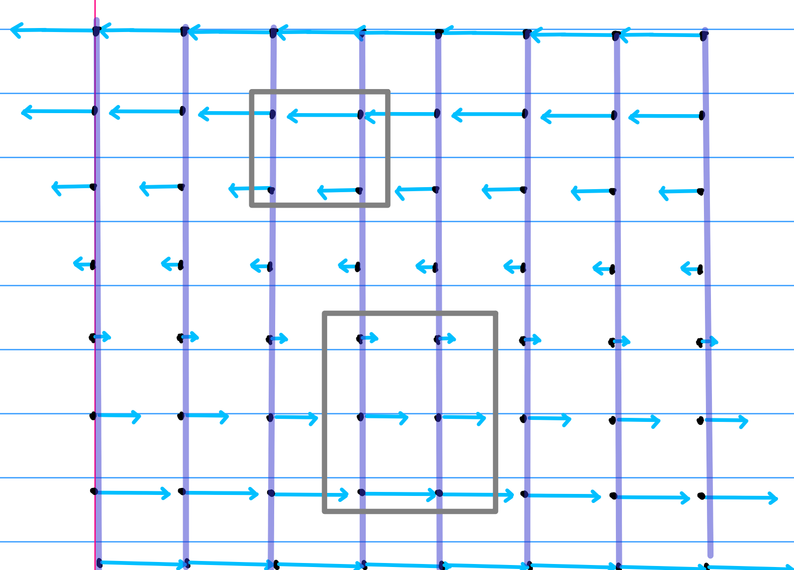
> 0 : ccw rotating

$$-\frac{\partial P}{\partial y} > 0$$



decreasing =
counter-clock-wise



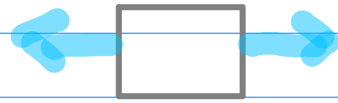


net result of $\frac{\partial P}{\partial y}$
over a region

< 0 : ccw rotating

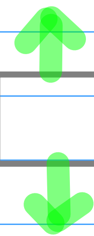
Physical Interpretation of Partial Derivatives

$\frac{\partial P}{\partial x} > 0$



increasing = outbound

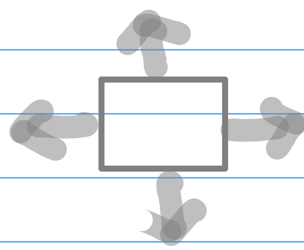
$\frac{\partial Q}{\partial y} > 0$



increasing = outbound

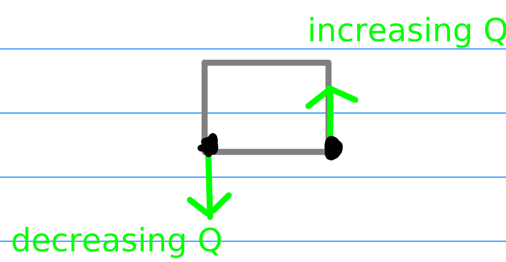
$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)$$

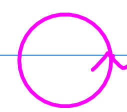
$\operatorname{div} \vec{F} > 0$



outbound

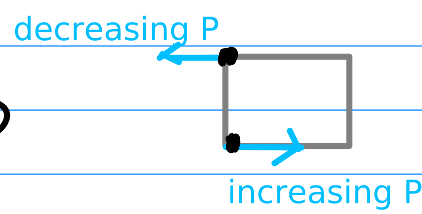
$\frac{\partial Q}{\partial x} > 0$

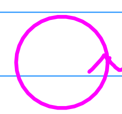




increasing =
counter-clock-wise

$-\frac{\partial P}{\partial y} > 0$

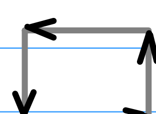




decreasing =
counter-clock-wise

$$\operatorname{Curl} \vec{F} = \nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

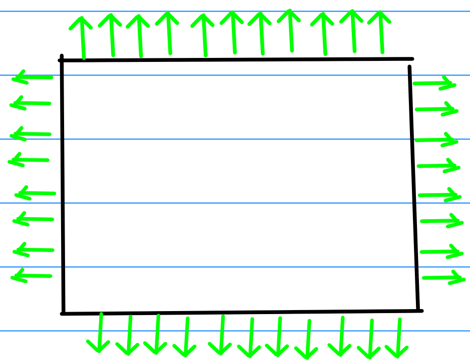
$\operatorname{Curl} \vec{F} > 0$

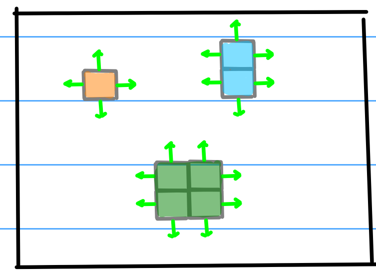


CCW

Stoke's Theorem

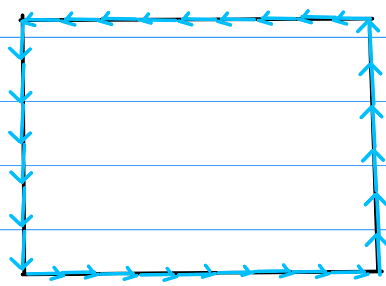
$$\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_D \left(\text{div } \vec{F} \right) dA = \iint_D \nabla \cdot \vec{F} \, dA$$

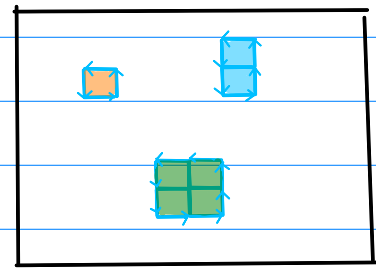




$\vec{n}$ normal vector
w.r.t the contour C

$$\oint_C \vec{F} \cdot \vec{T} \, ds = \iint_D \left(\text{curl } \vec{F} \right) \cdot \vec{k} \, dA = \iint_D \left(\nabla \times \vec{F} \right) \cdot \vec{k} \, dA$$





$\vec{T}$ tangent vector
w.r.t the contour C

Green's Theorem

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \nabla\cdot\vec{F}\,dA$$

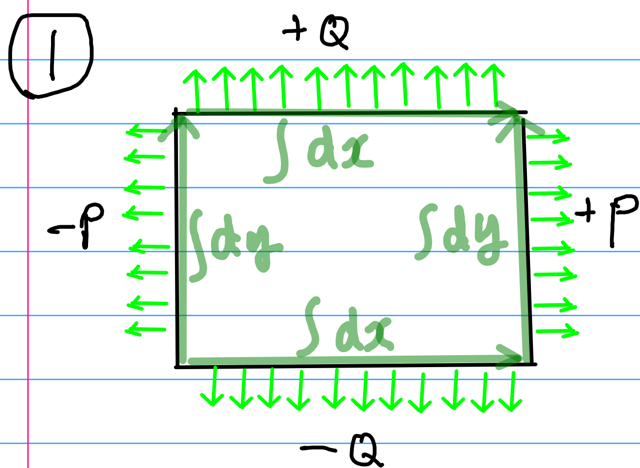
$$\boxed{\oint P\,dy - Q\,dx = \iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy}$$

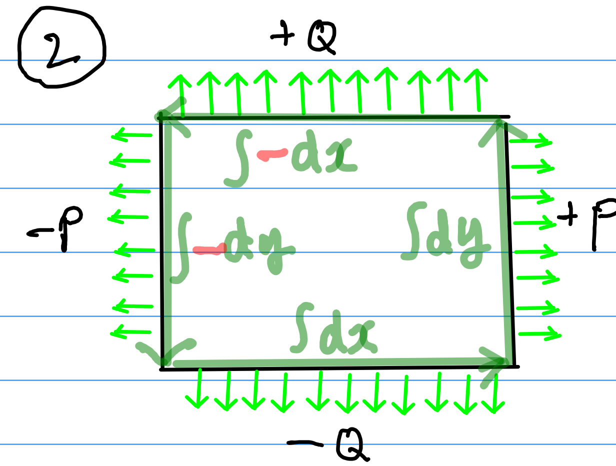
$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_D (\nabla\times\vec{F})\cdot\vec{k}\,dA$$

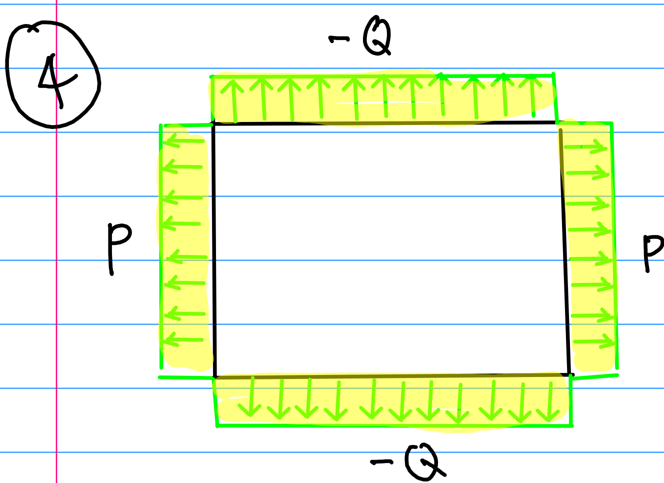
$$\boxed{\oint P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}$$

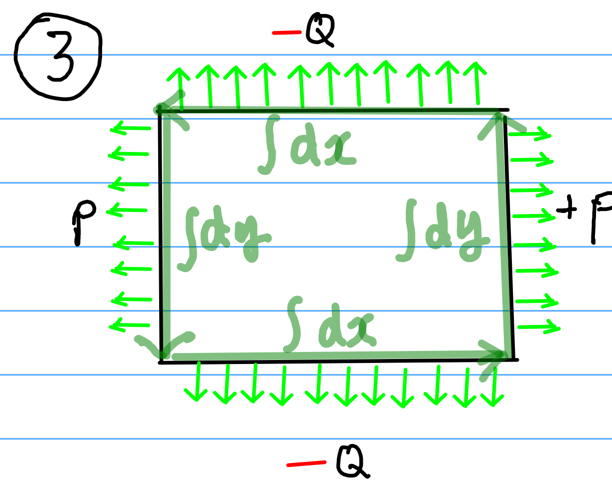
$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = P\vec{i} + Q\vec{j} = \vec{F}$$

$\oint (P\,dy - Q\,dx)$ Line Integral 1

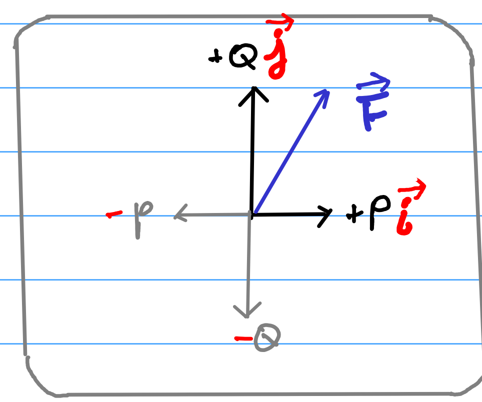








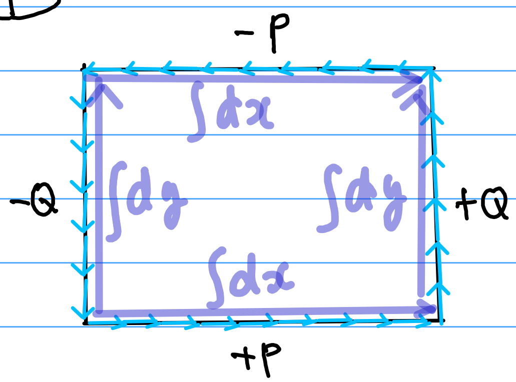
$$\oint (P\,dy - Q\,dx)$$



$$\oint (P\,dy + Q\,dx)$$

Line Integral 2

1



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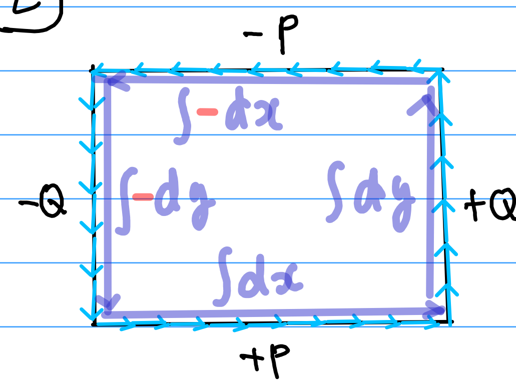

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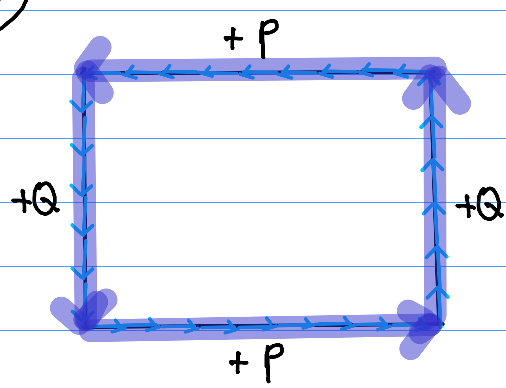

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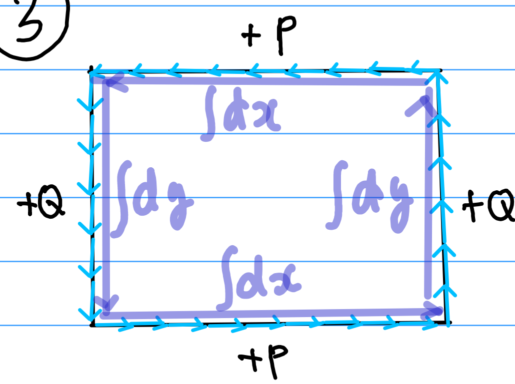
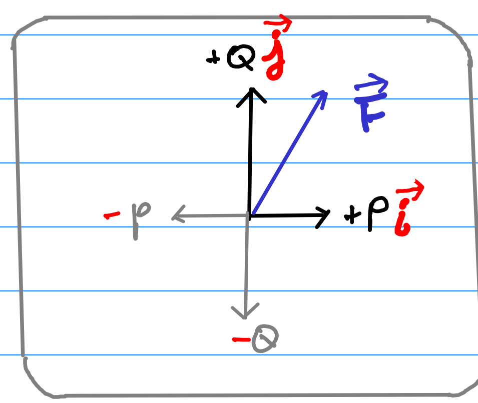


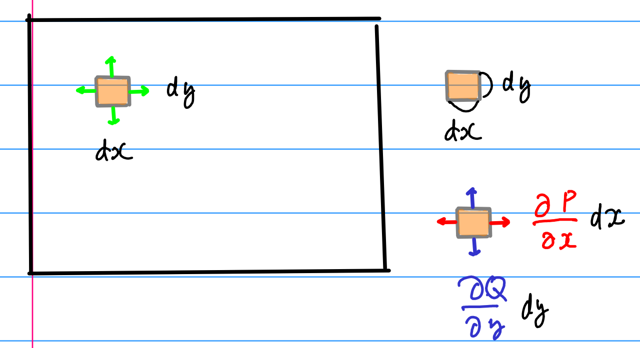
$$\oint (P\,dx + Q\,dy)$$



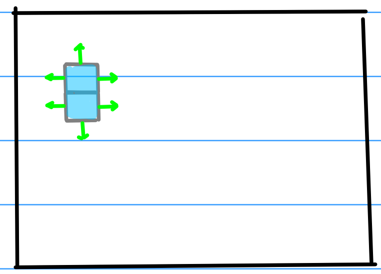
$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$$

Double Integral 1

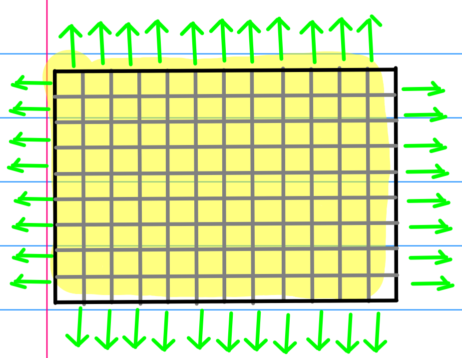
①



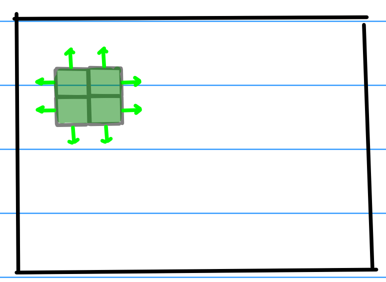
②



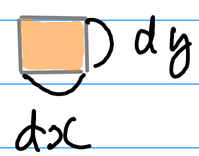
④



③



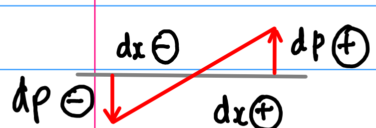
$$\iint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dx\,dy$$



$dA = dx\,dy$

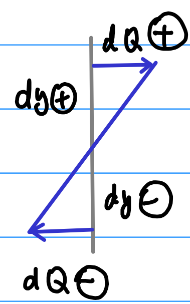
+

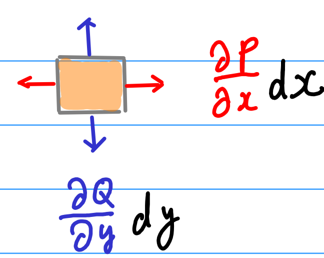
$dP = \frac{\partial P}{\partial x} dx$



+

$dQ = \frac{\partial Q}{\partial y} dy$

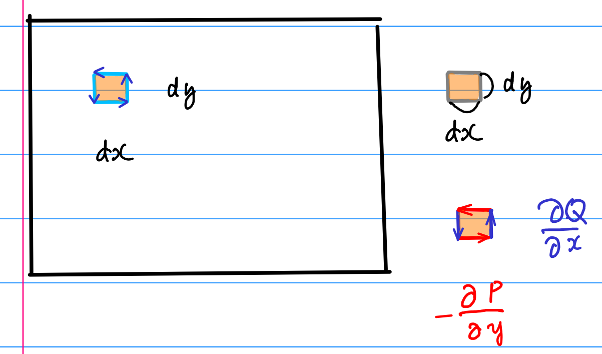




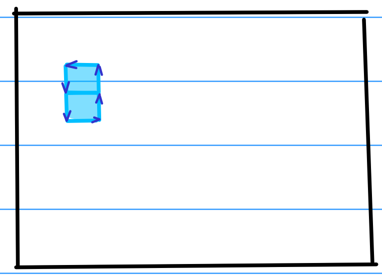
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

Double Integral 2

1



2



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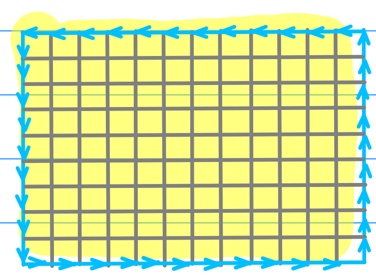
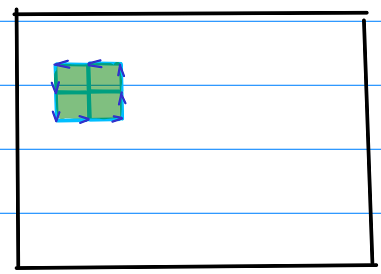
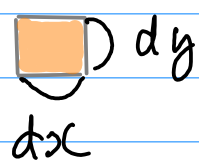


3



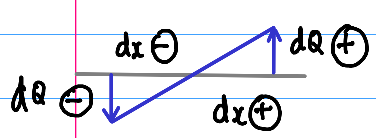
$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$



$dA = dx\,dy$

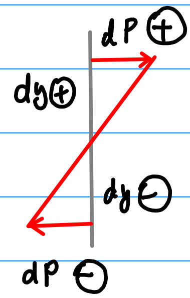
+

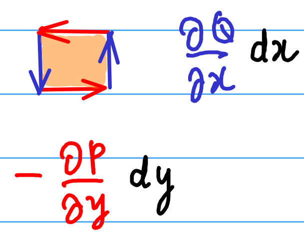
$dQ = \frac{\partial Q}{\partial x} dx$



+

$dP = \frac{\partial P}{\partial y} dy$



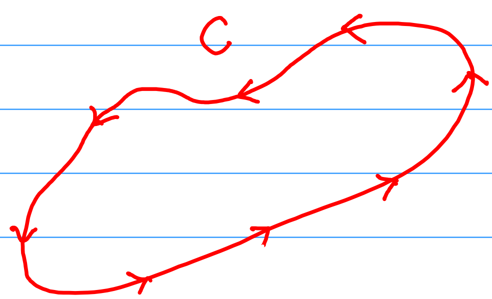


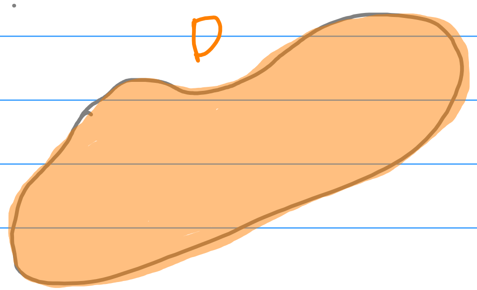
-

Green's Theorem

simple closed curve positively oriented piecewise smooth C, the region enclosed by C D

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$





$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

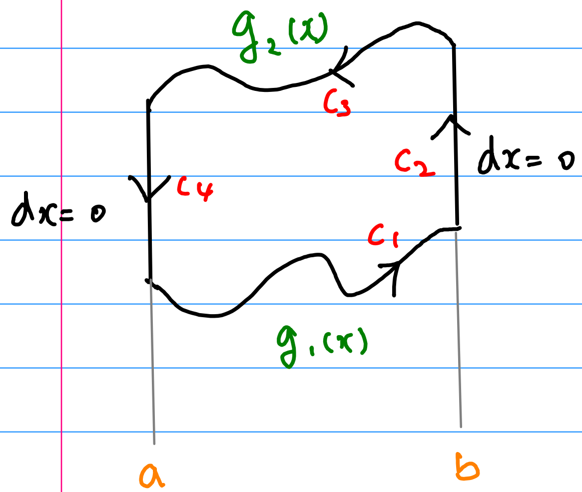
$$\int \frac{\partial P}{\partial y}\,dy \Rightarrow P \qquad dP = \frac{\partial P}{\partial y}\,dy$$

$$\int \frac{\partial Q}{\partial x}\,dx \Rightarrow Q \qquad dQ = \frac{\partial Q}{\partial x}\,dx$$

$$\boxed{\int f'(x)\,dx \Rightarrow f(x)} \qquad \int_a^b f'(x)\,dx = f(b) - f(a)$$

Fundamenta theorem

Type 1 regions and contours



double integral

$$\boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\, dy\, dx$$

$$= \int_a^b P(x, g_2(x)) - P(x, g_1(x))\, dx$$

line integral

$$\boxed{\int_C P\, dx} = \int_{C_1 + C_2 + C_3 + C_4} P\, dx$$

$$\boxed{C_2, C_4 : dx = 0 \qquad \int_{C_2} P\, dx = \int_{C_4} P\, dx = 0}$$

$$= \int_{C_1} P\, dx - \int_{-C_3} P\, dx$$

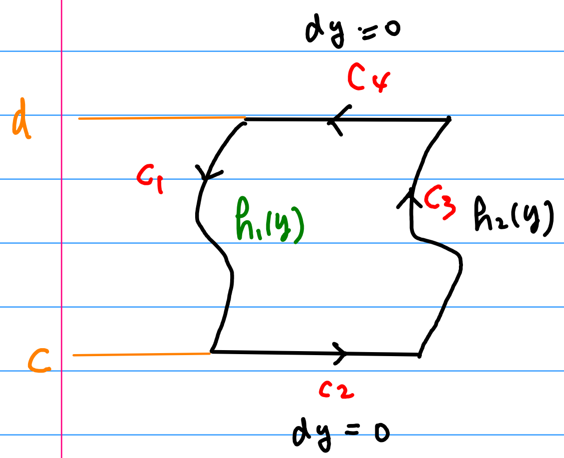
$$= \int_a^b P(x, g_1(x))\, dx - \int_a^b P(x, g_2(x))\, dx$$

$$\boxed{\int_C P\, dx} = - \boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$

line integral

double integral

Type 2 regions and contours



double integral

$$\boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial y}\, dx\, dy$$

$$= \int_c^d Q(h_2(y), y) - Q(h_1(y), y)\, dy$$

line integral

$$\boxed{\int_C Q\, d} = \int_{C_1 + C_2 + C_3 + C_4} Q\, dy$$

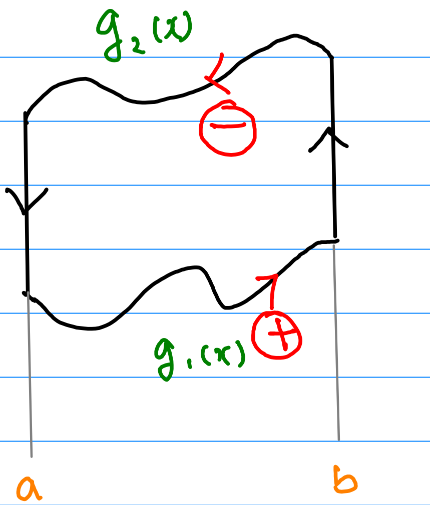
$$= -\int_{-C_1} Q\, dy + \int_{C_3} Q\, dy$$

$$\boxed{\begin{array}{l} C_2, C_4 : dy = 0 \\ \int_{C_2} Q\, dy = \int_{C_4} Q\, dy = 0 \end{array}}$$

$$= -\int_c^d Q(h_1(y), y)\, dy + \int_c^d Q(h_2(y), y)\, dy$$

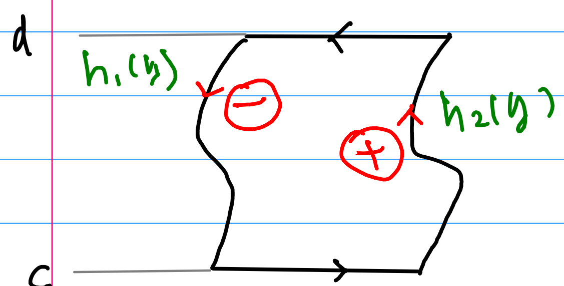
$$\boxed{\int_C Q\, dy} = + \boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

line integral double integral



$$\boxed{\int_C P\,dx} = -\boxed{\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$

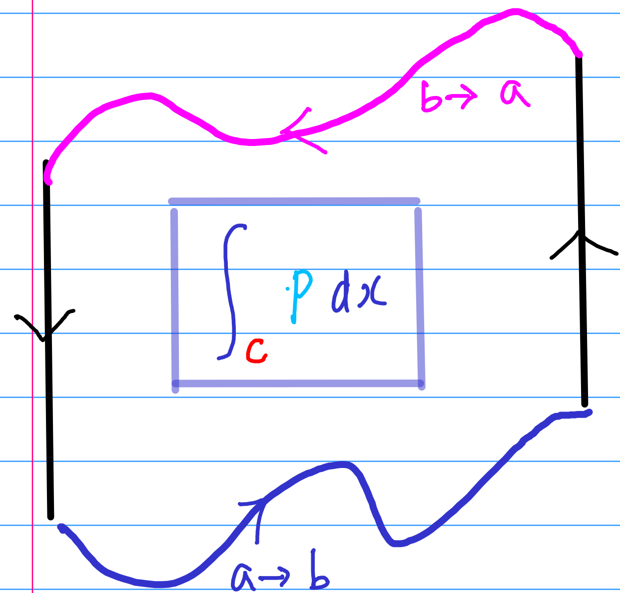
$+g_1 - g_2$



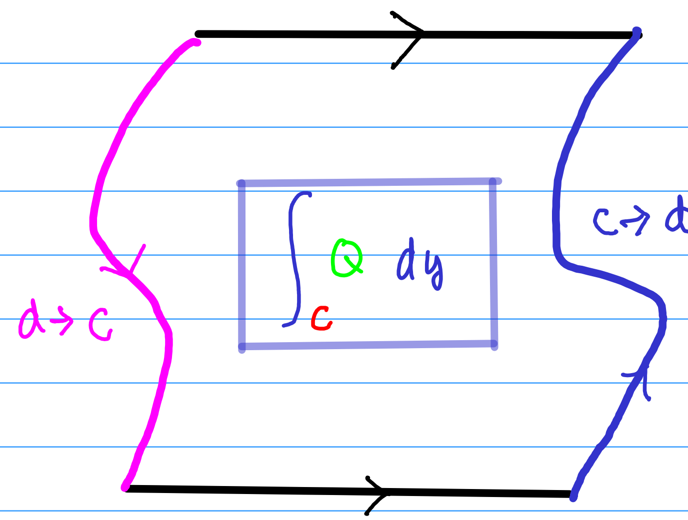
$$\boxed{\int_C Q\,dy} = +\boxed{\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

$+h_2 - h_1$

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$



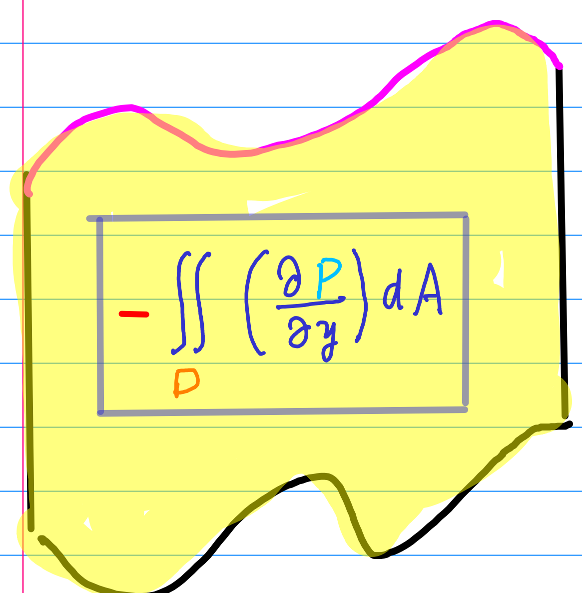
$$\int_C P\,dx$$



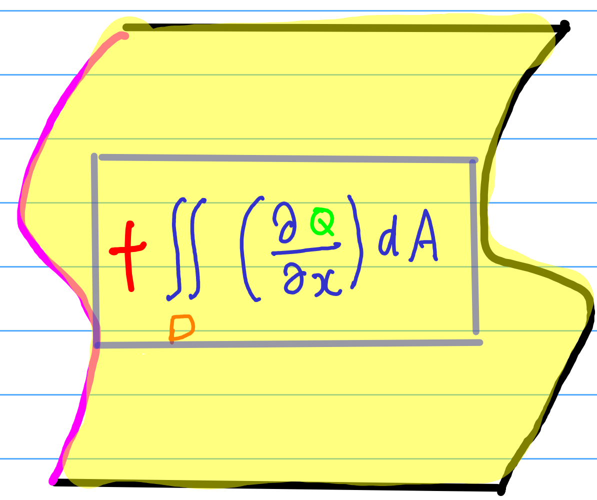
$$\int_C Q\,dy$$

$$\int_a^b \overset{P}{\boxed{\int \left(\frac{\partial P}{\partial y}\right) dy}} \; dx \qquad \int_c^d \overset{Q}{\boxed{\int \left(\frac{\partial Q}{\partial x}\right) dx}} \; dy$$

$$\int_a^b -(\text{TOP} - \text{BOTTOM})\,dx \qquad \int_c^d (\text{RIGHT} - \text{LEFT})\,dy$$

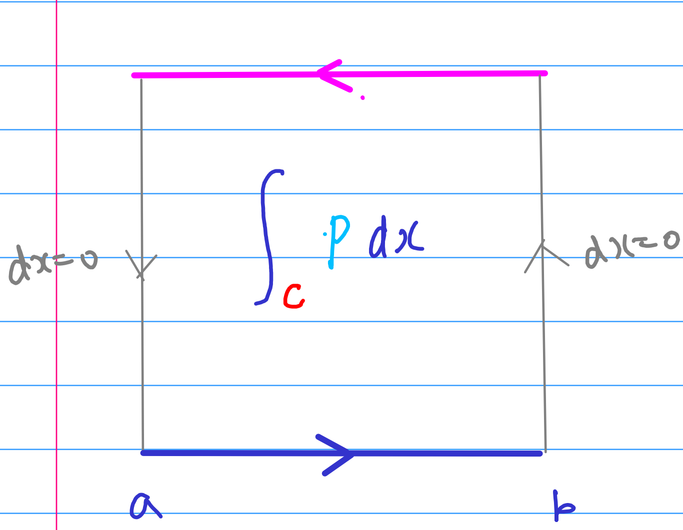


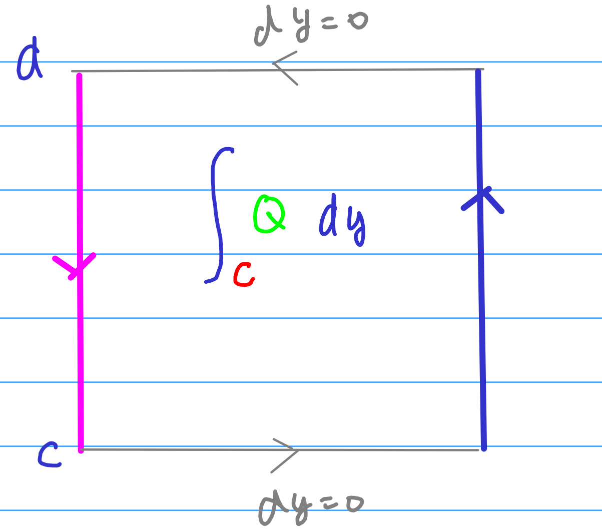
$$\boxed{-\iint_D \left(\frac{\partial P}{\partial y}\right) dA}$$



$$\boxed{+\iint_D \left(\frac{\partial Q}{\partial x}\right) dA}$$

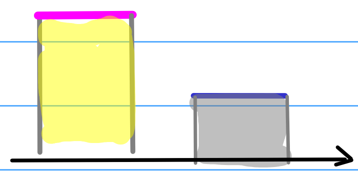
Rectangular regions and contours

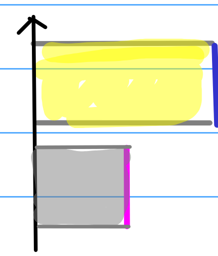




$$\int_a^b -(\text{TOP} - \text{BOTTOM})\,dx$$

$$\int_c^d (\text{RIGHT} - \text{LEFT})\,dy$$





$$\int_a^b \boxed{\int \left(\frac{\partial P}{\partial y}\right) dy} \rightarrow P \qquad dx$$

$$\int_c^d \boxed{\int \left(\frac{\partial Q}{\partial x}\right) dx} \rightarrow Q \qquad dy$$

$$-\iint_D \left(\frac{\partial P}{\partial y}\right) dA$$

$$+\iint_D \left(\frac{\partial Q}{\partial x}\right) dA$$

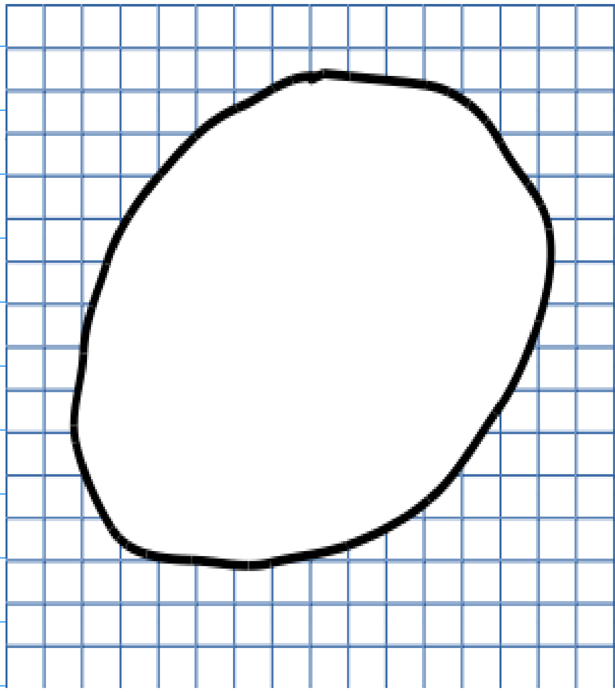
General Contour

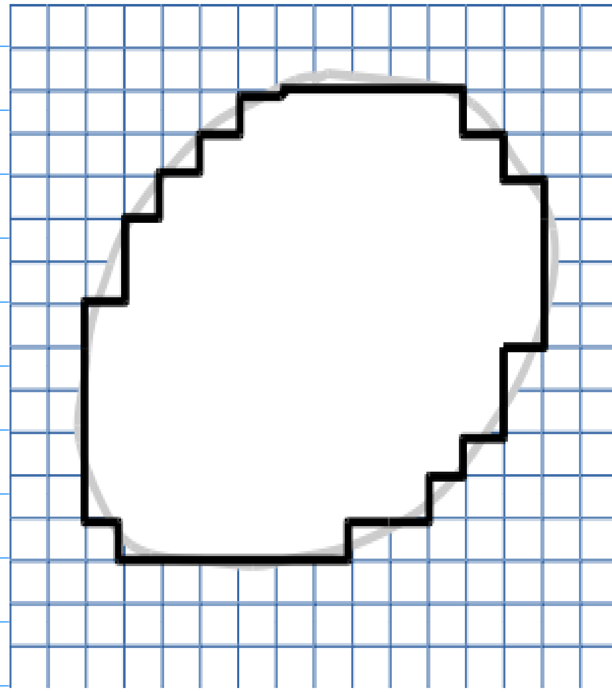
$$\int_C P\,dx \quad + \quad \int_C Q\,dy$$

$$\equiv \quad -\iint_D \left(\frac{\partial P}{\partial y}\right) dA \quad + \quad \iint_D \left(\frac{\partial Q}{\partial x}\right) dA$$

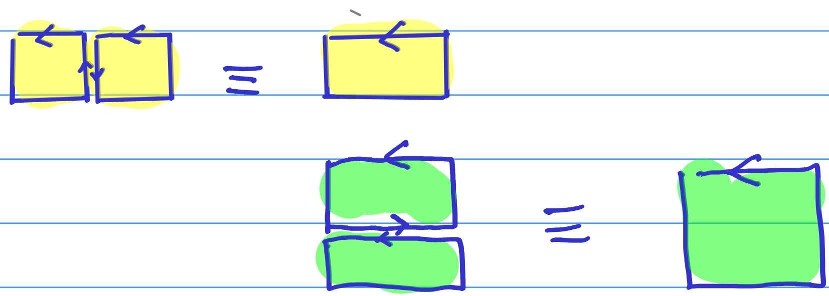
$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

Approximation (Digitization)

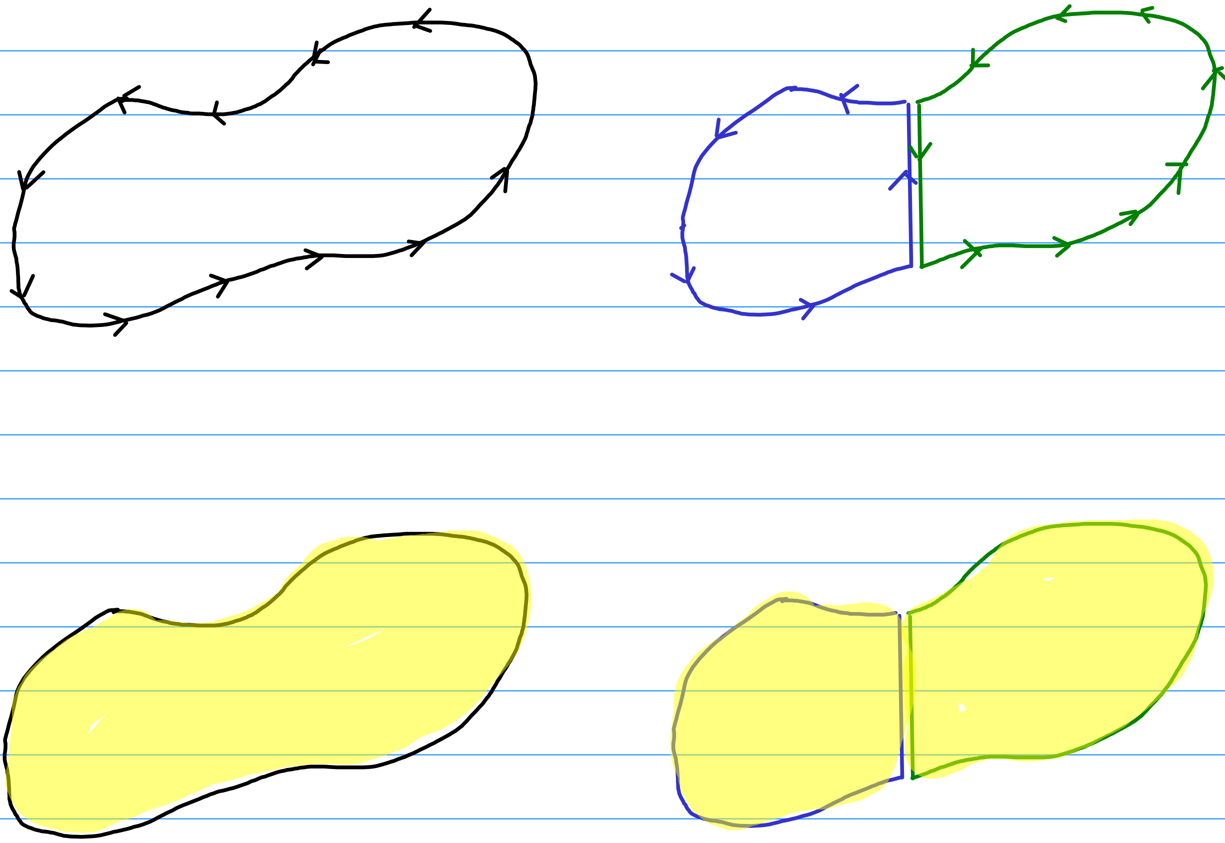




approximated by a collection of ~~rectangles~~

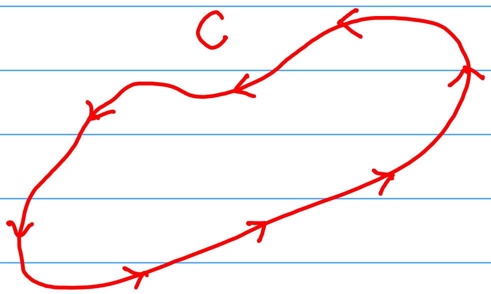


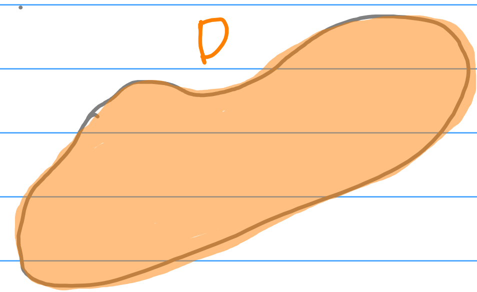
http://ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/
3.-double-integrals-and-line-integrals-in-the-plane/part-c-greens-theorem/
session-67-proof-of-greens-theorem/MIT18_02SC_notes_67.pdf



$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$





If P, Q are the x, y component of a <u>gradient vector field</u> then there exists a <u>potential function</u> f

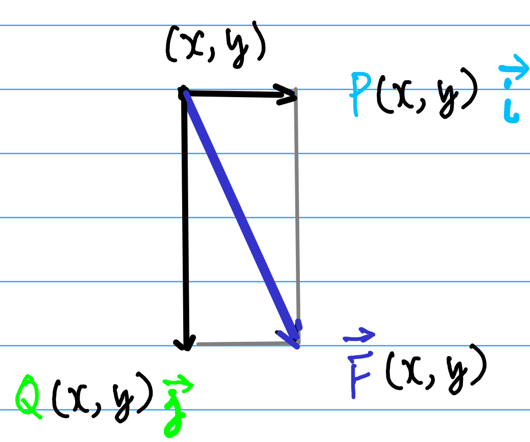
$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$= P\,\vec{i} + Q\,\vec{j}$$

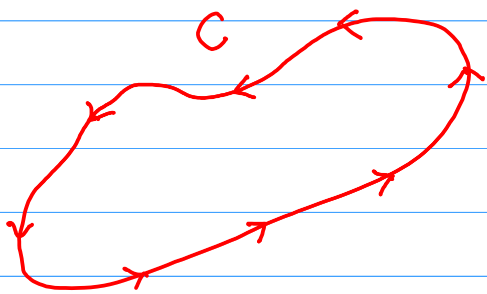
$$= \vec{F}$$

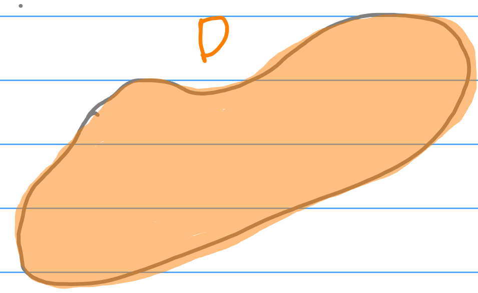
$$f(x,y) = \int P(x,y)\,dx$$

$$f(x,y) = \int Q(x,y)\,dy$$



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$





P, Q can be x, y component of a **gradient** vector field of $f(x.y)$ $\Leftrightarrow$

$$P = \frac{\partial f}{\partial x} \qquad \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x}$$

$$Q = \frac{\partial f}{\partial y} \qquad \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\,\partial y}$$

conservative field

Example

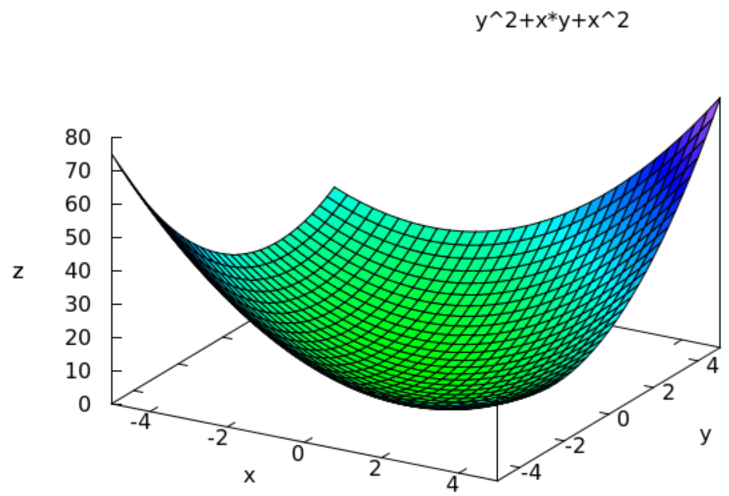
$$f(x, y) = x^2 + xy + y^2$$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = x + 2y$$

$$\nabla f = \underbrace{(2x+y)}_{\substack{\parallel \\ P}}\vec{i} + \underbrace{(x+2y)}_{\substack{\parallel \\ Q}}\vec{j}$$

$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} = 1$$ ∴ Conservative Vector Field

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \qquad \oint_C \vec{F} \cdot d\vec{r} = 0$$



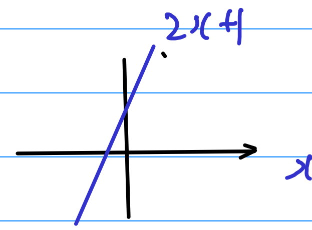
$F(x,y) = x^2 + xy + y^2$

$\frac{\partial f}{\partial x} = 2x+y$

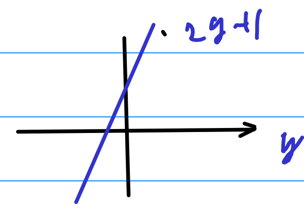
$\frac{\partial f}{\partial y} = x+2y$

$f(1,1) = 1+1+1=3$

$y=1 \Rightarrow \frac{\partial f}{\partial x} = 2x+1$

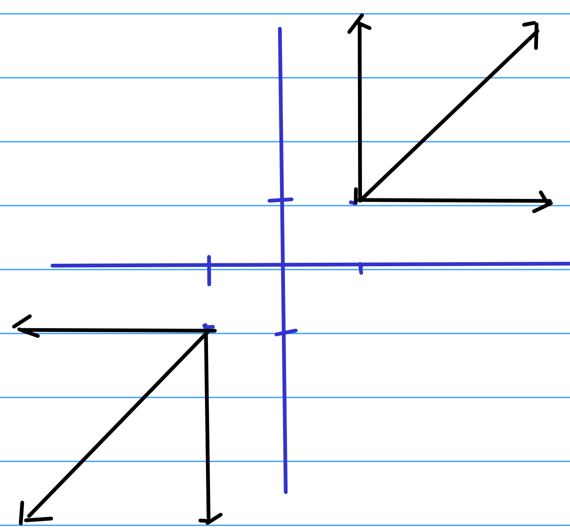


$x=1 \Rightarrow \frac{\partial f}{\partial y} = 1+2y$



$\frac{\partial f}{\partial x}(1,1)\vec{i} + \frac{\partial f}{\partial y}(1,1)\vec{y} = 3\vec{i} + 3\vec{y}$

$f(-1,-1) = 1+1+1=3$

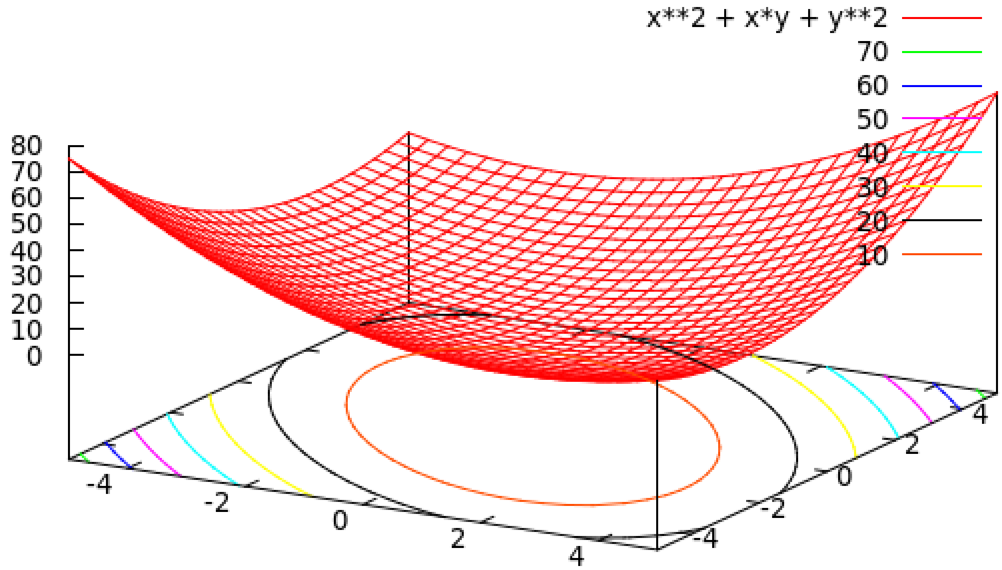


$y=-1 \Rightarrow \frac{\partial f}{\partial x} = 2x-1$

$x=-1 \Rightarrow \frac{\partial f}{\partial y} = -1+2y$

$\frac{\partial f}{\partial x}(-1,-1)\vec{i} + \frac{\partial f}{\partial y}(-1,-1)\vec{y} = -3\vec{i} - 3\vec{y}$

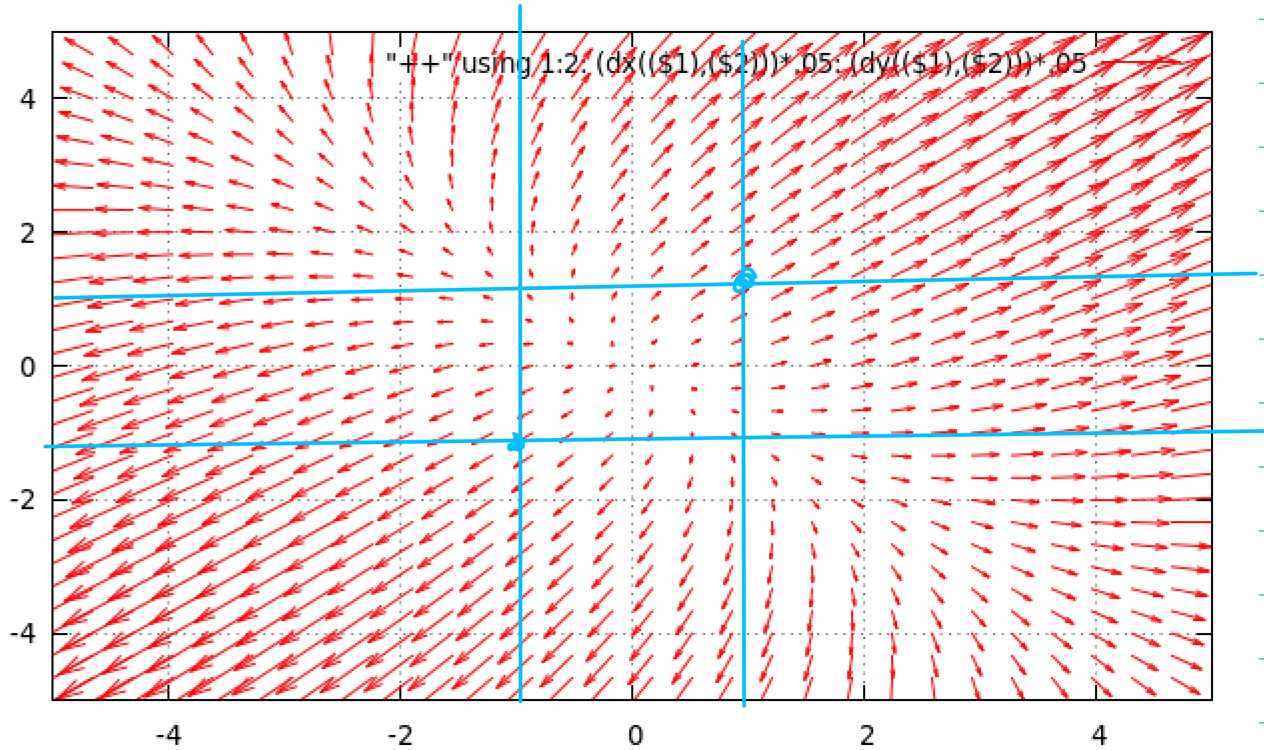
* Contour graphs in gnuplot



$$f(x,y) = x^2 + xy + y^2$$

```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> set contour base
gnuplot> set cntrparam levels 10
gnuplot> splot x**2 + x*y + y**2
```

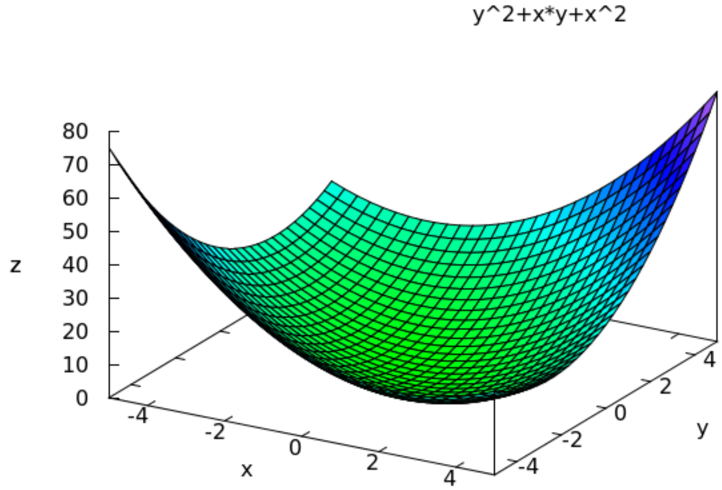
* Gradient field plot in gnuplot



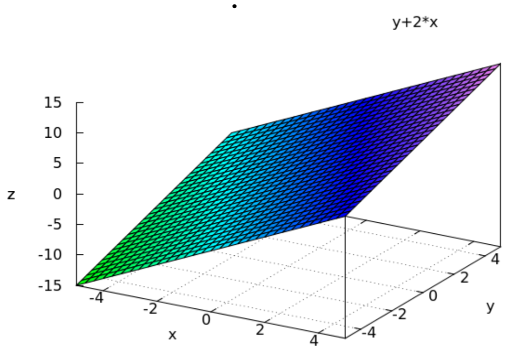
$\nabla f(x,y)$ $\quad$ $f(x,y) = x^2 + xy + y^2$

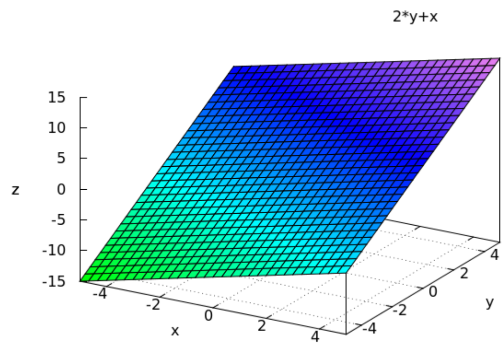
```
gnuplot> set xrange [-5:5]
gnuplot> set yrange [-5:5]
gnuplot> set sample 30
gnuplot> set isosample 31
gnuplot> dx(x,y) = 2*x + y
gnuplot> dy(x,y) = x + 2*y
gnuplot> plot "++" using 1:2: (dx(($1),($2)))*.05: (dy(($1),($2)))*.05 w vec
```

$$f(x,y) = x^2 + xy + y^2$$

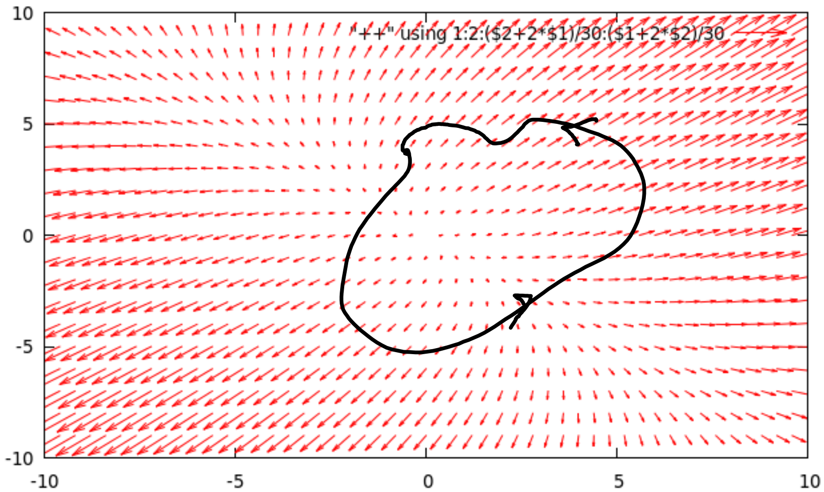


$$P(x,y) = \frac{\partial f}{\partial x}(x,y) = y + 2x \qquad Q(x,y) = \frac{\partial f}{\partial y}(x,y) = 2y + x$$





$$\nabla f(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = (y+2x)\,\vec{i} + (2y+x)\,\vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

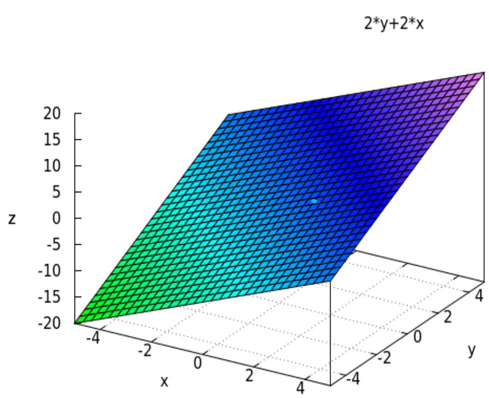
conservative

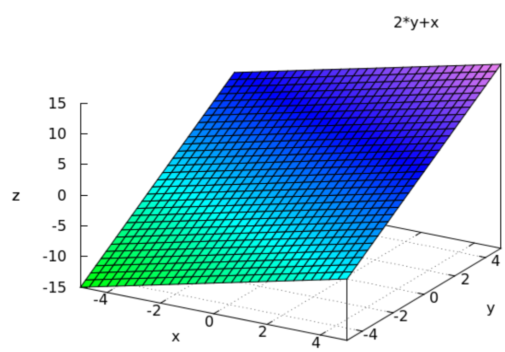
$f(x,y)$ · ✗ no potential function exists

$$\cancel{P(x,y) = \frac{\partial f}{\partial x}(x,y)}$$

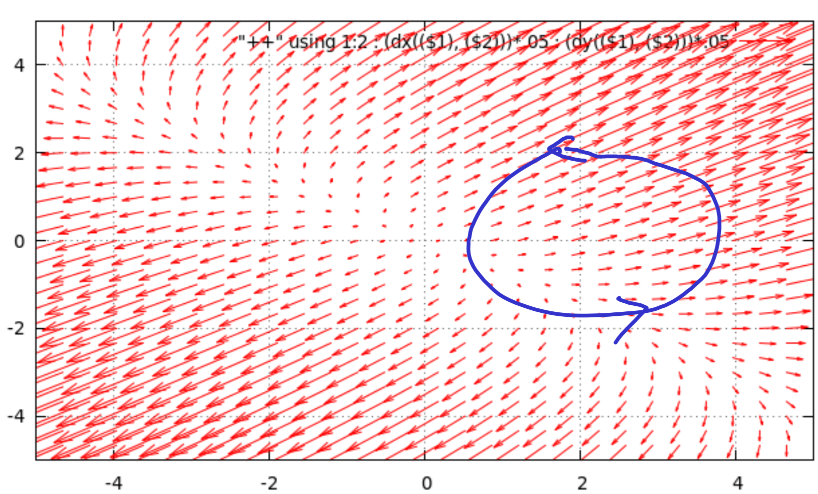
$$\cancel{Q(x,y) = \frac{\partial f}{\partial y}(x,y)}$$

$P(x,y) = 2y + 2x$ $\qquad$ $Q(x,y) = 2y + x$





$$P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} = (2y+2x)\,\vec{i} + (2y+x)\,\vec{j}$$



$$\oint_C \vec{F} \cdot d\vec{r} \neq 0$$

~~conservative~~

* Exact Differential

$\boxed{P(x,y)\,dx + Q(x,y)\,dy}$: a differential

if this is a total differential of a certain function f
then it is exact differential

$$\boxed{\Rightarrow \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = dz} \qquad z = f(x,y)$$

$$P(x,y) = \frac{\partial f}{\partial x} \qquad Q(x,y) = \frac{\partial f}{\partial y}$$

the necessary and sufficient condition

$\boxed{P(x,y)\,dx + Q(x,y)\,dy}$: an exact differential

$$\Leftrightarrow \quad \boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

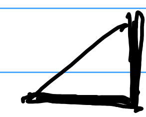
$$\oint_C xy\,dx = \int_{C_1} xy\,dx + \int_{C_2} xy\,dx + \int_{C_3} xy\,dx$$

$y=0$ $\qquad$ $y=1$ $\qquad$ $y=2x$

$$= \int_0^1 0\,dx + \int_1^1 x\,dx + \int_1^0 2x^2\,dx$$

$$= \left[\frac{2}{3}x^3\right]_1^0 = -\frac{2}{3}$$

$y=2x$



$$\oint_C x^2y^3\,dx = \int_{C_1} x^2y^3\,dy + \int_{C_2} x^2y^3\,dy + \int_{C_3} x^2y^3\,dy$$

$x=[0,1]$ $\qquad$ $x=1$ $\qquad$ $x=\frac{y}{2}$

$$= \int_0^0 x^2y^3\,dy + \int_0^2 y^3\,dy + \int_2^0 \frac{y^5}{4}\,dy$$

$$= \left[\frac{1}{4}y^4\right]_0^2 + \left[\frac{1}{24}y^6\right]_2^0$$

$$= 4 - \cancel{\frac{64}{24}}\ \frac{16}{6}\ \frac{8}{3} = \frac{12-8}{3} = +\frac{4}{3}$$

$$\frac{2}{3}$$