

# Random Process Background (1C)

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

- 1 Open Sets and Neighborhoods
  - Open Set
  - Neighborhood
  - Class
- 2 Filters
  - Filter
  - Proper Filter and Ultra Filter
  - Filter Example
- 3 Topological Space
  - Topological Space
  - A discrete topology
  - Examples of topology

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# Open set examples

- The *circle* represents the set of points  $(x, y)$  satisfying  $x^2 + y^2 = r^2$ .  
the *circle* set is its **boundary set**
- The *disk* represents the set of points  $(x, y)$  satisfying  $x^2 + y^2 < r^2$ .  
The *disk* set is an **open set**
- the **union** of the *circle* and *disk* sets is a **closed set**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point**  $P$ , contains all points that are sufficiently near to  $P$ 
  - all points whose **distance** to  $P$  is less than some value depending on  $P$

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Open set (2-1)

- more generally, an **open set** is a **member** of a **given collection** of **subsets** of a **given set**

- a given set
- subsets of a given set
- a given collection of subsets of a given set

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Open set (2-2)

- a **collection** has the following property of **containing**

- a **collection** contains
  - every **union** of its **members**
  - every **finite intersection** of its members
  - the **empty set**
  - the **whole set** itself

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



## Open set (3)

- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
  - every **subset** can be **open** (the **discrete topology**)
  - no **subset** can be **open** (the **indiscrete topology**) except
    - the space itself and
    - the empty set

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

## Open set (4)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
  - A **set** is a **collection** of distinct **objects**.
  - Given a **set**  $A$ , we say that  $a$  is an **element** of  $A$  if  $a$  is one of the distinct **objects** in  $A$ , and we write  $a \in A$  to denote this
  - Given two **sets**  $A$  and  $B$ , we say that  $A$  is a **subset** of  $B$  if every element of  $A$  is also an element of  $B$  write  $A \subseteq B$  to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

## Open set (5) Open Balls

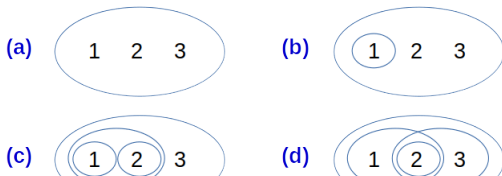
- An **open ball**  $B_r(\mathbf{a})$  in  $\mathbb{R}^n$   
centered at  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$  with radius  $r$   
is the set of all points  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$   
such that the distance between  $\mathbf{x}$  and  $\mathbf{a}$  is less than  $r$
- In  $\mathbb{R}^2$  an **open ball** is often called an **open disk**

We give these definitions in general, for when one is working in  $\mathbb{R}^n$   
since they are really not all that different to define in  $\mathbb{R}^n$  than in  $\mathbb{R}^2$

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

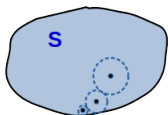
## Open set (6) Interior points

- Suppose that  $S \subseteq \mathbb{R}^n$
- A point  $p \in S$  is an **interior point** of  $S$  if there exists an **open ball**  $B_r(p) \subseteq S$
- Intuitively,  $p$  is an **interior point** of  $S$  if we can squeeze an entire **open ball** centered at  $p$  within  $S$

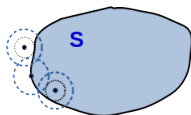


# Open set (7) Boundary points

- A point  $\mathbf{p} \in \mathbb{R}^n$  is a **boundary point** of  $S$  if all **open balls** centered at  $\mathbf{p}$  contain both **points** in  $S$  and **points** not in  $S$
- The **boundary** of  $S$  is the **set**  $\partial S$  that consists of all of the **boundary points** of  $S$ .



an interior point



a boundary point

# Open set (8) Open and Closed Sets

- A set  $O \subseteq \mathbb{R}^n$  is **open** if every point in  $O$  is an **interior point**.
- A set  $C \subseteq \mathbb{R}^n$  is **closed** if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

## Open set (9) Bounded and Unbounded

- A set  $S$  is **bounded** if there is an **open ball**  $B_M(0)$  such that

$$S \subseteq B.$$

intuitively, this means that we can enclose  
all of the **set**  $S$  within a large enough **ball**  
centered at the origin,  $B_M(0)$

- A **set** that is not **bounded** is called **unbounded**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

# Family of sets (1)

- a **collection**  $F$  of **subsets** of a given **set**  $S$  is called a **family** of **subsets** of  $S$ , or a **family** of **sets** over  $S$ .
- More generally, a **collection** of any **sets** whatsoever is called a **family** of **sets**, **set family**, or a **set system**

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)



## Family of sets (2)

- The term "**collection**" is used here because,
  - in some contexts,  
a **family** of **sets** may be allowed  
to contain repeated copies of any given **member**, and
  - in other contexts  
it may form a **proper class** rather than a **set**.

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

# Examples of family of sets (1)

- The **set** of all **subsets** of a given **set**  $S$  is called the **power set** of  $S$  and is denoted by  $\wp(S)$ .

The **power set**  $\wp(S)$  of a given **set**  $S$  is a **family** of **sets** over  $S$ .

- A **subset** of  $S$  having  $k$  elements is called a  **$k$ -subset** of  $S$ .

The  **$k$ -subset**  $S^{(k)}$  of a set  $S$  form a **family** of **sets**.

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

## Examples of family of sets (2)

- Let  $S = \{a, b, c, 1, 2\}$ .

An example of a **family** of **sets** over  $S$

(in the multiset sense) is given by  $F = \{A_1, A_2, A_3, A_4\}$ , where

$A_1 = \{a, b, c\}$ ,  $A_2 = \{1, 2\}$ ,  $A_3 = \{1, 2\}$ , and  $A_4 = \{a, b, 1\}$ .

[https://en.wikipedia.org/wiki/Family\\_of\\_sets](https://en.wikipedia.org/wiki/Family_of_sets)

# Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a **point**  $x$  is a **filter base** of the **neighbourhood filter**;
- this means that it is a **subset**  $\mathcal{B} \subseteq \mathcal{N}(x)$  such that for all  $V \in \mathcal{N}(x)$ , there exists some  $B \in \mathcal{B}$  such that  $B \subseteq V$ . That is, for any **neighbourhood**  $V$  we can find a **neighbourhood**  $B$  in the **neighbourhood basis** that is contained in  $V$ .

[https://en.wikipedia.org/wiki/Neighbourhood\\_system#Neighbourhood\\_basis](https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis)

## Neighbourhood basis (2)

- Equivalently,  $\mathcal{B}$  is a local basis at  $x$  if and only if the neighbourhood filter  $\mathcal{N}$  can be recovered from  $\mathcal{B}$  in the sense that the following equality holds:

$$\mathcal{N}(x) = \{V \subseteq X : B \subseteq V \text{ for some } B \in \mathcal{B}\}$$

- A family  $\mathcal{B} \subseteq \mathcal{N}(x)$  is a neighbourhood basis for  $x$  if and only if  $\mathcal{B}$  is a cofinal subset of  $(\mathcal{N}(x), \supseteq)$  with respect to the partial order  $\supseteq$  (importantly, this partial order is the superset relation and not the subset relation).

[https://en.wikipedia.org/wiki/Neighbourhood\\_system#Neighbourhood\\_basis](https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis)

# A collection of sets around $x$

- In general, one refers to the family of **sets** containing 0, used to approximate 0, as a **neighborhood basis**;
- a **member** of this **neighborhood basis** is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set ( $X$ ); rather than just the **real numbers**.
- In this case, given a **point** ( $x$ ) of that **set** ( $X$ ), one may define a **collection** of **sets** "**around**" (that is, containing)  $x$ , used to approximate  $x$ .

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Smaller sets containing $x$

- Of course, this **collection** would have to *satisfy* certain properties (known as **axioms**) for otherwise we may not have a *well-defined method* to measure **distance**.
- For example, every **point** in  $X$  should **approximate**  $x$  to some **degree** of **accuracy**.
- Thus  $X$  should be in this **family**.
- Once we begin to define "smaller" **sets** containing  $x$ , we tend to **approximate**  $x$  to a greater **degree** of **accuracy**.
- Bearing this in mind, one may define the remaining **axioms** that the **family** of **sets** about  $x$  is required to satisfy.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

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# Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;  
it is also called a **solid sphere**.
  - a **closed ball**  
includes the *boundary points* that constitute the sphere
  - an **open ball**  
excludes them

[https://en.wikipedia.org/wiki/Ball\\_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

## Open ball (2)

- A **ball** in  $n$  dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or  $(n - 1)$ -sphere
- One may talk about **balls** in any **topological space**  $X$ , not necessarily induced by a **metric**.
- An  $n$ -dimensional **topological ball** of  $X$  is any **subset** of  $X$  which is **homeomorphic** to an **Euclidean n-ball**.

[https://en.wikipedia.org/wiki/Ball\\_\(mathematics\)](https://en.wikipedia.org/wiki/Ball_(mathematics))

# Neighborhood (1)

- a **neighbourhood** is one of the basic *concepts* in a **topological space**.
- It is closely related to the *concepts* of **open set** and **interior**.
- Intuitively speaking, a **neighbourhood** of a **point** is a **set of points** containing that **point** where one can move some amount in any direction away from that **point** without leaving the **set**.

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

# Interior

- the **interior** of a **subset**  $S$  of a **topological space**  $X$  is the **union** of *all* **subsets** of  $S$  that are **open** in  $X$ .
- A **point** that is in the **interior** of  $S$  is an **interior point** of  $S$ .
- The **interior** of  $S$  is the **complement** of the **closure** of the complement of  $S$ .  
the closure of (boundary + exterior)
- In this sense, **interior** and **closure** are dual notions.

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

# Exterior

- The **exterior** of a set  $S$  is the **complement** of the **closure** of  $S$ ; the closure of  $S = \text{boundary} + \text{interior}$
- it consists of the **points** that are in neither the **set** nor its **boundary**.
- The **interior**, **boundary**, and **exterior** of a **subset** together partition the whole **space** into three **blocks**
- fewer when one or more of these is empty

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

# Interior Point (1)

- If  $S$  is a **subset** of a **Euclidean space**, then  $x$  is an **interior point** of  $S$  if there exists an **open ball** centered at  $x$  which is completely contained in  $S$ .
- This definition generalizes to any **subset**  $S$  of a **metric space**  $X$  with **metric**  $d$ :  
 $x$  is an **interior point** of  $S$  if there exists a real number  $r > 0$ , such that  $y$  is in  $S$  whenever the distance  $d(x, y) < r$ .

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

# Interior Point (2)

- This definition generalizes to **topological spaces** by replacing "**open ball**" with "**open set**".
  - if there exists an *open ball* centered at  $x$  which is completely contained in  $S$ .
  - if  $x$  is contained in an *open subset* of  $X$  that is completely contained in  $S$ .

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

# Interior Point (3)

- If  $S$  is a **subset** of a **topological space**  $X$  then  $x$  is an **interior point** of  $S$  in  $X$  if  $x$  is contained in an **open subset** of  $X$  that is completely contained in  $S$ .
- Equivalently,  $x$  is an **interior point** of  $S$  if  $S$  is a **neighbourhood** of  $x$ .

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)



# Interior of a Set (1)

- The **interior** of a **subset**  $S$  of a **topological space**  $X$ , can be defined in any of the following equivalent ways:
  - the largest **open subset** of  $X$  contained in  $S$ .
  - the union of all **open sets** of  $X$  contained in  $S$ .
  - the **set** of all **interior points** of  $S$ .

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

## Interior of a Set (2)

- The **interior** of a **subset**  $S$  of a **topological space**  $X$ , denoted by  $\mathit{int}_X S$  or  $\mathit{int} S$  or  $S^\circ$
- If the **space**  $X$  is understood from **context** then the shorter notation  $\mathit{int} S$  is usually preferred to  $\mathit{int}_X S$ .

[https://en.wikipedia.org/wiki/Interior\\_\(topology\)#Interior\\_point](https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point)

# Neighborhood of a point (1-1)

- If  $X$  is a **topological space** and  $p$  is a **point** in  $X$ , then a **neighbourhood** of  $p$  is a **subset**  $V$  of  $X$  that includes an **open set**  $U$  containing  $p$ ,

$$p \in U \subseteq V \subseteq X.$$

- $X$  : a **topological space**
- $V$  : a **subset** of  $X$
- $U$  : an **open set** containing  $p$
- $p$  : a **point** in  $X$
- $V$  : a **neighbourhood** of  $p$

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

# Neighborhood of a point (1-2)

- This is also equivalent to the **point**  $p \in X$  belonging to the **topological interior** of  $V$  in  $X$ .
- The **neighbourhood**  $V$  need not be an **open subset** of  $X$ , but when  $V$  is **open** in  $X$  then it is called an **open neighbourhood**.
- Some authors have been known to require **neighbourhoods** to be **open**, so it is important to note conventions.

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

## Neighborhood of a point (2)

- A set that is a neighbourhood of each of its points is open since it can be expressed as the union of open sets containing each of its points.
- A closed rectangle, as illustrated in the figure, is not a neighbourhood of all its points;
  - points on the edges or corners of the rectangle are not contained in any open set that is contained within the rectangle.
- The collection of all neighbourhoods of a point is called the neighbourhood system at the point.

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

# Neighborhood of a set (1-1)

- If  $S$  is a **subset** of a **topological space**  $X$ , then a **neighbourhood** of  $S$  is a **set**  $V$  that includes an **open set**  $U$  containing  $S$ ,

$$S \subseteq U \subseteq V \subseteq X.$$

- It follows that a **set**  $V$  is a **neighbourhood** of  $S$  if and only if it is a **neighbourhood** of all the **points** in  $S$ .
- Furthermore,  $V$  is a **neighbourhood** of  $S$  if and only if  $S$  is a **subset** of the **interior** of  $V$ .

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

## Neighborhood of a set (1-2)

- A **neighbourhood** of  $S$  that is also an **open subset** of  $X$  is called an **open neighbourhood** of  $S$ .
- The **neighbourhood** of a **point** is just a special case of this definition.

[https://en.wikipedia.org/wiki/Neighbourhood\\_\(mathematics\)](https://en.wikipedia.org/wiki/Neighbourhood_(mathematics))

# Neighborhood definition (1)

- the **open set axioms** are often taken as the definition of a **topology**, when they are quite *unintuitive*, though extremely useful in the long run.
- the **neighbourhood** definition, while somewhat *cumbersome*, has the advantage of being closely related to ideas from **analysis**, and has a *historical basis*

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>



# Neighborhood definition (2-1)

- A **neighbourhood topology** on a **set**  $X$  assigns to each element  $x \in X$  a non empty set  $N(x)$  of **subsets** of  $X$ , called **neighbourhoods** of  $x$
- with the following properties:

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

## Neighborhood definition (2-2)

- the properties of a **neighbourhood topology**:
  - If  $N$  is a **neighbourhood** of  $x$  then  $x \in X$
  - If  $M$  is a **neighbourhood** of  $x$  and  $M \subseteq N \subseteq X$ , then  $N$  is a **neighbourhood** of  $x$
  - The **intersection** of two **neighbourhoods** of  $x$  is a **neighbourhood** of  $x$
  - If  $N$  is a **neighbourhood** of  $x$ , then  $N$  contains a **neighbourhood**  $M$  of  $x$  such that  $N$  is a **neighbourhood** of each **point** of  $M$ .

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

## Neighborhood definition (3-1)

- Then one says a **function**  $f : X \rightarrow Y$  is **continuous** wrt **neighbourhoods** on  $X$  and  $Y$  if for each  $x \in X$  and **neighbourhood**  $N$  of  $f(x)$  there is a **neighbourhood**  $M$  of  $x$  such that  $f(M) \subseteq N$ .
- The **open set** definition of **continuity** is then justified as being equivalent to this definition in terms of **neighbourhoods**.

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

## Neighborhood definition (3-2)

- One also says a set  $U$  in  $X$  is **open** if  $U$  is a **neighbourhood** of all of its **points**. THEN one can develop the **open set axioms** and show that one can recover the **neighbourhoods**.

<https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology>

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# Class (1)

- a **class** is a **collection** of **sets**  
(or sometimes other **mathematical objects**)  
that can be unambiguously defined  
by a **property** that all its members share.
- **Classes** act as a way to have **set-like collections**  
while **differing** from **sets** so as to **avoid Russell's paradox**

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

## Class (2)

- A class *that is not a set* is called a **proper class**, and
- a class *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
  - the **class** of all sets
  - the **class** of all ordinal numbers
  - the **class** of all cardinal numbers

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

## Class (3)

- consider "the **set** of all **sets** with **property**  $X$ ."
- especially when dealing with **categories**, since the **objects** of a **concrete category** are all **sets** with certain additional **structure**.
- However, **if** we're not *careful* about this we can get into serious *trouble* –

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>



## Class (4)

- let  $X$  be the **set** of all **sets** which do not contain *themselves*
- Since  $X$  is a **set**, we can ask whether  $X$  is an element of *itself*.
- But then we run into a **paradox** – **if**  $X$  contains *itself* as an element, **then** it does not, and vice versa.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

## Class (5)

- In order to avoid this **paradox**, we cannot consider the **collection** of all **sets** to be itself a **set**.
- This means we have to *throw out* the whole "the **set** of all **sets** with **property** X" construction. But we wanted that.
- So the way we get around it is to say that there's something called a **class**, which is like a **set** but not a **set**.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

## Class (6)

- Then we can talk about "the class  $X$  of all sets with property  $Y$ ."
- Since  $X$  is not a set, it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes, we need to create another name that goes another step back, and so forth.

<https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects>

# Class Examples (1)

- The **collection** of all **algebraic structures** of a given type will usually be a **proper class**.  
(a **class** *that is not a set* is called a **proper class**)
  - the **class** of all **groups**
  - the **class** of all **vector spaces**
  - and many others.
- Within set theory, many **collections** of **sets** turn out to be **proper classes**.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

## Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in **bijection** with the **class** of all ordinal numbers.
  - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
  - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))  
<https://editarians.com/cardinals-ordinals/>

# Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all **classes** are **sets**".
- These **paradoxes** do not arise with **classes** because there is no notion of **classes** containing **classes**.
- Otherwise, one could, for example, define a **class** of all **classes** that do not contain themselves, which would lead to a **Russell paradox** for classes.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

# Class Paradoxes (2)

- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain **classes** are **proper** (i.e., that they are not **sets**).
  - **Russell's paradox** *suggests a proof* that the **class** of all **sets** which do not contain themselves is **proper**
  - the **Burali-Forti paradox** *suggests* that the **class** of all ordinal numbers is **proper**.

[https://en.wikipedia.org/wiki/Class\\_\(set\\_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

# Russell's Paradox (1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined **property**, there is the **set** of **all** and only the **objects** that have that **property**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)



## Russell's Paradox (2)

- Let  $R$  be the **set of all sets** ( $R = \{x \mid x \notin x\}$ )  
that are not members of themselves ( $R \notin R$ ).
  - *if*  $R$  is not a **member** of itself ( $R \notin R$ ),  
*then* its definition (the **set of all sets**) entails  
that it is a **member** of itself ( $R \in R$ );
  - yet, *if* it is a **member** of itself ( $R \in R$ ),  
*then* it is not a **member** of itself ( $R \notin R$ ),  
since it is the **set of all sets**  
that are not members of themselves ( $R \notin R$ )
- the resulting **contradiction** is **Russell's paradox**.
- Let  $R = \{x \mid x \notin x\}$ , then  $R \in R \iff R \notin R$

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (3)

- Most *sets* commonly encountered are not *members* of themselves.
- For example, consider the *set* of all squares in a plane.
- This *set* is not itself a square in the plane, thus it is not a *member* of itself.
- Let us call a *set* "**normal**" if it is not a *member* of itself, and "**abnormal**" if it is a *member* of itself.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (4)

- Clearly every **set** must be either **normal** or **abnormal**.
- The **set** of squares in the plane is **normal**.
- In contrast, the **complementary set** that contains everything which is not a square in the plane is itself not a square in the plane, and so it is one of its own **members** and is therefore **abnormal**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

# Russell's Paradox (5)

- Now we consider the **set** of all **normal sets**,  $R$ , and try to determine whether  $R$  is **normal** or **abnormal**.
  - *If*  $R$  were **normal**, it would be contained in the **set** of all **normal sets** (itself), and therefore be **abnormal**;
  - on the other hand *if*  $R$  were **abnormal**, it would not be contained in the **set** of all **normal sets** (itself), and therefore be **normal**.
- This leads to the conclusion that  $R$  is neither **normal** nor **abnormal**: **Russell's paradox**.

[https://en.wikipedia.org/wiki/Russell%27s\\_paradox](https://en.wikipedia.org/wiki/Russell%27s_paradox)

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# Binary Relation (1)

- a **binary relation** associates **elements** of one **set**, called the **domain**, with **elements** of another set, called the **codomain**.
- A **binary relation** over sets  $X$  and  $Y$  is a new set of **ordered pairs**  $(x,y)$  consisting of **elements**  $x$  from  $X$  and  $y$  from  $Y$ .

[https://en.wikipedia.org/wiki/Binary\\_relation](https://en.wikipedia.org/wiki/Binary_relation)

## Binary Relation (2)

- It is a generalization of a unary function.
- It encodes the common concept of relation:
- an element  $x$  is related to an element  $y$ ,  
if and only if the pair  $(x, y)$  belongs  
to the set of ordered pairs that defines the binary relation.
- A binary relation is the most studied special case  $n = 2$   
of an  $n$ -ary relation over sets  $X_1, \dots, X_n$ ,  
which is a subset of the Cartesian product  $X_1 \times \dots \times X_n$ .

[https://en.wikipedia.org/wiki/Binary\\_relation](https://en.wikipedia.org/wiki/Binary_relation)

# Homogeneous Relation

- a **homogeneous relation** (also called endorelation) on a set  $X$  is a **binary relation** between  $X$  and itself, i.e. it is a **subset** of the **Cartesian product**  $X \times X$ .
- This is commonly phrased as "a **relation** on  $X$ " or "a **(binary) relation** over  $X$ ".
- An example of a **homogeneous relation** is the relation of **kinship**, where the relation is between people.

[https://en.wikipedia.org/wiki/Homogeneous\\_relation](https://en.wikipedia.org/wiki/Homogeneous_relation)



## Partially Ordered Set (1-1)

- a **partial order** on a **set** is an arrangement such that, for certain **pairs** of elements, one precedes the other.
- The word **partial** is used to indicate that not every **pair** of elements needs to be comparable; that is, there may be **pairs** for which neither element precedes the other.
- **Partial orders** thus generalize **total orders**, in which every pair is comparable.

[https://en.wikipedia.org/wiki/Partially\\_ordered\\_set](https://en.wikipedia.org/wiki/Partially_ordered_set)

# Partially Ordered Set (1-2)

- Formally, a **partial order** is a **homogeneous binary relation** that is **reflexive**, **transitive** and **antisymmetric**.
- A **partially ordered set** (**poset** for short) is a set on which a **partial order** is defined.
- A **reflexive**, **weak**, or **non-strict partial order**, commonly referred to simply as a **partial order**, is a **homogeneous relation**  $\leq$  on a **set**  $P$  that is **reflexive**, **antisymmetric**, and **transitive**.

[https://en.wikipedia.org/wiki/Partially\\_ordered\\_set](https://en.wikipedia.org/wiki/Partially_ordered_set)

## Partially Ordered Set (2)

- a **homogeneous relation**  $\leq$  on a **set**  $P$  that is **reflexive**, **antisymmetric**, and **transitive**.
- That is, for all  $a, b, c \in P$ , it must satisfy:
  - **Reflexivity**:  
 $a \leq a$ , i.e. every element is related to itself.
  - **Antisymmetry**:  
if  $a \leq b$  and  $b \leq a$  then  $a = b$ ,  
i.e. no two distinct elements precede each other.
  - **Transitivity**:  
if  $a \leq b$  and  $b \leq c$  then  $a \leq c$ .
- A **non-strict partial order** is also known as an **antisymmetric preorder**.

[https://en.wikipedia.org/wiki/Partially\\_ordered\\_set](https://en.wikipedia.org/wiki/Partially_ordered_set)

# Filter in Set Theory (1-1)

- A **filter** on a **set** may be thought of as representing a "**collection** of *large subsets*", one intuitive example being the **neighborhood filter**.
- keep *large* grains excluding *small* impurities

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

## Filter in Set Theory (1-2)

- When you put a filter in your sink, the idea is that you filter out the *big* chunks of food, and let the water and the *smaller* chunks go through (which can, in principle, be washed through the pipes) .
- You filter out the *larger parts*.
- A filter filters out the *larger sets*.
- It is a way to say "these *sets* are '*large*'"

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

# Filter in Set Theory (1-3)

- a **filter** on a set  $X$  is a family  $\mathcal{B}$  of subsets such that:

- 1  $X \in \mathcal{B}$  and  $\emptyset \notin \mathcal{B}$
- 2 if  $A \in \mathcal{B}$  and  $B \in \mathcal{B}$ ,  
then  $A \cap B \in \mathcal{B}$
- 3 If  $A, B \subset X, A \in \mathcal{B}$ , and  $A \subset B$ ,  
then  $B \in \mathcal{B}$

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)

## Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

$$X \in \mathcal{B}$$

- and "nothing" is definitely not;

$$\emptyset \notin \mathcal{B}$$

- if something is *larger* than a *large set*, then it is also *large*;

$$\text{If } A, B \subset X, A \in \mathcal{B}, \text{ and } A \subset B, \text{ then } B \in \mathcal{B}$$

- and two *large sets intersect* on a *large set*.

$$\text{If } A \in \mathcal{B} \text{ and } B \in \mathcal{B}, \text{ then } A \cap B \in \mathcal{B}$$

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)



# Filter in Set Theory (1-5)

- you can think about this as
  - being **co-finite**,
  - or being of **measure 1** on the **unit interval**,
  - or having a **dense open subset** (again on the unit interval).
- These are examples of ways where a [set](#) can be thought of as "almost everything". and that is the idea behind a filter.

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)



# Co-finite

- a **cofinite subset** of a set  $X$  is a subset  $A$  whose complement in  $X$  is a finite set.
- a subset  $A$  contains all but *finitely many* elements of  $X$
- If the complement is not finite, but is countable, then one says the set is **countable**.
- These arise naturally when generalizing structures on finite sets to infinite sets, particularly on infinite products, as in the **product topology** or direct sum.
- This use of the prefix "co" to describe a property possessed by a set's complement is *consistent* with its use in other terms such as "comeagre set".

<https://en.wikipedia.org/wiki/Cofiniteness>

# Unit interval

- the **unit interval** is the **closed interval**  $[0,1]$ , that is, the set of all real numbers that are greater than or equal to 0 and less than or equal to 1.
- It is often denoted  $I$  (capital letter I).
- In addition to its role in **real analysis**, the **unit interval** is used to study **homotopy theory** in the field of **topology**.
- the term "**unit interval**" is sometimes applied to the other shapes that an interval from 0 to 1 could take:  $(0,1]$ ,  $[0,1)$ , and  $(0,1)$ .
- However, the notation  $I$  is most commonly reserved for the **closed interval**  $[0,1]$ .

# Dense set

- In **topology**, a **subset**  $A$  of a topological space  $X$  is said to be **dense** in  $X$  if every **point** of  $X$  either belongs to  $A$  or else is arbitrarily "close" to a **member** of  $A$ 
  - for instance, the **rational numbers** are a **dense** subset of the **real numbers** because every **real number** either is a **rational number** or has a rational number arbitrarily close to it (see **Diophantine approximation**).
- Formally,  $A$  is **dense** in  $X$  if the *smallest* **closed subset** of  $X$  containing  $A$  is  $X$  itself.
- The **density** of a **topological space**  $X$  is the **least cardinality** of a **dense subset** of  $X$ .

[https://en.wikipedia.org/wiki/Dense\\_set](https://en.wikipedia.org/wiki/Dense_set)

# Proper Subset

- a **set**  $A$  is a **subset** of a set  $B$   
if all **elements** of  $A$  are also **elements** of  $B$ ;
- $B$  is then a **superset** of  $A$ .
- It is possible for  $A$  and  $B$  to be equal;
- if they are unequal, then  $A$  is a **proper subset** of  $B$ .
- The relationship of one **set** being a **subset** of another  
is called **inclusion** (or sometimes **containment**).
- $A$  is a **subset** of  $B$  may also be expressed  
as  $B$  includes (or contains)  $A$  or  $A$  is included (or contained) in  $B$ .
- A  **$k$ -subset** is a **subset** with  $k$  **elements**.

<https://en.wikipedia.org/wiki/Subset>

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# Proper Filter (1-1)

- Fix a **partially ordered set (poset)**  $P$ .
- Intuitively, a **filter**  $F$  is a **subset** of  $P$  whose members are **elements large enough** to satisfy some *criterion*.
- For instance, if  $x \in P$ , then the **set of elements above**  $x$  is a **filter**, called the **principal filter** at  $x$ .

[https://en.wikipedia.org/wiki/Filter\\_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

# Proper Filter (1-2)

- If  $x$  and  $y$  are **incomparable elements** of  $P$ , then neither the **principal filter** at  $x$  nor  $y$  is contained in the other
  - two **elements**  $x$  and  $y$  of a set  $P$  are said to be **comparable** with respect to a **binary relation**  $\leq$  if at least one of  $x \leq y$  or  $y \leq x$  is **true**. They are called **incomparable** if they are not **comparable**.
  - Hasse diagram of the natural numbers, partially ordered by " $x \leq y$  if  $x$  divides  $y$ ". The numbers 4 and 6 are **incomparable**, since neither divides the other.

[https://en.wikipedia.org/wiki/Filter\\_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

<https://en.wikipedia.org/wiki/Comparability>

# Proper Filter (1-3)

- Similarly, a **filter** on a **set**  $S$  contains those **subsets** that are sufficiently large to contain some given *thing*.
- For example, if  $S$  is the *real line* and  $x \in S$ , then the **family** of **sets** including  $x$  *in their interior* is a **filter**, called the **neighborhood filter** at  $x$ .
- The *thing* in this case is slightly larger than  $x$ , but it still does not contain any other specific point of the line.

[https://en.wikipedia.org/wiki/Filter\\_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))



## Proper Filter (2)

- The above considerations motivate the **upward closure** requirement in the definition below: "large enough" **objects** can always be made larger.
- To understand the other two conditions, reverse the roles and instead consider  $F$  as a "locating scheme" to find  $x$ .
- In this interpretation, one searches in some **space**  $X$ , and expects  $F$  to describe those **subsets** of  $X$  that contain the **goal**.
- The **goal** must be located somewhere; thus the empty set  $\emptyset$  can never be in  $F$ .
- And if two **subsets** both contain the **goal**, then should "zoom in" to their common region.

[https://en.wikipedia.org/wiki/Filter\\_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

## Proper Filter (3)

- An **ultrafilter** describes a "perfect locating scheme" where each scheme component gives new information (either "look here" or "look elsewhere").
- **Compactness** is the property that "every search is fruitful," or, to put it another way, "every locating scheme ends in a search result."
- A common use for a **filter** is to define properties that are satisfied by "generic" elements of some topological space.
- This application generalizes the "locating scheme" to find **points** that might be hard to write down explicitly.

[https://en.wikipedia.org/wiki/Filter\\_\(mathematics\)](https://en.wikipedia.org/wiki/Filter_(mathematics))

# Neighborhood Filter (1-1)

- Let  $X$  be a **set**;
- the **elements** of  $X$  are usually called **points**
- We allow  $X$  to be **empty**.
  
- Let  $\mathcal{N}$  be a **function**  
*assigning* to each  $x$  (**point**) in  $X$   
a non-empty **collection**  $\mathcal{N}(x)$  of **subsets** of  $X$ .
  
- The **elements** of  $\mathcal{N}(x)$  will be called  
**neighbourhoods** of  $x$  with respect to  $\mathcal{N}$   
(or, simply, **neighbourhoods** of  $x$ ).

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Neighborhood Filter (1-2)

- Let  $X$  be a [set](#);
- $\mathcal{N}$  : a [function](#) assigning to each [point](#)  $x$  in  $X$
- $\mathcal{N}(x)$  : a non-empty [collection](#) of [subsets](#) of  $X$ .
- The [elements](#) of  $\mathcal{N}(x)$ 
  - [subsets](#) of  $X$
  - [neighbourhoods](#) of  $x$  with respect to  $\mathcal{N}$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Neighborhood Filter (1-3)

- The function  $\mathcal{N}$  is called a neighbourhood topology if *some axioms* are satisfied;
- then  $X$  with  $\mathcal{N}$  is called a topological space –  $(X, \mathcal{N})$

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Neighborhood Filter (1-4)

- If  $(X, \mathcal{T})$  is a **topological space** and  $p \in X$ , a **neighbourhood** of  $p$  is a **subset**  $V$  of  $X$ , in which  $p \in U \subseteq V$ , and  $U$  is open.
- We say that  $V$  is a  $\mathcal{T}$ - **neighbourhood** of  $x \in X$  or that  $V$  is a **neighborhood** of  $x$
- The **set** of all **neighbourhoods** of  $x \in X$ , denoted  $\mathcal{N}_x$  is called the **neighbourhood filter** of  $x$

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

# Neighborhood Filter (1-4)

- An example of **Neighborhood Filters** on a **Topological space**.
- Let  $X = \{a, b, c\}$  and let  $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{b, c\}, \{a, b\}, X\}$
- Let
$$\mathcal{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, X\}$$
$$\mathcal{N}_b = \{\{b\}, \{a, b\}, \{b, c\}, X\}$$
$$\mathcal{N}_c = \{\{b, c\}, X\}.$$
- In this example  $a, c$  is a **neighborhood** of  $a$  but not of  $c$ .
- Thus a **set** does not have to be a **neighborhood** of all of its points.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

## Neighborhood Filter (2)

- One can specify a **topology** in more than four different ways.
  - ① The standard definition specifies the **open sets**, what we usually call a "**topology**."
  - ② to specify the **close sets** - this is of course only a *trivial difference*.
  - ③ to specify a **closure operation** on **subsets** of your **space**
  - ④ to specify a **neighborhood filter** for every **point** satisfying the **natural axiom** that every **neighborhood** of  $x$  is a **neighborhood** of every **point** of one of its **subsets**
- So in this sense **neighborhood filters** tell you everything they possibly could about a **topological space**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>



## Neighborhood Filter (3-1)

- Probably the best way to think about the **neighborhood filter** of  $x$ : is that it contains all information regarding **convergence** to  $x$ .
- In the first **topological spaces** one encounters, **convergence** is usually of **sequences**.
- But this isn't enough to describe the **topology** in arbitrary **spaces**, for instance the infinite-dimensional **spaces** of **functional analysis**.
- It becomes important to speak of **convergence** of **nets**, or of **filters**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

## Neighborhood Filter (3-2)

- A **filter** on  $X$  is just a nontrivial subset of the **powerset** of  $X$  **closed** under finite intersection and **superset**, and a **filter converges** to a **point**  $x$  if and only if it contains the **neighborhood filter** of  $x$ .
- In contrast to the case with **sequences**, this is enough to specify a **topology**: in fact it's enough to describe how **ultrafilters**, that is, **maximal filters**, **converge**.
- So in this sense the **neighborhood filter** encapsulates the viewpoint that **topology generalizes** the study of **convergent sequences**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

# Neighborhood Filter (4-1)

- in a sense the **neighborhood filter** describes the smallest **neighborhood** of a **point**
  - except that there is no **smallest neighborhood**!
- That's true, at least, in many of the most interesting **spaces**, and is the main reason to worry about a whole **filter** of **neighborhoods**
  - if there were a smallest **neighborhood** then in any hypothesis requiring something to hold on a sufficiently small **neighborhood** of  $x$  we could just pick the smallest **neighborhood**.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

# Neighborhood Filter (4-2)

- But the smallest neighborhood of a point must be contained in the intersection of all its neighborhoods, and in, say, a Hausdorff space the intersection of all neighborhoods of  $x$  is  $x$ , which is not a neighborhood of  $x$  when  $x$  is not isolated.
- So the filter functions as a virtual smallest neighborhood of  $x$ : it doesn't converge to a neighborhood of  $x$ , so we can't think about its limit, but functionally we do just that.

<https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter>

# Ultrafilter (1)

- an **ultrafilter** on a given **partially ordered set** (or "**poset**")  $P$  is a certain **subset** of  $P$ , namely a **maximal filter** on  $P$ ; that is, a **proper filter** on  $P$  that cannot be enlarged to a bigger **proper filter** on  $P$ .
- If  $X$  is an arbitrary **set**, its **power set**  $\mathcal{P}(X)$ , ordered by **set inclusion**, is always a Boolean algebra and hence a **poset**, and **ultrafilters** on  $\mathcal{P}(X)$  are usually called **ultrafilter** on the **set**  $X$ .

<https://en.wikipedia.org/wiki/Ultrafilter>

## Ultrafilter (2)

- In **order theory**, an **ultrafilter** is a **subset** of a **partially ordered set** that is **maximal** among all proper filters.
- This implies that any **filter** that properly contains an **ultrafilter** has to be equal to the whole **poset**.

<https://en.wikipedia.org/wiki/Ultrafilter>

## Ultrafilter (3)

- An **ultrafilter** on a **set**  $X$  may be considered as a **finitely additive measure** on  $X$ .
- In this view, every **subset** of  $X$  is either considered "*almost everything*" (has measure 1) or "*almost nothing*" (has measure 0), depending on whether it belongs to the given **ultrafilter** or not

<https://en.wikipedia.org/wiki/Ultrafilter>

# Ultrafilter (4)

- Formally, if  $P$  is a **set**, **partially ordered** by  $\leq$  then
- a **subset**  $F \subseteq P$  is called a **filter** on  $P$  if  $F$  is nonempty, for every  $x, y \in F$ , there exists some **element**  $z \in F$  such that  $z \leq x$  and  $z \leq y$ , and for every  $x \in F$  and  $y \in P$ ,  $x \leq y$  implies that  $y$  is in  $F$  too;
- a **proper subset**  $U$  of  $P$  is called an ultrafilter on  $P$  if  $U$  is a filter on  $P$ , and there is no **proper filter**  $F$  on  $P$  that properly extends  $U$  (that is, such that  $U$  is a **proper subset** of  $F$ ).

<https://en.wikipedia.org/wiki/Ultrafilter>



# Outline

- 1 Open Sets and Neighborhoods
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  - Class
- 2 Filters
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  - Proper Filter and Ultra Filter
  - **Filter Example**
- 3 Topological Space
  - Topological Space
  - A discrete topology
  - Examples of topology

# Filter Examples (1)

- Let  $X = 1, 2, 3$   
Choose some element from  $X$  say  $F = 1, 1, 2, 1, 3, 1, 2, 3$
- Then every **intersection** of an element of  $F$  with another element in  $F$  is in  $F$  again.  
Examples:  $1 \cap 1, 2, 3 = 1$     $1, 2 \cap 1, 2, 3 = 1, 2$   
 $1, 3 \cap 1, 2, 3 = 1, 3$     $1, 2, 3 \cap 1, 2, 3 = 1, 2, 3$
- Also the original  $X = 1, 2, 3$  is also in  $F$ .  
Here  $F = 1, 1, 2, 1, 3, 1, 2, 3$  is called the **filter** on  $X = 1, 2, 3$

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

## Filter Examples (2)

- Suppose we have the collection  $G = \{1, 1, 2, 1, 3, 2, 3, 1, 2, 3\}$
- Then we have  $1, 3 \cap 2, 3 = 3$  but 3 isn't in  $G$ .  
So this  $G$  is not called a filter.
- Now with  $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$   
can we put as any other element in it  
so that after placing the extra element it is still a filter?  
Probably not in this case.  
So on  $X = \{1, 2, 3\}$ ,  $F = \{1, 1, 2, 1, 3, 1, 2, 3\}$  is an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter Examples (3)

- If we have started say with  $H = 1, 1, 2, 1, 2, 3$   
this is still a **filter** on  $X = 1, 2, 3$   
but we can still add  $1, 3$   
and it will still be classified as **filter**.
- So on  $X = 1, 2, 3$   
 $F = 1, 1, 2, 1, 3, 1, 2, 3$  is an **Ultrafilter**  
but  $H = 1, 1, 2, 1, 2, 3$  is a filter but not an **Ultrafilter**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter Examples (4)

- Now suppose we have  $X = 1, 2, 3, 4$   
Let  $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- Every in intersection of element of  $F$  is in  $F$  again.  
We have as examples  $1, 4 \cap 1, 4 = 1, 4$     $1, 4 \cap 1, 2, 4 = 1, 4$   
 $1, 4 \cap 1, 3, 4 = 1, 4$     $1, 2, 4 \cap 1, 2, 4 = 1, 2, 4$     $1, 2, 4 \cap 1, 3, 4 = 1, 4$   
 $1, 3, 4 \cap 1, 3, 4 = 1, 3, 4$     $1, 2, 3, 4 \cap 1, 2, 3, 4 = 1, 2, 3, 4$
- Also  $X = 1, 2, 3, 4$  is also in  $F = 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$   
and the null element  $\emptyset =$  is not in  $F$ .

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter Examples (5)

- We call  $F$  a **filter** but not an **Ultrafilter** on  $X = 1, 2, 3, 4$
- We can still add element in it and it will still be a **filter** for instance by adding the element 1 from  $X = 1, 2, 3, 4$  we can have the filter  $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- This is an **Ultrafilter** on  $X = 1, 2, 3, 4$  as we cannot add any further element from  $X = 1, 2, 3, 4$  that satisfies **closures** on **intersection**.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter Examples (6)

- There is another collection of sets taken from  $X = 1, 2, 3, 4$
- which is the powerset  
 $P = \{1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$
- This contain the **null element**  $\emptyset =$  so we cannot call this as **Ultrafilter**.
- This is not a **proper filter** according to the article in Wikipedia.
- In the **powerset** every **intersection** of element is again in the **powerset** again but it contains  $t$
- he **null element**  $\emptyset =$  and isn't classified as proper filter.

<https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory>

# Filter Examples (7)

- There is another collection of sets taken from  $X = 1, 2, 3, 4$
- which is the powerset  
 $P = \{1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$
- This contain the **null element**  $\emptyset =$  so we cannot call this as **Ultrafilter**.
- This is not a **proper filter** according to the article in Wikipedia.
- In the **powerset** every **intersection** of element is again in the **powerset** again but it contains  $t$
- he **null element**  $\emptyset =$  and isn't classified as proper filter.

[https://en.wikipedia.org/wiki/Filter\\_\(set\\_theory\)#filter\\_base](https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base)



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# Topology

- **topology**  
from the Greek words  
τόπος, 'place, location',  
and λόγος, 'study'

<https://en.wikipedia.org/wiki/Topology>

## Topology (2)

- **topology** is concerned with the properties of a **geometric object** that are preserved
  - under **continuous deformations** such as
    - stretching
    - twisting
    - crumpling
    - bending
  - that is, without
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

<https://en.wikipedia.org/wiki/Topology>

# Topological space (1)

- a **topological space** is, roughly speaking,  
a **geometrical space**  
in which **closeness** is defined  
but cannot necessarily be **measured**  
by a **numeric distance**.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

## Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,
- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms  
formalizing the concept of closeness.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

## Topological space (3)

- There are several *equivalent definitions* of a **topology**, the most commonly used of which is the **definition** through **open sets**, which is easier than the others to manipulate.

[https://en.wikipedia.org/wiki/Borel\\_set](https://en.wikipedia.org/wiki/Borel_set)

## Topological space (4)

- A **topological space** is the most **general type** of a **mathematical space** that allows for the definition of
  - **limits**
  - **continuity**
  - **connectedness**
- Although very **general**, the concept of **topological spaces** is **fundamental**, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called **point-set topology** or **general topology**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Topological space (5)

- Common types of **topological spaces** include
  - **Euclidean spaces** : a **set** of **points** satisfying certain **relationships**, expressible in terms of **distance** and **angles**.
  - **metric spaces** : a **set** together with a notion of **distance** between **points**. The **distance** is measured by a function called a **metric** or **distance function**.
  - **manifolds** : a topological space that *locally* resembles **Euclidean space** near each point. More precisely, an **n-manifold** is a topological space with the property that each **point** has a **neighborhood** that is **homeomorphic** to an **open subset** of **n-dimensional Euclidean space**.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)



# Discrete Topology

- a **discrete space** is a **topological space**,  
in which the **points** form a **discontinuous sequence**,  
meaning they are isolated from each other in a certain sense.
- The **discrete topology** is  
the finest **topology** that can be given on a **set**.
  - every **subset** is **open**
  - every **singleton subset** is an **open set**

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# Singleton

- a **singleton**, also known as a **unit set** or **one-point set**, is a **set** with exactly one element.
- for example, the **set**  $\{0\}$  is a **singleton** whose single element is 0

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# Indiscrete Space (1)

- a **topological space** with the **trivial topology** is one where the only **open sets** are the **empty set** and the **entire space**.
- Such spaces are commonly called **indiscrete**, **anti-discrete**, **concrete** or **codiscrete**.
  - every **subset** can be **open** (the **discrete topology**), or
  - no **subset** can be **open** (the **indiscrete topology**) except the space itself and the empty set .

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

## Indiscrete Space (2)

- Intuitively, this has the consequence that all **points** of the **space** are "**lumped together**" and cannot be **distinguished** by topological means (not topologically **distinguishable** points)
- Every **indiscrete space** is a **pseudometric space** in which the **distance** between any two **points** is zero.

[https://en.wikipedia.org/wiki/Discrete\\_space](https://en.wikipedia.org/wiki/Discrete_space)

# $T_0$ Space

- a **topological space**  $X$  is a  $T_0$  **space** or **if** for every **pair** of distinct points of  $X$ , at least one of them has a **neighborhood** not containing the other.
- In a  $T_0$  **space**, all **points** are topologically distinguishable.
- This condition, called the  $T_0$  **condition**, is the weakest of the **separation axioms**.
- Nearly all topological spaces *normally* studied in mathematics are  $T_0$  **space**.

[https://en.wikipedia.org/wiki/Kolmogorov\\_space](https://en.wikipedia.org/wiki/Kolmogorov_space)

# Topologically distinguishable points

- Intuitively, an **open set** provides a *method* to *distinguish* two **points**.
- two points in a **topological space**, there exists an **open set**
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)

# Topologically distinguishable points

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# Metric spaces

- In this manner, one may speak of whether two **points**, or more generally two **subsets**, of a **topological space** are "**near**" without concretely defining a **distance**.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of **distance**, which are called **metric spaces**.

[https://en.wikipedia.org/wiki/Open\\_set](https://en.wikipedia.org/wiki/Open_set)



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# Why called a discrete topology? (1)

- the **discrete topology** is the **finest** topology  
- it cannot be subdivided further.
- if you think of the elements of the set  
as indivisible "discrete" atoms,  
each one appears as a **singleton set**.
- can effectively "see" the **individual points**  
in the topology itself.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

## Why called a discrete topology? (2)

- the **indiscrete topology** consists only of  $X$  itself and  $\emptyset$ .
- This topology obscures everything about *how many points* were in the original set.
- It fully agglomerates the points of the set together.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

# Why called a discrete topology? (3)

- helpful to think of **topologies** as **obscuring** or **blurring** together the *underlying points* of the set.
- **topologies** are all about **nearness relations**: points in an **open set** are in the vicinity of one another.
- **topologically indistinguishable** points points that never appear alone in an **open set**,
  - they are so **close** as to be **identical**, from the perspective of the topology,

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

## Why called a discrete topology? (4)

- the **discrete topology**
  - has no **indistinguishable points**.
  - **obscures** nothing about the underlying set.
  - each **point** in the set is
    - clearly highlighted
    - distinguishable
    - recoverable as an **open singleton set** in the topology.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

# Why called a discrete topology? (5)

- If you think of **topologies** that can arise from **metrics**, the **discrete topology** arises from **metrics** such as

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

- This metric "shatters" the **points**  $X$ , isolating each one within its own **unit ball**.
  - In such a space, the only **convergent sequences** are the ones that are eventually constant;
  - you can't find points **arbitrarily close** to any other points.
  - because points are **isolated** in this way,
  - it makes sense to call the space "**discrete**".

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

## Why called a discrete topology? (6-1)

- Every **function** from a **discrete space** is automatically continuous.
- for this reason, the **discrete topology** is the one that best "represents"  $X$  in **topological space**.
- the nature of a **set** is characterized by its **functions**,
- the nature of a **topological space** is characterized by its continuous functions.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

## Why called a discrete topology? (6-2)

- So, note that if  $T$  is any **topological space**, there's a natural bijjective correspondence between **functions**  $f : X \rightarrow \text{set}(T)$  and continuous morphisms  $g : \text{discrete}(X) \rightarrow T$ .
- For every **function** on  $X$ , you can find a continuous function on  $\text{discrete}(X)$ , and given any continuous function on  $\text{discrete}(X)$ , you can uniquely recover a **function** on  $X$
- The **discrete topology** best represents the structure of the **set**  $X$  which, as you say, is discretized into individual points.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>



## Why called a discrete topology? (7-1)

- Throughout **abstract algebra**, **isomorphisms** describe which structures are "the same".
- A **topological isomorphism** (a **homeomorphism**) between two **topologies** says that they are essentially the same topology.
- An **isomorphism** of **sets** is just a **bijection**;
- it says that the **sets** contain the same number of **elements**.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

# Why called a discrete topology? (7-3)

- Continuing the discussion of functions above, two **discrete topologies** are **topologically isomorphic** (**homeomorphic**) if and only if their underlying **sets** are **isomorphic** as sets (**bijective**).
- Put casually, this means that the discrete-topology-creating process maintains the similarity and differences between the underlying **sets**: **discrete topologies** are the same if and only if their underlying **sets** are the same.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

# Why called a discrete topology? (8)

- This is all the more important when we realize that **sets** are the same when they have the same number of **points**.
- Hence **discrete topologies** are the same when (and only when) their underlying **sets** have "**discrete points**" in the same quantity.
- You can count the points in a **discrete topology** through **isomorphisms**, and the **discrete topology** is the only topology for which this is possible.

<https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...>

# Characterization (1)


- In mathematics, a characterization of an object is a set of conditions that, while different from the definition of the object, is logically equivalent to it.[1] To say that "Property P characterizes object X" is to say that not only does X have property P, but that X is the only thing that has property P (i.e., P is a defining property of X). Similarly, a set of properties P is said to characterize X, when these properties distinguish X from all other objects. Even though a characterization identifies an object in a unique way, several characterizations can exist for a single object. Common mathematical expressions for a characterization of X in terms of P include "P is necessary and sufficient for X", and "X holds if and only if P".

[https://en.wikipedia.org/wiki/Characterization\\_\(mathematics\)](https://en.wikipedia.org/wiki/Characterization_(mathematics))

## Characterization (2)

- It is also common to find statements such as "Property Q characterizes Y up to isomorphism". The first type of statement says in different words that the extension of P is a singleton set, while the second says that the extension of Q is a single equivalence class (for isomorphism, in the given example — depending on how up to is being used, some other equivalence relation might be involved).
- A reference on mathematical terminology notes that characteristic originates from the Greek term kharax, "a pointed stake":
- From Greek kharax came kharakhter, an instrument used to mark or engrave an object. Once an object was marked, it became distinctive, so the character of something came to mean its distinctive nature. The Late Greek suffix -istikos converted the noun character into the adjective characteristic, which, in addition to maintaining its adjectival meaning, later became a noun as well.[2]

## Characterization (4)

- A rational number, generally defined as a ratio of two integers, can be characterized as a number with finite or repeating decimal expansion.[1] A parallelogram is a quadrilateral whose opposing sides are parallel. One of its characterizations is that its diagonals bisect each other. This means that the diagonals in all parallelograms bisect each other, and conversely, that any quadrilateral whose diagonals bisect each other must be a parallelogram. "Among probability distributions on the interval from 0 to  $\infty$  on the real line, memorylessness characterizes the exponential distributions." This statement means that the exponential distributions are the only probability distributions that are memoryless, provided that the distribution is continuous as defined above (see Characterization of probability distributions for more). "According to Bohr–Mollerup theorem, among all functions  $f$  such that  $f(1) = 1$  and  $x f(x) = f(x + 1)$  for  $x > 0$ , log-convexity characterizes the gamma function." 

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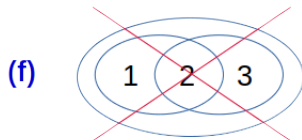
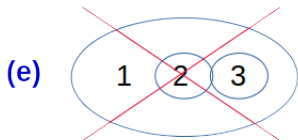
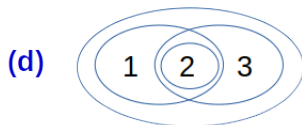
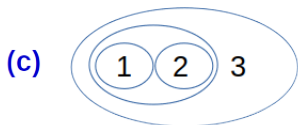
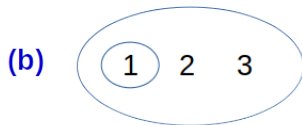
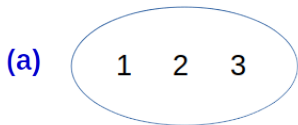
# Examples of topology (1)

- Let  $\tau$  be denoted with the circles, here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and two non-examples **(e)**, **(f)** of topologies on the three-point set  $\{1, 2, 3\}$ .
- **(e)** is not a topology because the union of  $\{2\}$  and  $\{3\}$  [i.e.  $\{2, 3\}$ ] is missing;
- **(f)** is not a topology because the intersection of  $\{1, 2\}$  and  $\{2, 3\}$  [i.e.  $\{2\}$ ], is missing.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)



# Examples of topology (2)



Every union of  $(c)$ 

$(c)$  is a topology  $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$   
every union of  $(c)$

| $\cup$      | $\{\}$      | $\{1\}$     | $\{2\}$     | $\{1,2\}$   | $\{1,2,3\}$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| $\{\}$      | $\{\}$      | $\{1\}$     | $\{2\}$     | $\{1,2\}$   | $\{1,2,3\}$ |
| $\{1\}$     | $\{1\}$     | $\{1\}$     | $\{1,2\}$   | $\{1,2\}$   | $\{1,2,3\}$ |
| $\{2\}$     | $\{2\}$     | $\{1,2\}$   | $\{2\}$     | $\{1,2\}$   | $\{1,2,3\}$ |
| $\{1,2\}$   | $\{1,2\}$   | $\{1,2\}$   | $\{1,2\}$   | $\{1,2\}$   | $\{1,2,3\}$ |
| $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ |

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

Every intersection of  $(c)$ 

$(c)$  is a topology  $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$   
every intersection of  $(c)$

| $\cap$      | $\{\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ | $\{1,2,3\}$ |
|-------------|--------|---------|---------|-----------|-------------|
| $\{\}$      | $\{\}$ | $\{\}$  | $\{\}$  | $\{\}$    | $\{\}$      |
| $\{1\}$     | $\{\}$ | $\{1\}$ | $\{\}$  | $\{1\}$   | $\{1\}$     |
| $\{2\}$     | $\{\}$ | $\{\}$  | $\{2\}$ | $\{2\}$   | $\{2\}$     |
| $\{1,2\}$   | $\{\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ | $\{1,2\}$   |
| $\{1,2,3\}$ | $\{\}$ | $\{1\}$ | $\{2\}$ | $\{1,2\}$ | $\{1,2,3\}$ |

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Every union of (f)

(f) is not a topology  $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$   
**every union of (f)**

|             |             |             |             |             |
|-------------|-------------|-------------|-------------|-------------|
| $\cup$      | $\{\}$      | $\{1,2\}$   | $\{2,3\}$   | $\{1,2,3\}$ |
| $\{\}$      | $\{\}$      | $\{1,2\}$   | $\{2,3\}$   | $\{1,2,3\}$ |
| $\{1,2\}$   | $\{1,2\}$   | $\{1,2\}$   | $\{1,2,3\}$ | $\{1,2,3\}$ |
| $\{2,3\}$   | $\{2,3\}$   | $\{1,2,3\}$ | $\{2,3\}$   | $\{1,2,3\}$ |
| $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ | $\{1,2,3\}$ |

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Every intersection of (f)

(f) is not a topology  $\{\{\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$   
every intersection of (f)

| $\cap$      | $\{\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|-------------|--------|-----------|-----------|-------------|
| $\{\}$      | $\{\}$ | $\{\}$    | $\{\}$    | $\{\}$      |
| $\{1,2\}$   | $\{\}$ | $\{1,2\}$ | $\{2\}$   | $\{1,2\}$   |
| $\{2,3\}$   | $\{\}$ | $\{2\}$   | $\{2,3\}$ | $\{2,3\}$   |
| $\{1,2,3\}$ | $\{\}$ | $\{1,2\}$ | $\{2,3\}$ | $\{1,2,3\}$ |

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

## Examples of topology (3)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the *trivial* or *indiscrete topology* on  $X$  is  
the family  $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\emptyset, X\}$   
consisting of only the two subsets of  $X$   
required by the axioms  
forms a topology of  $X$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Examples of topology (4)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the family  $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$   
 $= \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$   
of six **subsets** of  $X$  forms another **topology** of  $X$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)

# Examples of topology (5)

- Given  $X = \{1, 2, 3, 4\}$ ,  
the *discrete topology* on  $X$  is  
the *power set* of  $X$ , which is the family  $\tau = \wp(X)$   
consisting of *all possible subsets* of  $X$ .  
the family

$$\begin{aligned}\tau = & \{ \{\}, \{1\}, \{2\}, \{3\}, \{4\} \\ & \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \\ & \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\} \end{aligned}$$

- In this case the topological space  $(X, \tau)$   
is called a *discrete space*.

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)



# Examples of topology (6)

- Given  $X = \mathbb{Z}$ , the set of integers, the family  $\tau$  of all finite subsets of the integers plus  $\mathbb{Z}$  itself is not a topology, because (for example) the union of all finite sets not containing zero is not finite but is also not all of  $\mathbb{Z}$ , and so it cannot be in  $\tau$ .

[https://en.wikipedia.org/wiki/Topological\\_space](https://en.wikipedia.org/wiki/Topological_space)