

Euclidean Vector Space (1A)

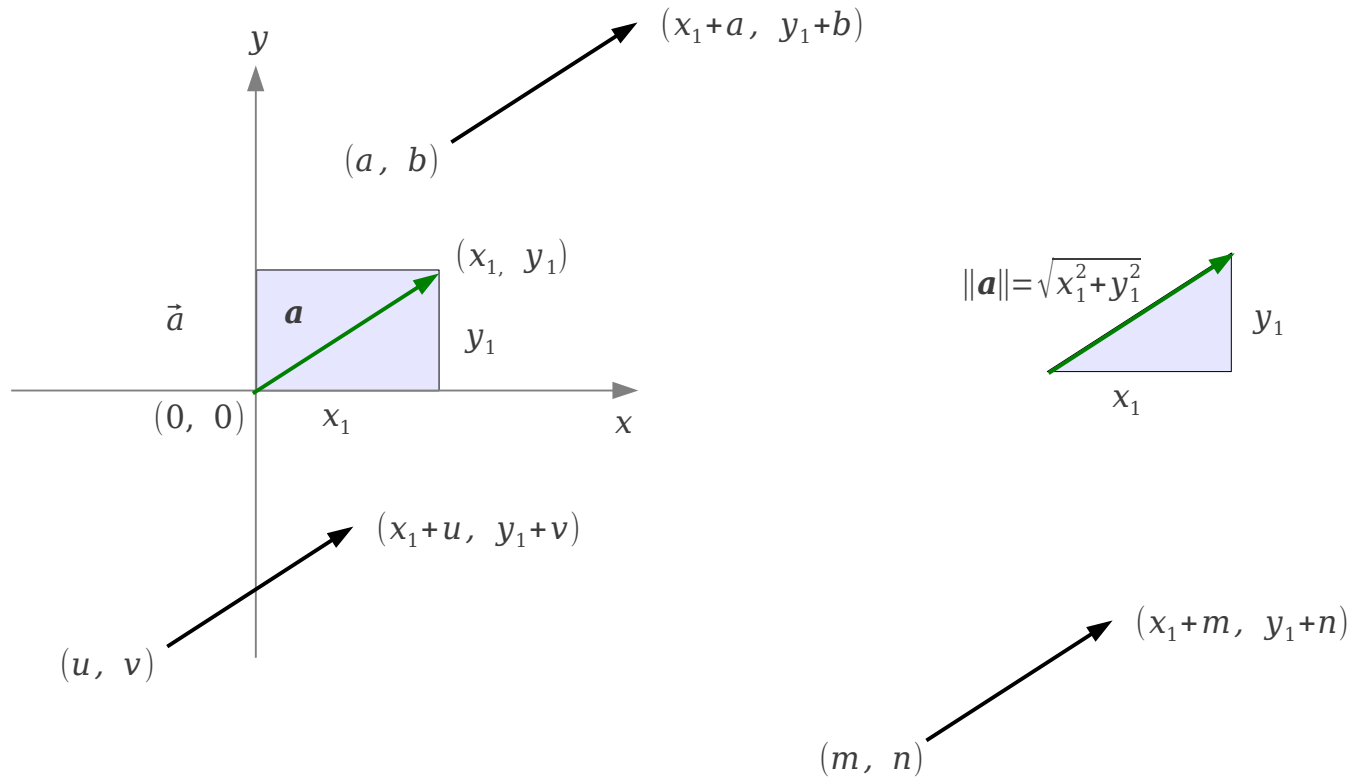
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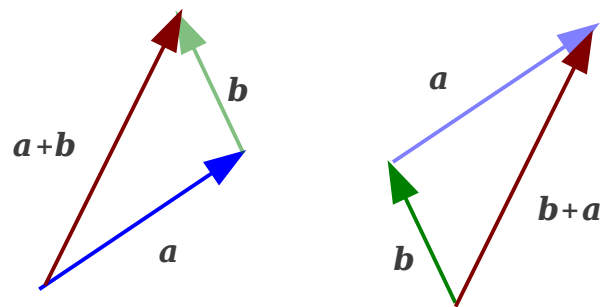
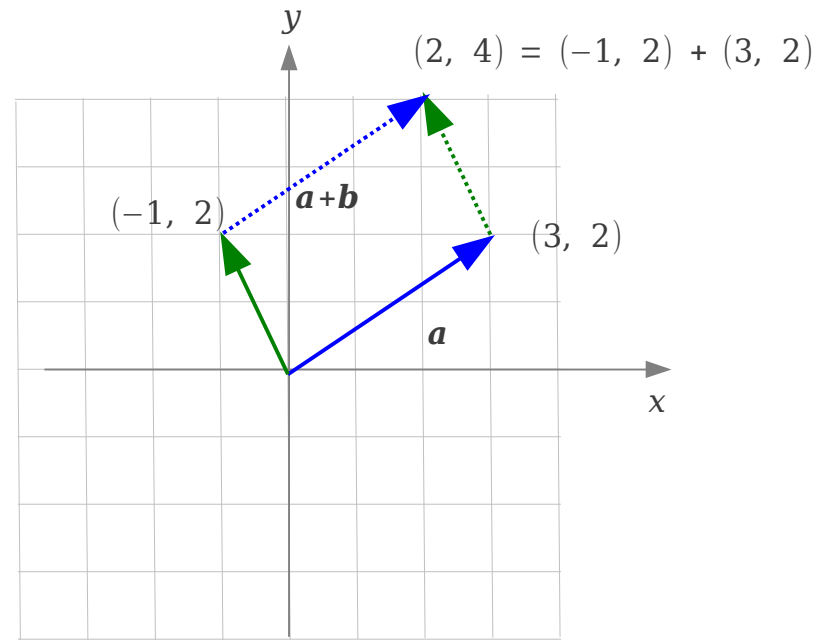
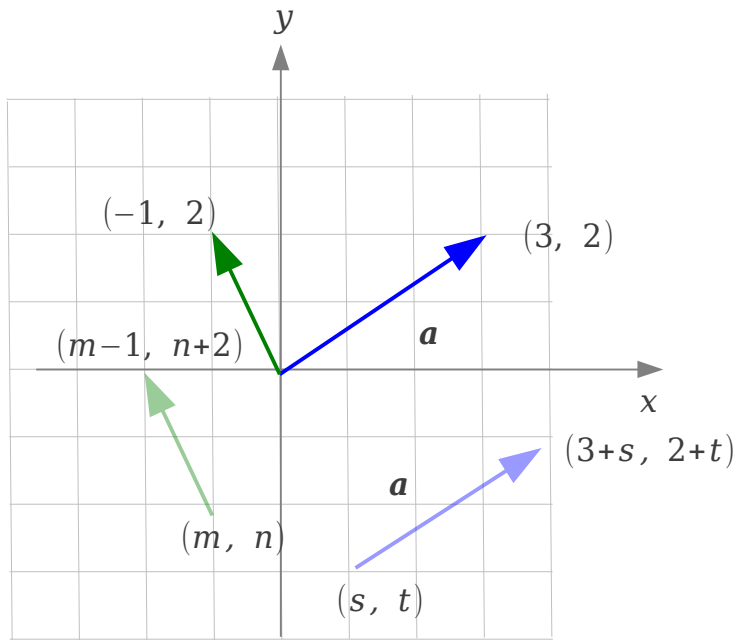
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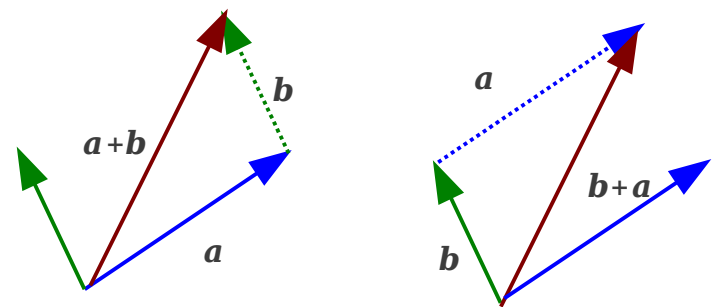
Vectors



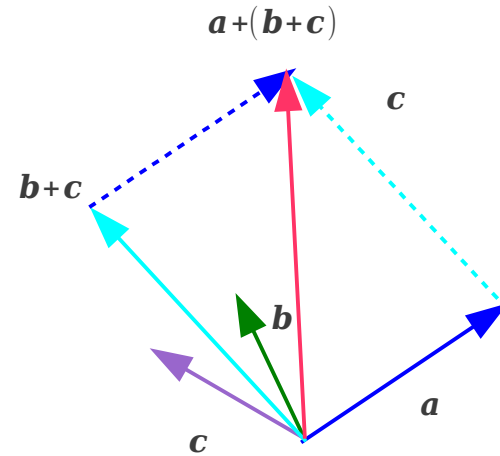
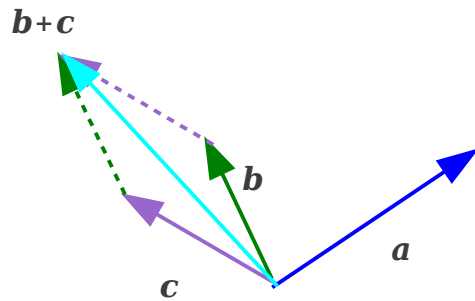
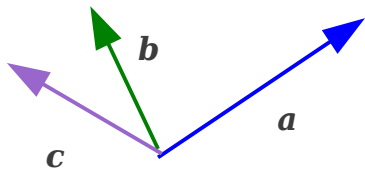
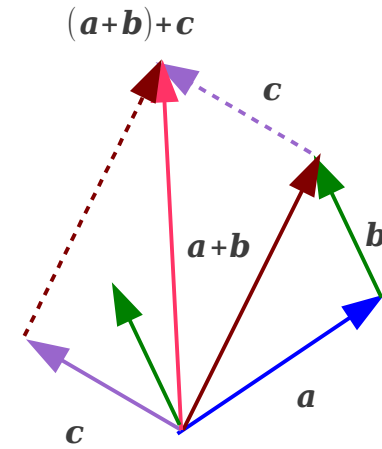
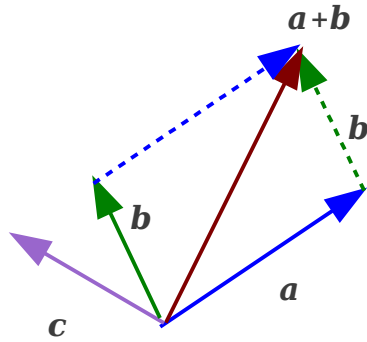
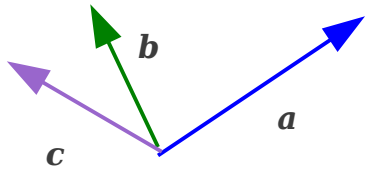
Vector Addition



$$a+b = b+a$$

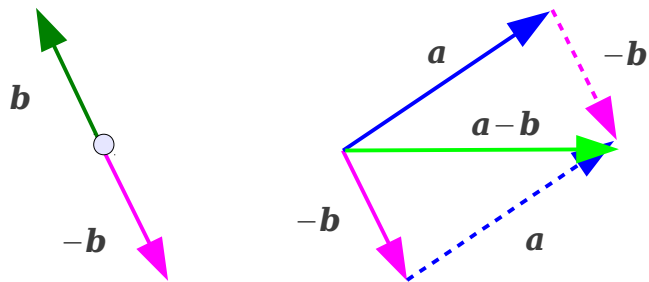
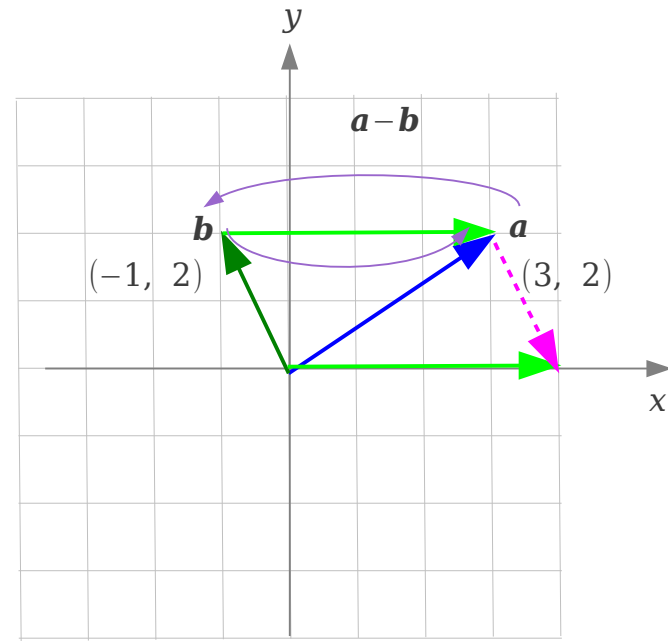
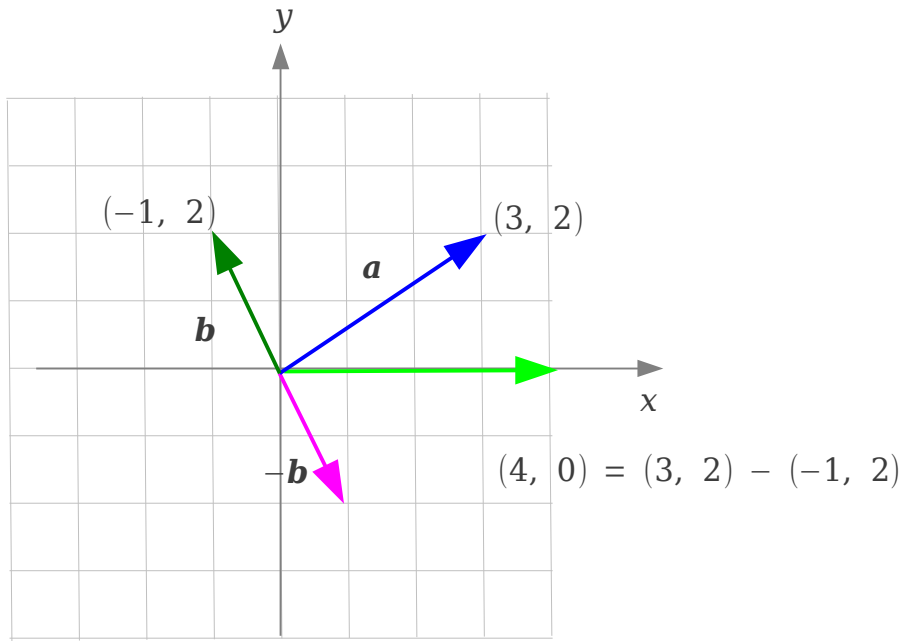


Vector Addition



$$(a+b)+c = a+(b+c)$$

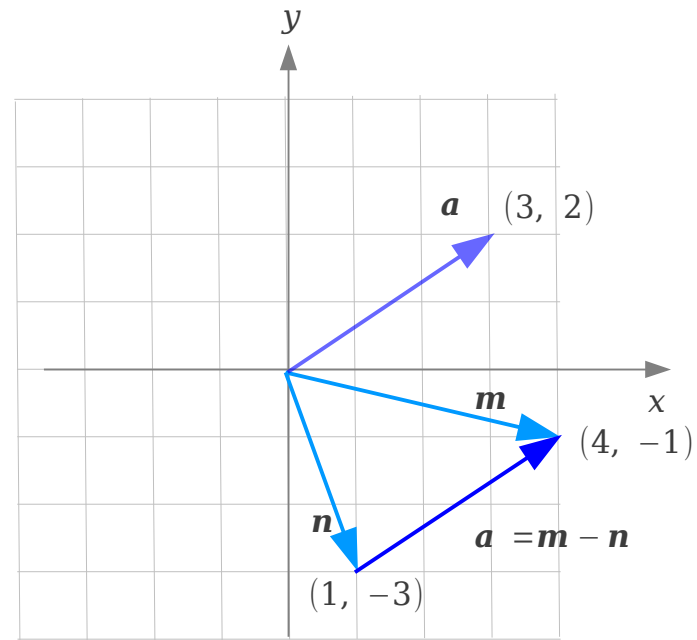
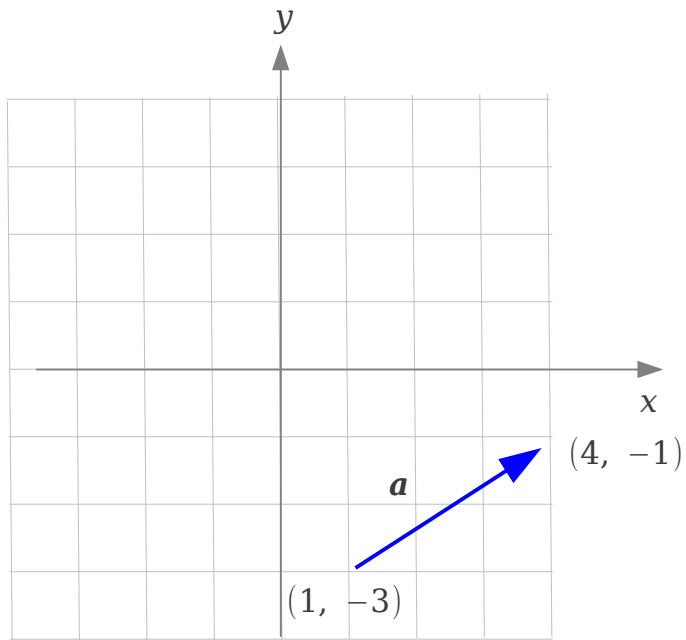
Vector Subtraction



$a - b$

subtract a from b
arrow from b to a

Component Form



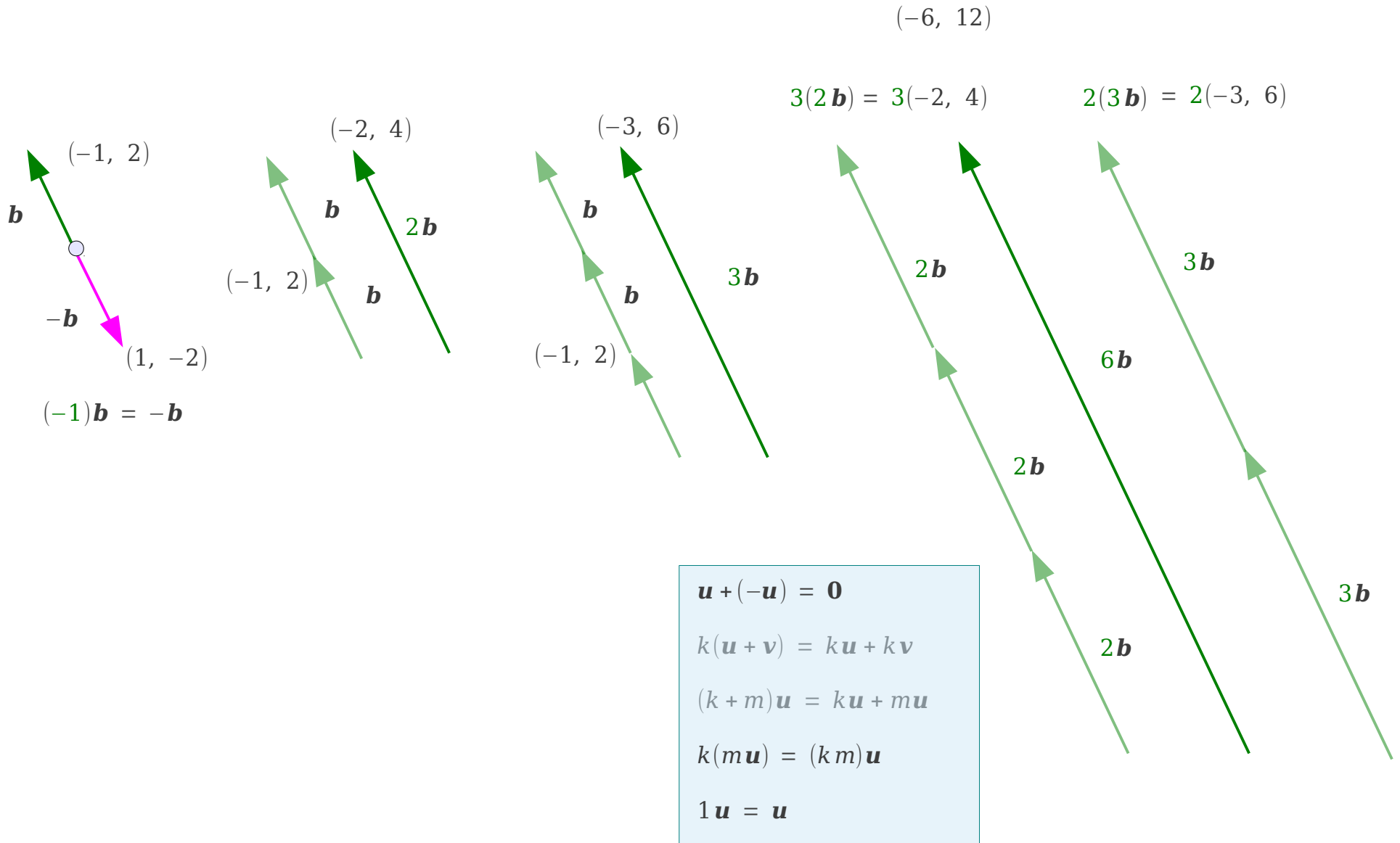
$$m = (4, -1)$$

$$n = (1, -3)$$

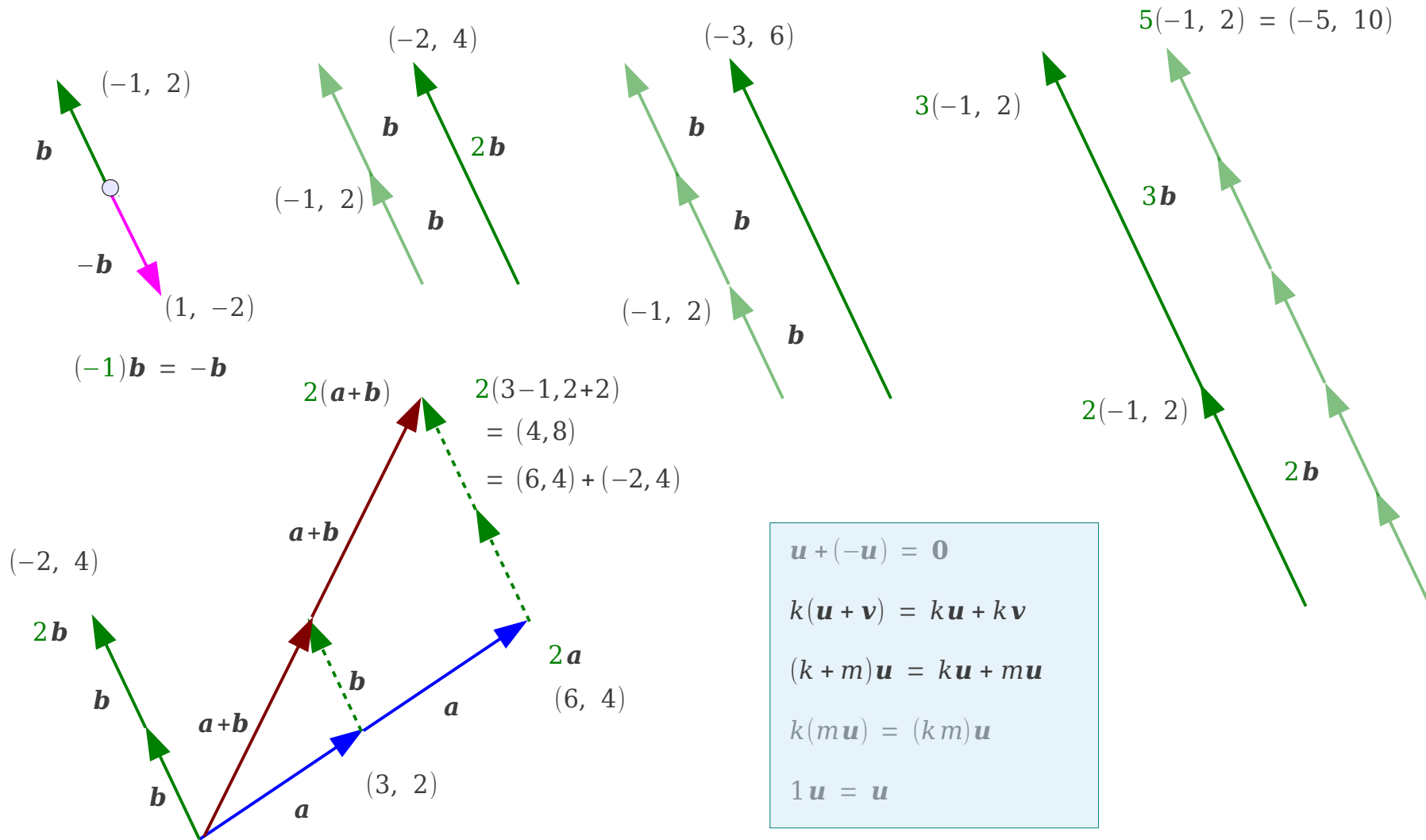
$$a = m - n = (4, -1) - (1, -3) = (3, 2)$$

Finding the **Component Form** of a vector

Scalar Multiplication (1)



Scalar Multiplication (2)



n-Space

Ordered 2-tuples (v_1, v_2) $v_1, v_2 \in R$

$R^2 \iff$ 2-space \iff { all ordered 2-tuples (v_1, v_2) }

Ordered 3-tuples (v_1, v_2, v_3) $v_1, v_2, v_3 \in R$

$R^3 \iff$ 3-space \iff { all ordered 3-tuples (v_1, v_2, v_3) }

Ordered n-tuples (v_1, v_2, \dots, v_n) $v_1, v_2, \dots, v_n \in R$

$R^n \iff$ n-space \iff { all ordered n-tuples (v_1, v_2, \dots, v_n) }

set



Properties of Real Vector Spaces

$\mathbf{u}, \mathbf{v}, \mathbf{w}$ vectors in \mathbb{R}^n

k, m scalars

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$$

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

$$k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$$

$$(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$$

$$k(m\mathbf{u}) = (km)\mathbf{u}$$

$$1\mathbf{u} = \mathbf{u}$$

Linear Combination

$\mathbf{w}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ vectors in \mathbb{R}^n
 k_1, k_2, \dots, k_r scalars

\mathbf{w} is a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$

$$\mathbf{w} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + \dots + k_r \mathbf{v}_r$$

$$\begin{aligned} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} &= k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= m_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + m_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= n_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + n_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + n_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{aligned}$$

Vector Magnitude

- Norm
 - Length
 - Magnitude
- $$\|\mathbf{v}\|$$

$$R^2 \quad \mathbf{v} = (v_1, v_2) \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2} \geq 0$$

$$R^3 \quad \mathbf{v} = (v_1, v_2, v_3) \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \geq 0$$

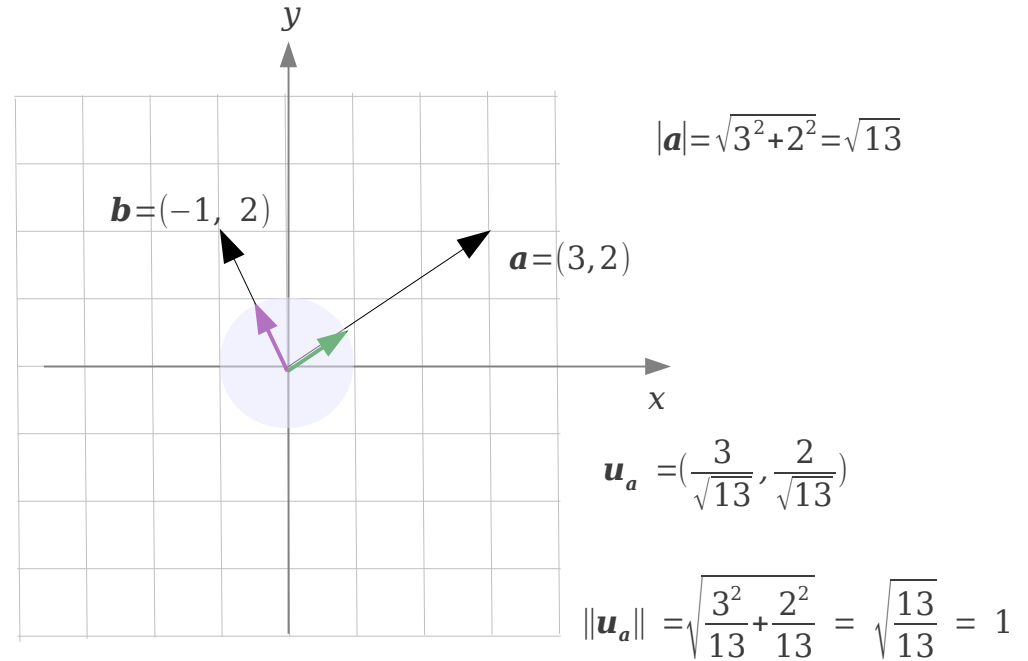
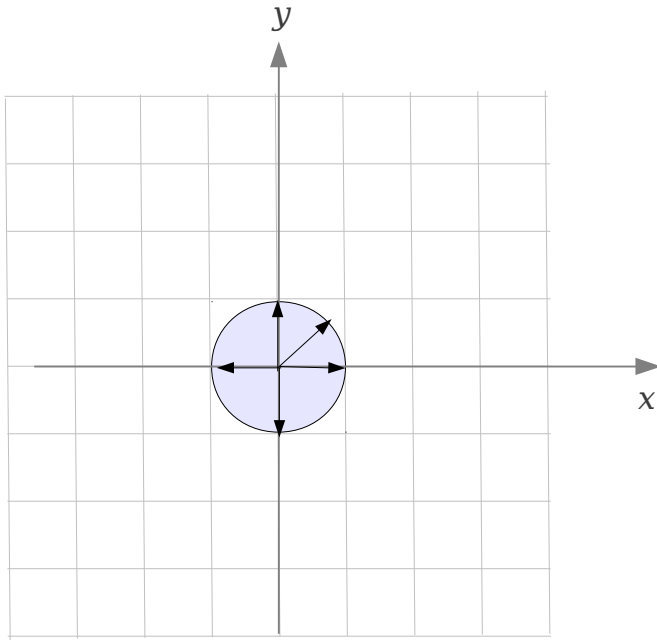
$$R^n \quad \mathbf{v} = (v_1, v_2, \dots, v_n) \quad \|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} \geq 0$$

$$\|\mathbf{v}\| \geq 0$$

$$\|\mathbf{v}\| = 0 \quad \mathbf{v} = \mathbf{0}$$

$$\|k\mathbf{v}\| = |k| \|\mathbf{v}\| \geq 0$$

Unit Vector



$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

$$\|\mathbf{u}\| = \left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\|$$

$$k = \|\mathbf{v}\| \geq 0$$

$$= \left\| \frac{\mathbf{v}}{k} \right\| = \frac{\|\mathbf{v}\|}{k}$$

$$= \frac{\|\mathbf{v}\|}{\|\mathbf{v}\|} = 1$$

Standard Unit Vectors

R^2 \leftrightarrow 2-space \leftrightarrow { all ordered 2-tuples (v_1, v_2) }

$$\mathbf{i} = (1, 0) \qquad \mathbf{v} = (v_1, v_2) \qquad = v_1(1, 0)$$

$$\mathbf{j} = (0, 1) \qquad \qquad \qquad + v_2(0, 1)$$

R^3 \leftrightarrow 3-space \leftrightarrow { all ordered 3-tuples (v_1, v_2, v_3) }

$$\mathbf{i} = (1, 0, 0) \qquad \mathbf{v} = (v_1, v_2, v_3) \qquad = v_1(1, 0, 0)$$

$$\mathbf{j} = (0, 1, 0) \qquad \qquad \qquad + v_2(0, 1, 0)$$

$$\mathbf{k} = (0, 0, 1) \qquad \qquad \qquad + v_3(0, 0, 1)$$

R^n \leftrightarrow n-space \leftrightarrow { all ordered n-tuples (v_1, v_2, \dots, v_n) }

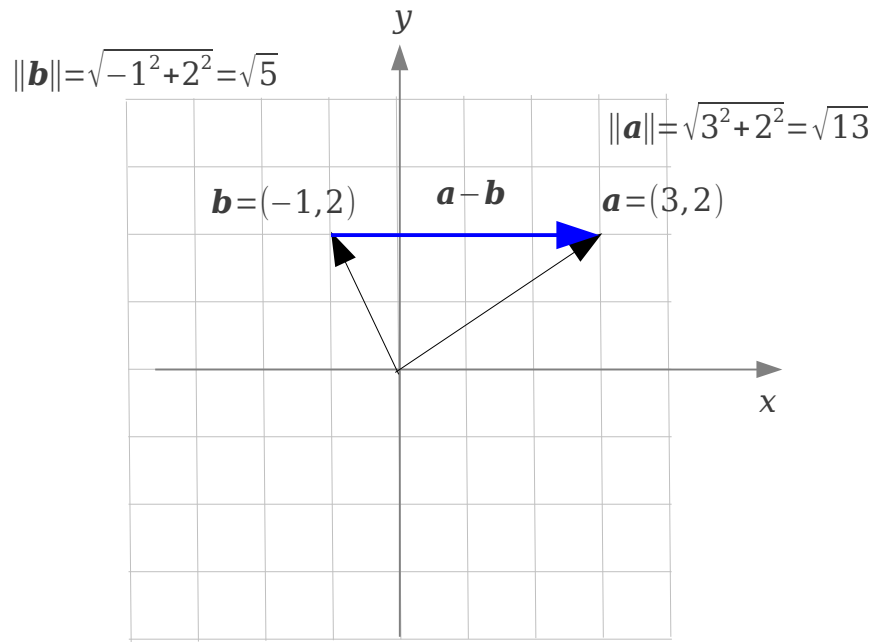
$$\mathbf{e}_1 = (1, 0, \dots, 0) \qquad \mathbf{v} = (v_1, v_2, \dots, v_n) \qquad = v_1(1, 0, \dots, 0)$$

$$\mathbf{e}_2 = (0, 1, \dots, 0) \qquad \qquad \qquad + v_2(0, 1, \dots, 0)$$

$$\dots \qquad \qquad \qquad + \dots$$

$$\mathbf{e}_n = (0, 0, \dots, 1) \qquad \qquad \qquad + v_n(0, 0, \dots, 1)$$

Distance between Vectors

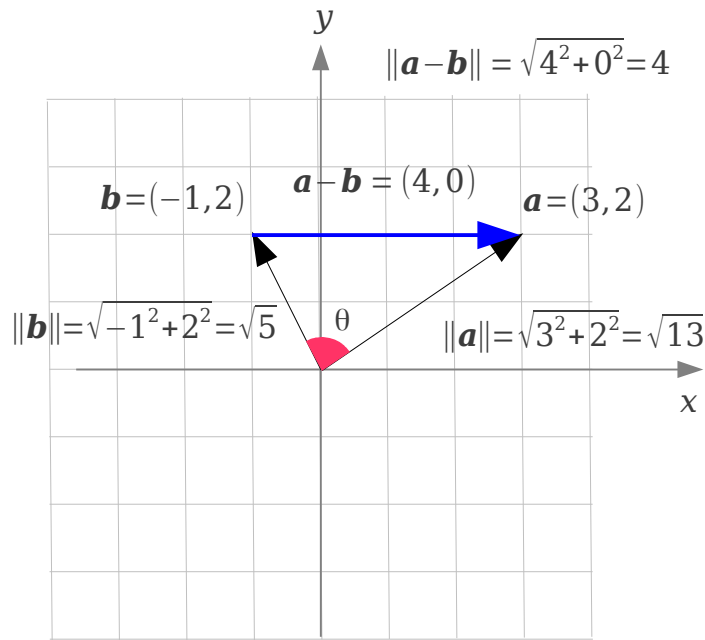


$$\begin{aligned} R^2 \quad d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \geq 0 \end{aligned}$$

$$\begin{aligned} R^3 \quad d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2} \geq 0 \end{aligned}$$

$$\begin{aligned} R^n \quad d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ &= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2} \geq 0 \end{aligned}$$

Law of Cosine



$$4^2 = \sqrt{13}^2 + \sqrt{5}^2 - 2\sqrt{13}\sqrt{5}\cos\theta$$

$$2 = 2\sqrt{13}\sqrt{5}\cos\theta \quad \cos\theta = \frac{1}{\sqrt{65}}$$

$$\mathbf{a} \cdot \mathbf{b} = 3 \cdot (-1) + 2 \cdot 2 = -3 + 4 = 1$$

Inner Product

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$R^2 \quad d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \geq 0$$

Law of Cosine

$$\|\mathbf{a} - \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

$$\begin{aligned} \mathbf{a} - \mathbf{b} \\ = (a_1 - b_1, a_2 - b_2) \end{aligned}$$

$$\begin{aligned} \mathbf{a} &= (a_1, a_2) \\ \mathbf{b} &= (b_1, b_2) \end{aligned}$$

$$(a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$a_1^2 + a_2^2$$

$$b_1^2 + b_2^2$$

$$-2(a_1 b_1 + a_2 b_2)$$

-2

$$a_1^2 + a_2^2$$

$$b_1^2 + b_2^2$$

$$\sqrt{a_1^2 + a_2^2}$$

$$\sqrt{b_1^2 + b_2^2}$$

cosθ

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

Dot Product

Dot Product
Euclidean Inner Product

\mathbf{a}, \mathbf{b} Vectors in R^n
 θ Angle between \mathbf{a}, \mathbf{b}

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

 $\mathbf{b} \leftarrow \mathbf{a}$

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\| \|\mathbf{a}\| \cos 0^\circ$$



$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$$



$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

$$R^2 \quad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \\ = a_1 b_1 + a_2 b_2$$

$$R^3 \quad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \\ = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$R^n \quad \mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \\ = a_1 b_1 + a_2 b_2 + \cdots a_n b_n$$

Properties of Dot Products

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

$$k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a}) \cdot \mathbf{b}$$

$$\mathbf{a} \cdot \mathbf{a} \geq 0 \quad \mathbf{a} \cdot \mathbf{a} = 0 \iff \mathbf{a} = \mathbf{0}$$

$$\mathbf{0} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{0} = 0$$

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$$

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$$

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c}$$

$$k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{v})$$

$$\begin{aligned} R^n \quad \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

Cauchy-Schwarz Inequality (1)

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

$$-1 \leq \cos\theta \leq +1$$

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \leq +1$$

$$\frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\| \|\mathbf{b}\|} \leq 1$$

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\|$$

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned}$$

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

$$\mathbf{b} = (b_1, b_2, \cdots, b_n)$$

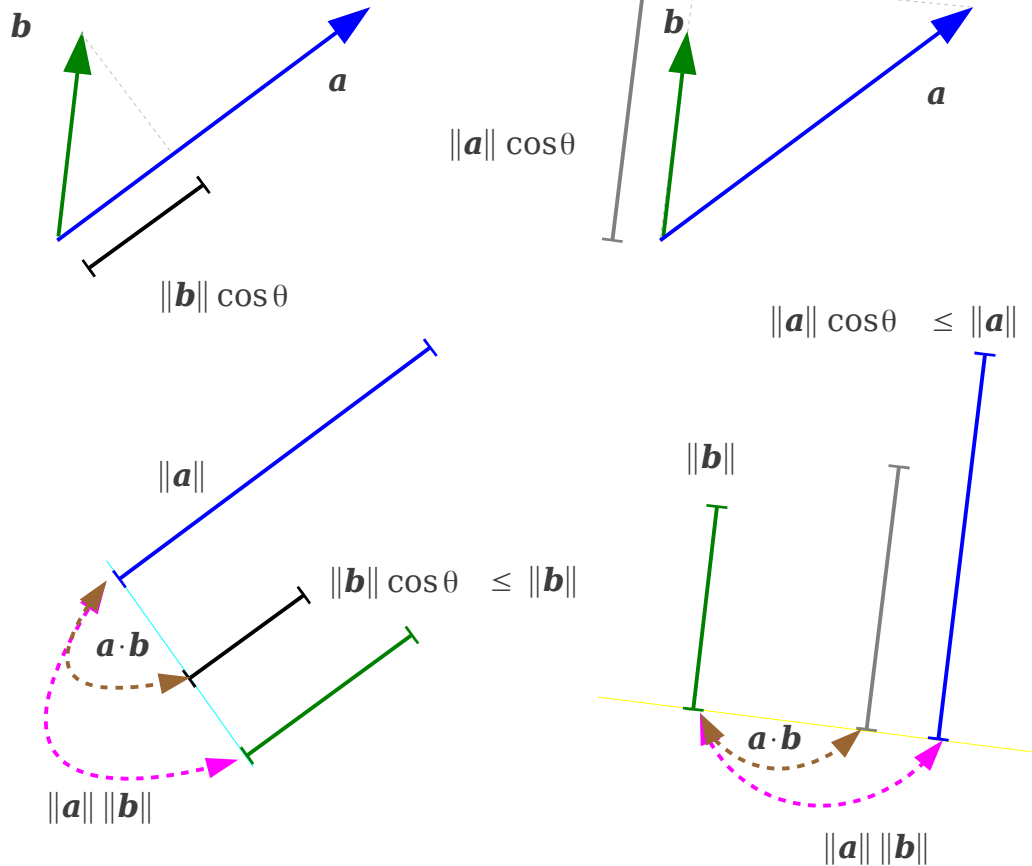
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$\|\mathbf{b}\| = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2}$$

Cauchy-Schwarz Inequality (2)

Assume $0 \leq \theta \leq 90^\circ$



$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta \\ &= a_1 b_1 + a_2 b_2 + \dots + a_n b_n \end{aligned}$$

$$\mathbf{a} = (a_1, a_2, \dots, a_n)$$

$$\mathbf{b} = (b_1, b_2, \dots, b_n)$$

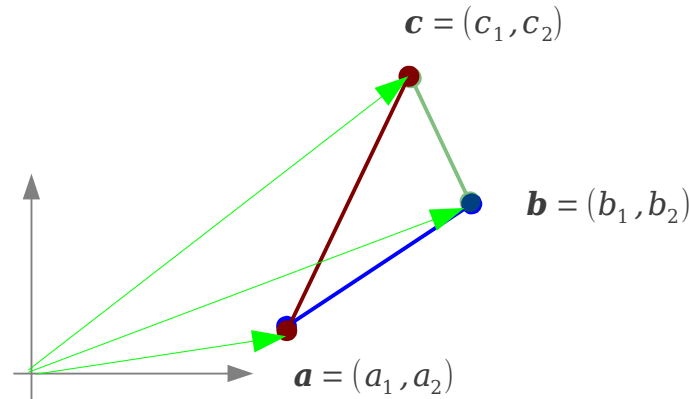
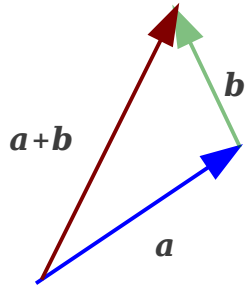
$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

$$\|\mathbf{b}\| = \sqrt{b_1^2 + b_2^2 + \dots + b_n^2}$$

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\| \|\mathbf{b}\| \quad \text{for } 0 \leq \theta \leq 180^\circ$$

Triangular Inequalities



$$\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$$

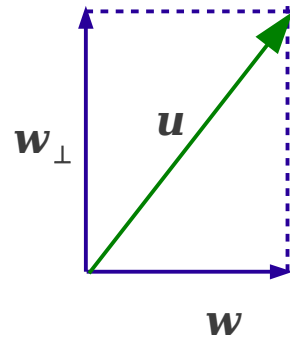
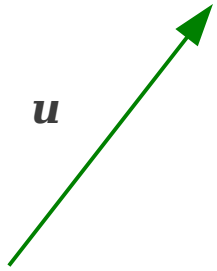
$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\ &= (\mathbf{a} \cdot \mathbf{a}) + 2(\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{b}) \\ &\leq \|\mathbf{a}\|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + \|\mathbf{b}\|^2 \\ &\leq \|\mathbf{a}\|^2 + 2\|\mathbf{a}\| \|\mathbf{b}\| + \|\mathbf{b}\|^2 \\ &\leq (\|\mathbf{a}\| + \|\mathbf{b}\|)^2 \end{aligned}$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} &= \|\mathbf{a}\|^2 \\ \mathbf{b} \cdot \mathbf{b} &= \|\mathbf{b}\|^2 \\ \mathbf{a} \cdot \mathbf{b} &\leq |\mathbf{a} \cdot \mathbf{b}| \\ |\mathbf{a} \cdot \mathbf{b}| &\leq \|\mathbf{a}\| \|\mathbf{b}\| \end{aligned}$$

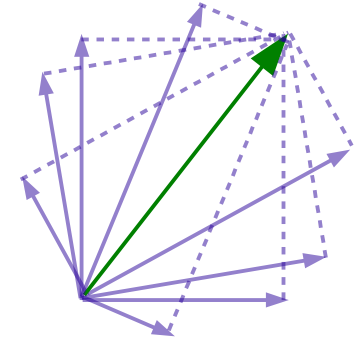
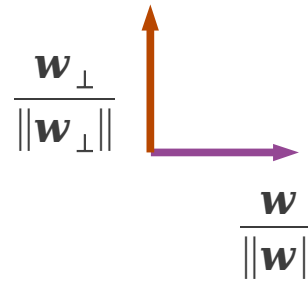
$$d(\mathbf{a}, \mathbf{b}) \leq d(\mathbf{a}, \mathbf{c}) + d(\mathbf{c}, \mathbf{b})$$

$$\begin{aligned} d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ &= \|\mathbf{a} - \mathbf{c} + \mathbf{c} - \mathbf{b}\| \\ &= \|(\mathbf{a} - \mathbf{c}) + (\mathbf{c} - \mathbf{b})\| \\ &\leq \|\mathbf{a} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{b}\| \\ &\leq d(\mathbf{a}, \mathbf{c}) + d(\mathbf{c}, \mathbf{b}) \end{aligned}$$

Vector Decomposition (1)



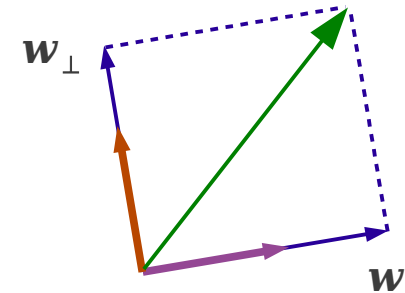
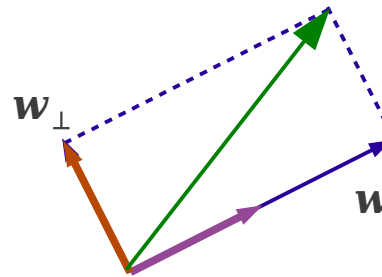
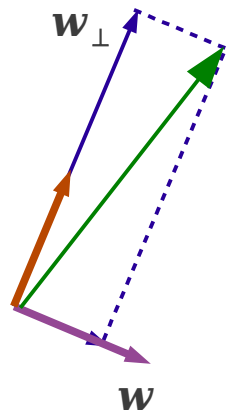
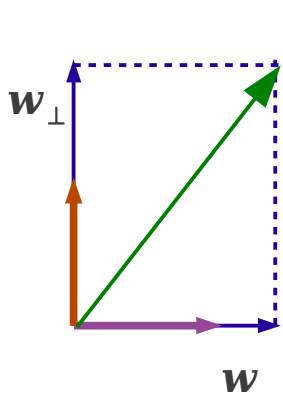
unit vectors



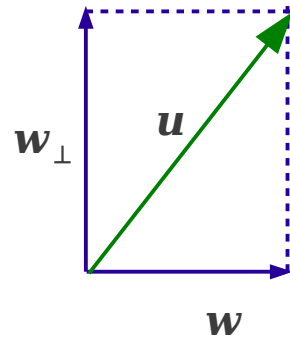
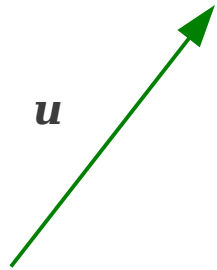
$$\mathbf{u} = \mathbf{w} + \mathbf{w}_\perp$$

$$\mathbf{u} - \mathbf{w} = \mathbf{w}_\perp$$

Various Decompositions

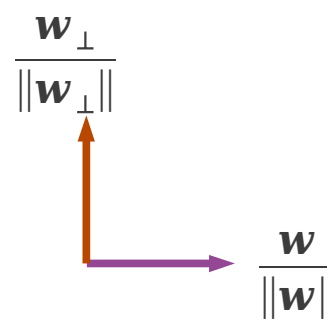


Vector Decomposition (2)

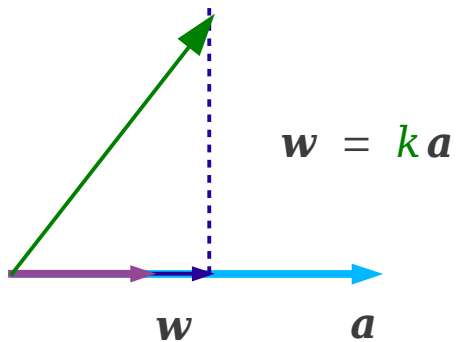
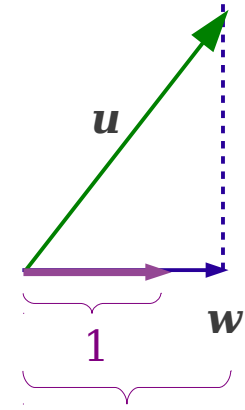


$$\mathbf{u} = \mathbf{w} + \mathbf{w}_\perp$$

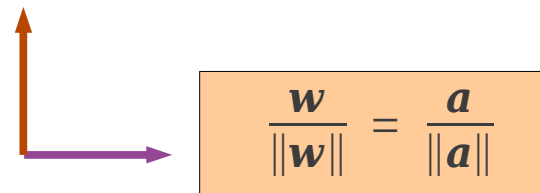
$$\mathbf{u} - \mathbf{w} = \mathbf{w}_\perp$$



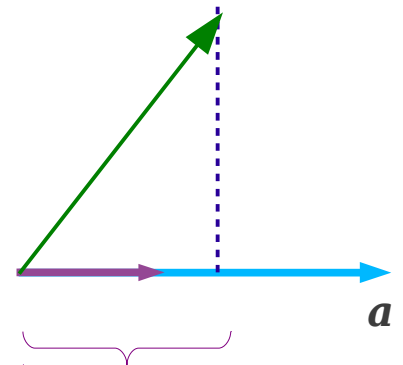
$$\mathbf{u} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} = \|\mathbf{u}\| \frac{\|\mathbf{w}\|}{\|\mathbf{w}\|} \cos \theta = \|\mathbf{u}\| \cos \theta$$



unit vectors



$$\mathbf{u} \cdot \frac{\mathbf{a}}{\|\mathbf{a}\|} = \|\mathbf{u}\| \frac{\|\mathbf{a}\|}{\|\mathbf{a}\|} \cos \theta = \|\mathbf{u}\| \cos \theta$$



Projection

unit vectors

scale factor

another scale factor w.r.t. a

$$u \cdot \frac{a}{\|a\|} = \|u\| \cos \theta$$

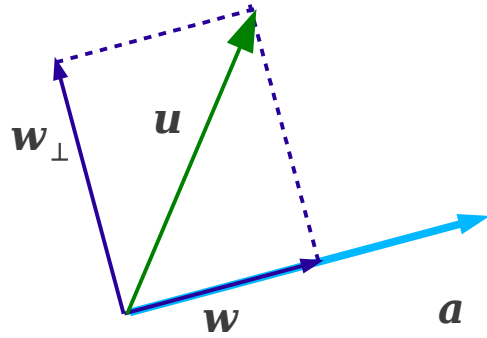
$$\left(\frac{u \cdot a}{\|a\|} \right) \frac{a}{\|a\|} = \left(\frac{u \cdot a}{\|a\|} \right) \frac{a}{\|a\|} = \frac{u \cdot a}{\|a\|^2} a$$

$$w = k a = \frac{u \cdot a}{\|a\|^2} a$$

$$w_{\perp} = u - w = u - \frac{u \cdot a}{\|a\|^2} a$$

$u = w + w_{\perp}$
 $u - w = w_{\perp}$

Projection (2)



$$\mathbf{w} = k\mathbf{a}$$

vector component of \mathbf{u} along \mathbf{a}

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}$$

vector component of \mathbf{u} orthogonal \mathbf{a}

$$\mathbf{w}_{\perp} = \mathbf{u} - \mathbf{w}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}$$

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|} = \|\mathbf{u}\| \cos\theta$$

Orthogonality

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

$$-1 \leq \cos\theta \leq +1$$

$$-1 \leq \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \leq +1$$

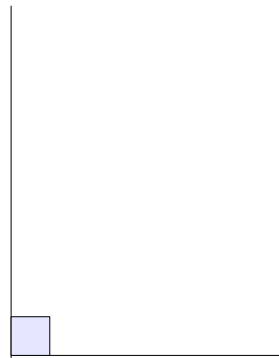
$$\frac{|\mathbf{a} \cdot \mathbf{b}|}{\|\mathbf{a}\| \|\mathbf{b}\|} \leq 1$$

$$\cos 90^\circ = 0$$

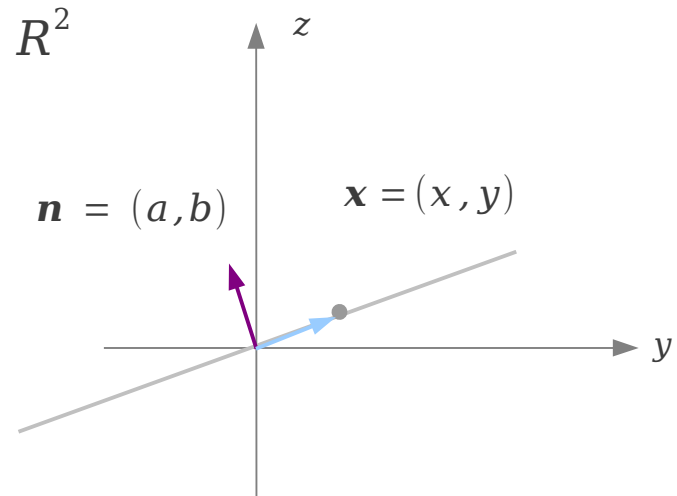
$$\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = 0$$

$$\mathbf{a} \cdot \mathbf{b} = 0$$

$$\begin{aligned} R^n \quad \mathbf{a} \cdot \mathbf{b} &= \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta \\ &= a_1 b_1 + a_2 b_2 + \dots + a_n b_n \end{aligned}$$

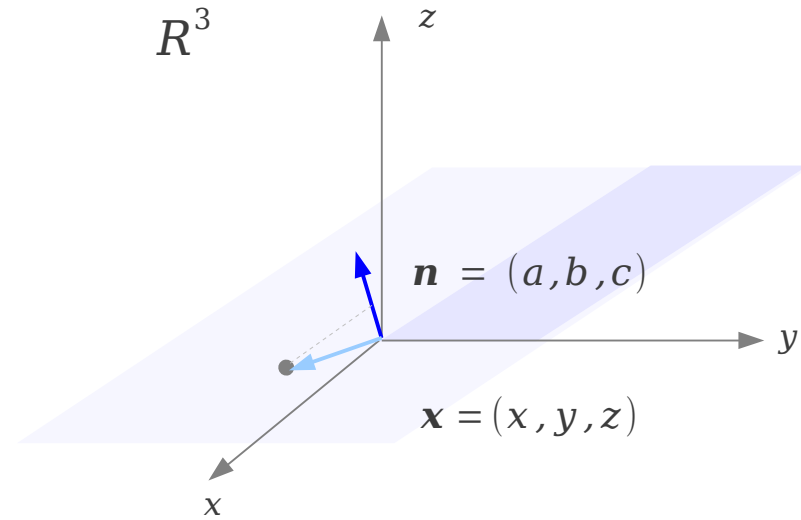


Point-Normal Equation (1)



$$\mathbf{n} \cdot \mathbf{x} = 0$$

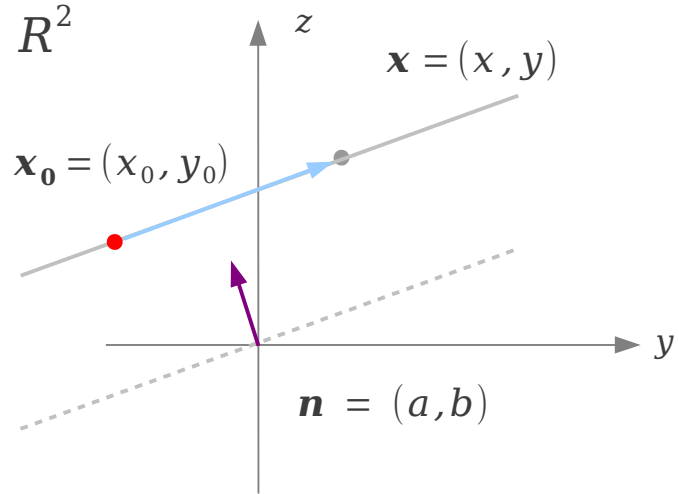
$$ax + by = 0$$



$$\mathbf{n} \cdot \mathbf{x} = 0$$

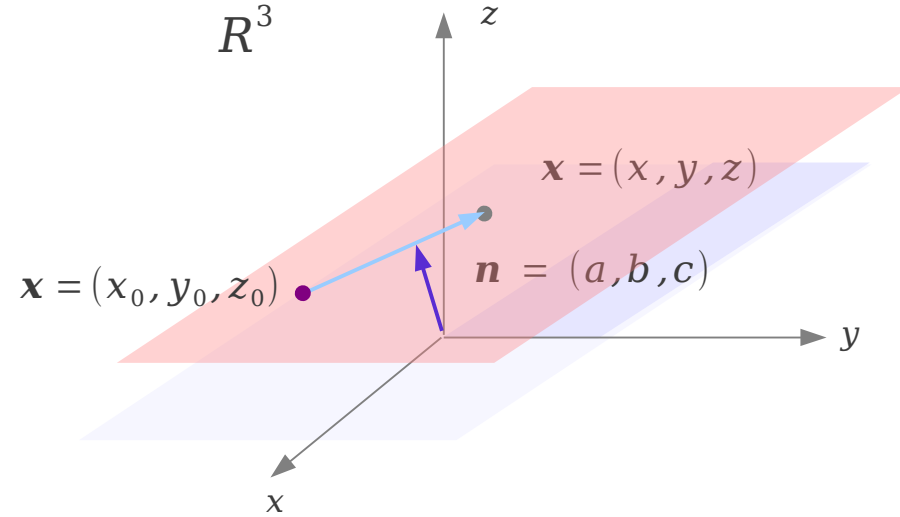
$$ax + by + cz = 0$$

Point-Normal Equation (2)



$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

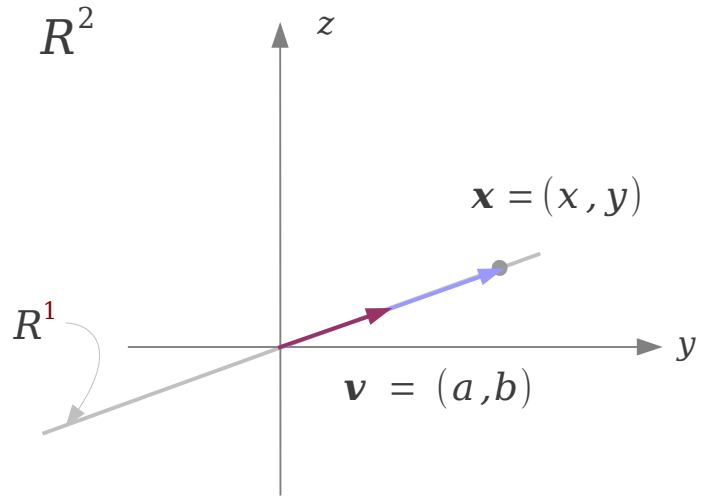
$$a(x - x_0) + b(y - y_0) = 0$$



$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

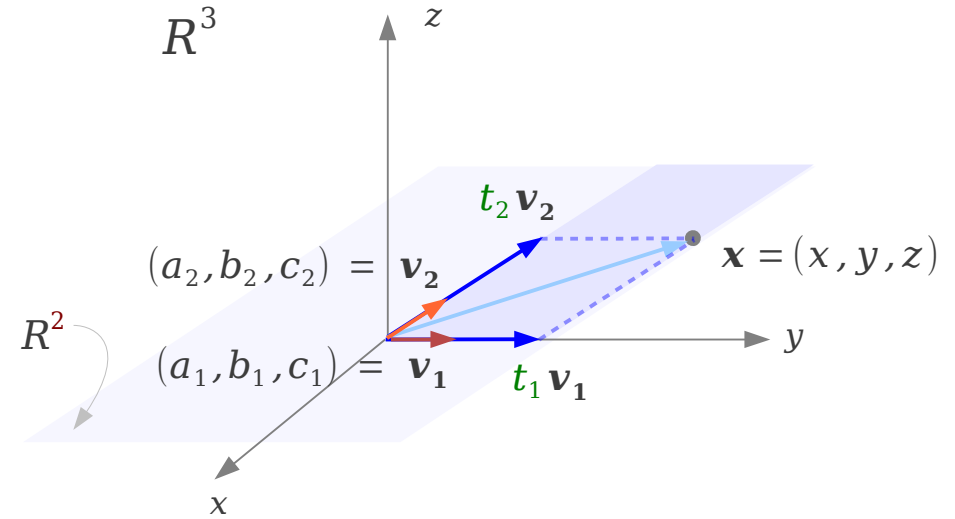
Parametric Equation (1)



$$\mathbf{x} = t\mathbf{v}$$

$$x = ta$$

$$y = tb$$



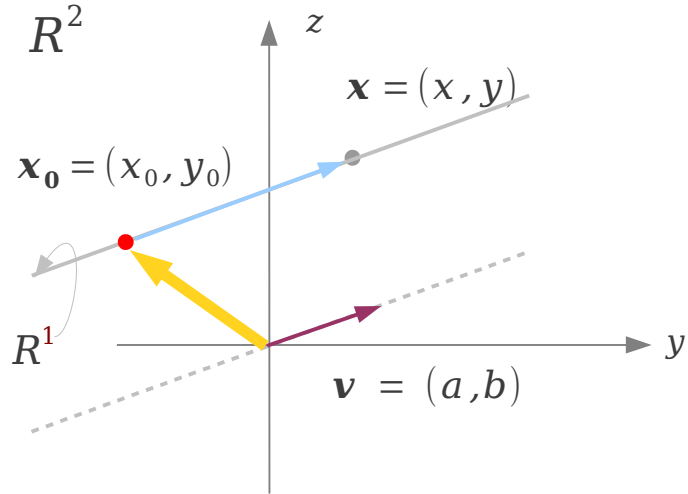
$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

$$x = t_1a_1 + t_2a_2$$

$$y = t_1b_1 + t_2b_2$$

$$z = t_1c_1 + t_2c_2$$

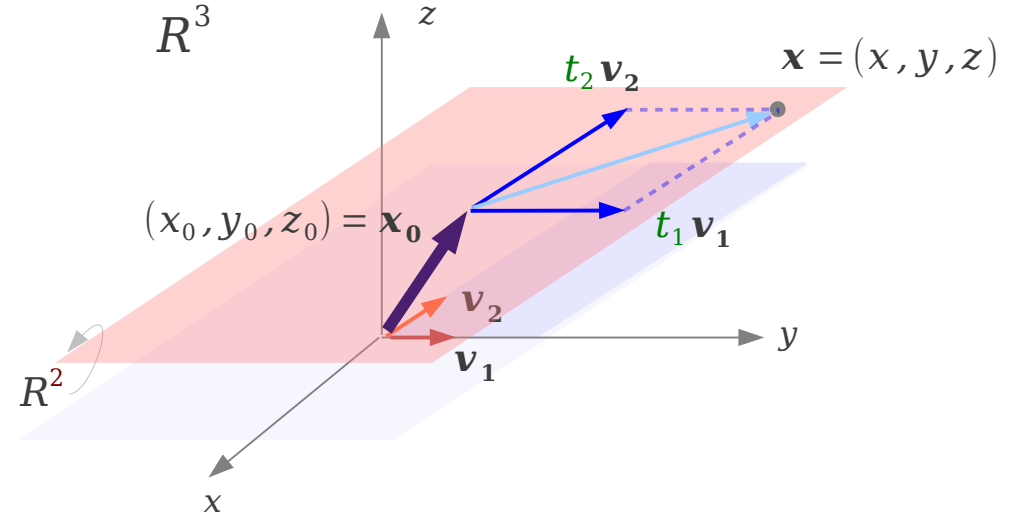
Parametric Equation (2)



$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

$$x = x_0 + ta$$

$$y = y_0 + tb$$



$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

$$x = x_0 + t_1a_1 + t_2a_2$$

$$y = y_0 + t_1b_1 + t_2b_2$$

$$z = z_0 + t_1c_1 + t_2c_2$$

Parametric Equation (3)

Line Equation in \mathbb{R}^n

\mathbf{x}_0 , \mathbf{v} vectors in \mathbb{R}^n

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

Plane Equation in \mathbb{R}^n

\mathbf{x}_0 , \mathbf{v}_1 , \mathbf{v}_2 vectors in \mathbb{R}^n

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

\mathbf{x}_0 , \mathbf{x}_1 vectors in \mathbb{R}^n

$$\mathbf{v} = (\mathbf{x}_1 - \mathbf{x}_0)$$

$$\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)$$

$$\mathbf{x} = (1 - t)\mathbf{x}_0 + t\mathbf{x}_1$$

Linear System (1)

Linear Equations

Corresponding Homogeneous Equation

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$$

$$\mathbf{a} = (a_1, a_2, \cdots, a_n)$$

$$\mathbf{x} = (x_1, x_2, \cdots, x_n)$$

normal vector

$$\mathbf{a} \cdot \mathbf{x} = b$$

$$\mathbf{a} \cdot \mathbf{x} = 0$$

each **solution** vector \mathbf{x} of a **homogeneous** equation
orthogonal to the coefficient vector \mathbf{a}

Homogeneous Linear System

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = 0$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = 0$$

... ..

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n = 0$$

$$\mathbf{r}_1 \cdot \mathbf{x} = 0$$

$$\mathbf{r}_2 \cdot \mathbf{x} = 0$$

...

$$\mathbf{r}_m \cdot \mathbf{x} = 0$$

Linear System (2)

Homogeneous Linear System

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array} \qquad \begin{array}{l} \mathbf{r}_1 \cdot \mathbf{x} = 0 \\ \mathbf{r}_2 \cdot \mathbf{x} = 0 \\ \cdots \\ \mathbf{r}_m \cdot \mathbf{x} = 0 \end{array}$$

each **solution** vector \mathbf{x} of a **homogeneous** equation
orthogonal to the row vector \mathbf{r}_i of the coefficient matrix

Homogeneous Linear System $\mathbf{A} \cdot \mathbf{x} = 0$ $\mathbf{A} : m \times n$

solution set consists of all vectors in R^n
that are **orthogonal** to every row vector of \mathbf{A}

Linear System (3)

Non-Homogeneous Linear System

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

$$\mathbf{A} : m \times n$$

Homogeneous Linear System

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{0}$$

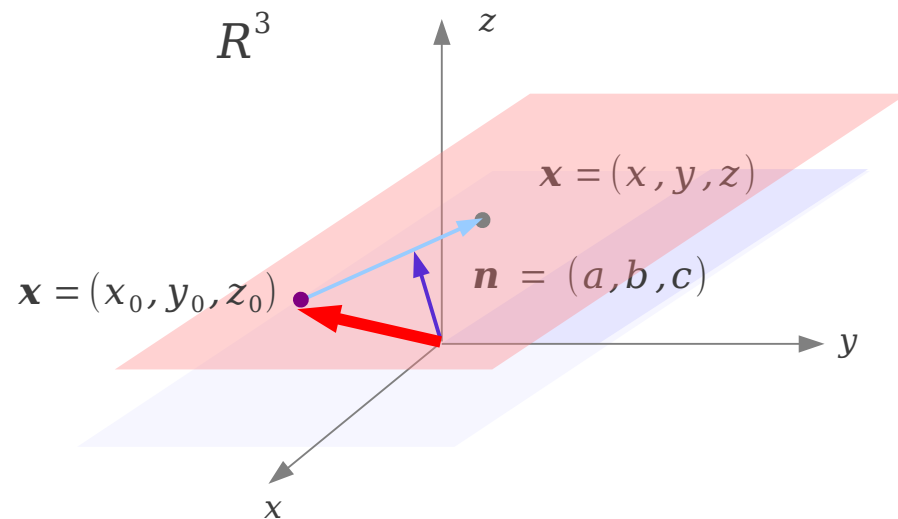
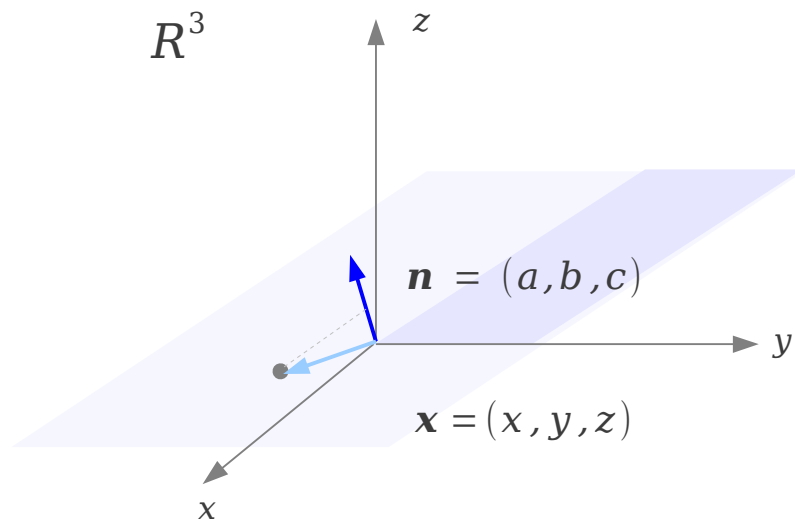
a particular solution

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

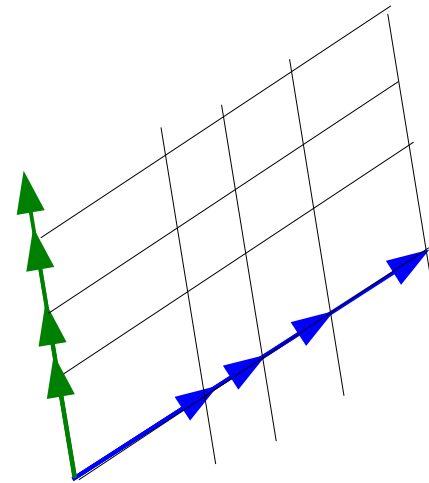
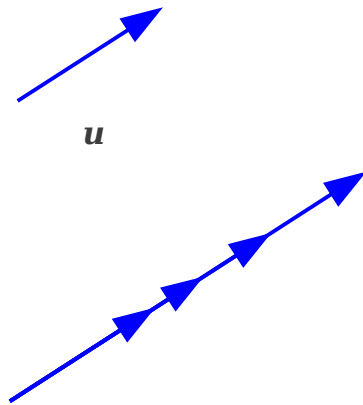
solution set consists of all vectors in R^n that are **orthogonal** to every row vector of \mathbf{A}

+

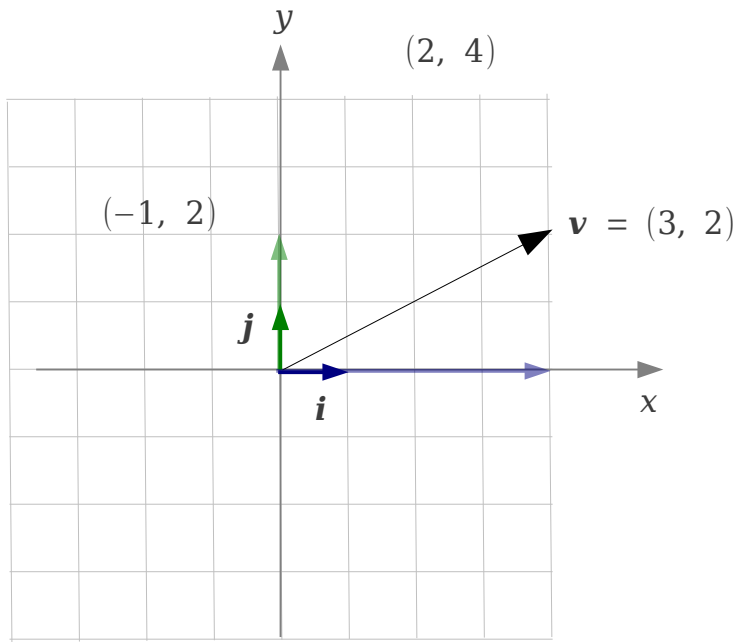
a particular solution \mathbf{x}_0 $\mathbf{A} \cdot \mathbf{x}_0 = \mathbf{b}$



Vector Addition



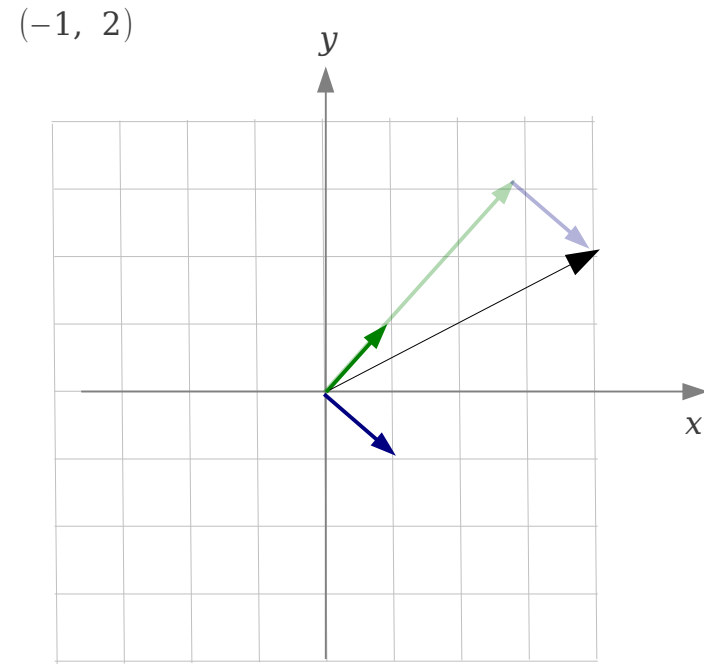
Basis



$$\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$$

$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0} \implies a = b = 0$$

basis



$$a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \mathbf{0} \implies a = b = 0$$

basis

Cross Product (1)

Determinant of order 3

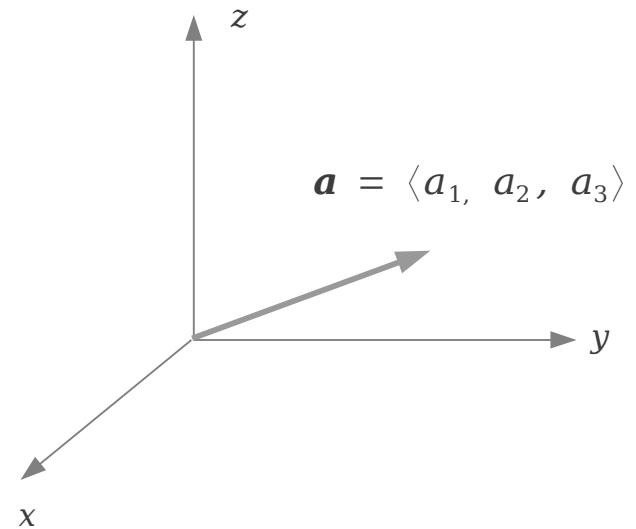
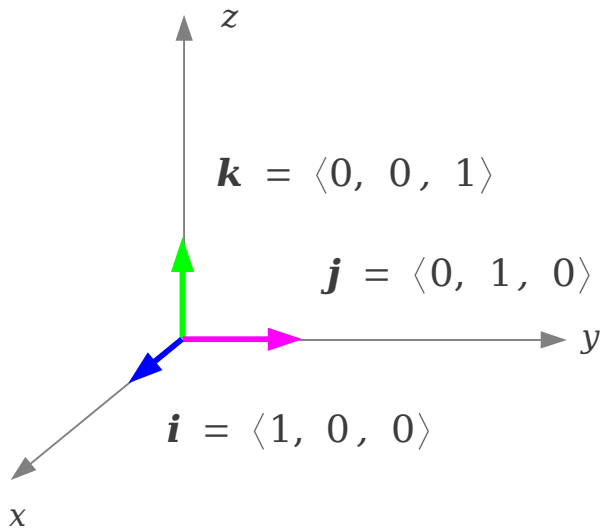
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \langle a_1, a_2, a_3 \rangle$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} = \langle b_1, b_2, b_3 \rangle$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Cross Product (2)



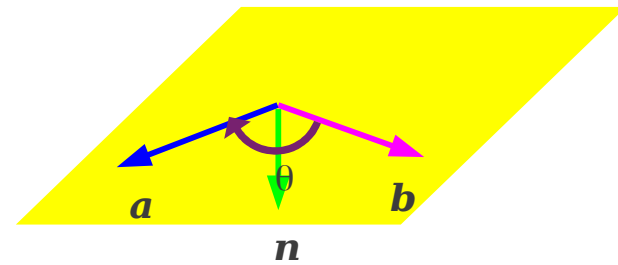
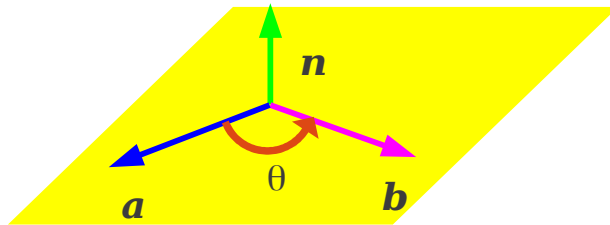
$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}$	\leftarrow normal to \mathbf{i} & \mathbf{j}	$\Rightarrow \mathbf{j} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\mathbf{k}$
$\mathbf{j} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i}$	\leftarrow normal to \mathbf{j} & \mathbf{k}	$\Rightarrow \mathbf{k} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\mathbf{i}$
$\mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j}$	\leftarrow normal to \mathbf{k} & \mathbf{i}	$\Rightarrow \mathbf{i} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\mathbf{j}$

Right Hand Rule

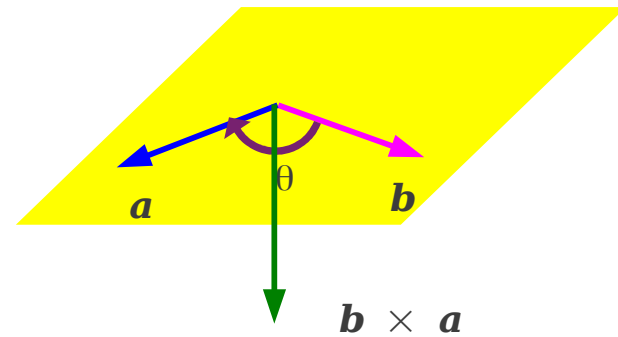
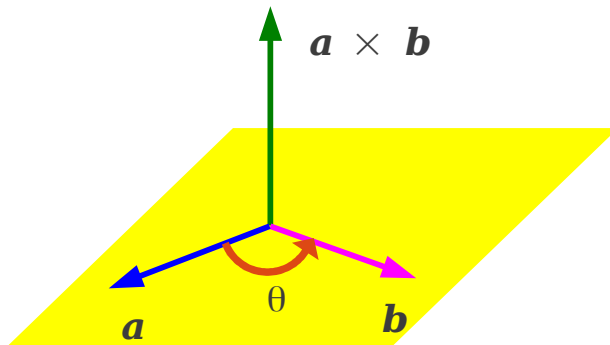
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{b} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Normal direction \mathbf{n}



Magnitude = $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$



Line Equations (1)

Vector Equation

$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

Parameter

Direction Vector

$$\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Parametric Equation

$$x = x_2 + ta_1$$

$$y = y_2 + ta_2$$

$$z = z_2 + ta_3$$

Component

$$ta_1 = x - x_2$$

$$ta_2 = y - y_2$$

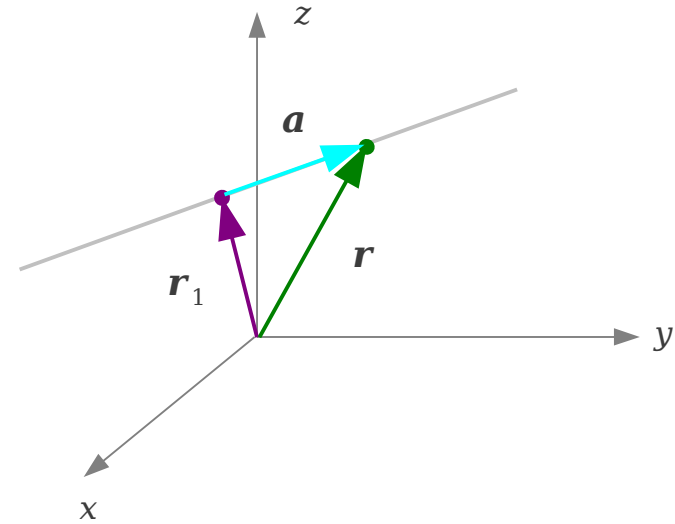
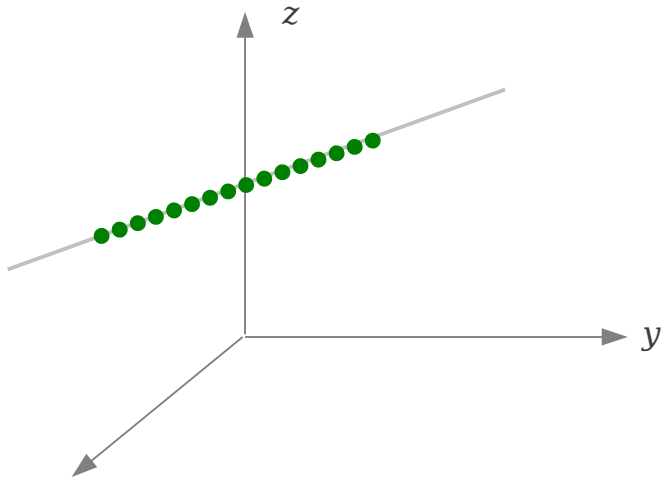
$$ta_3 = z - z_2$$

Symmetric Equation

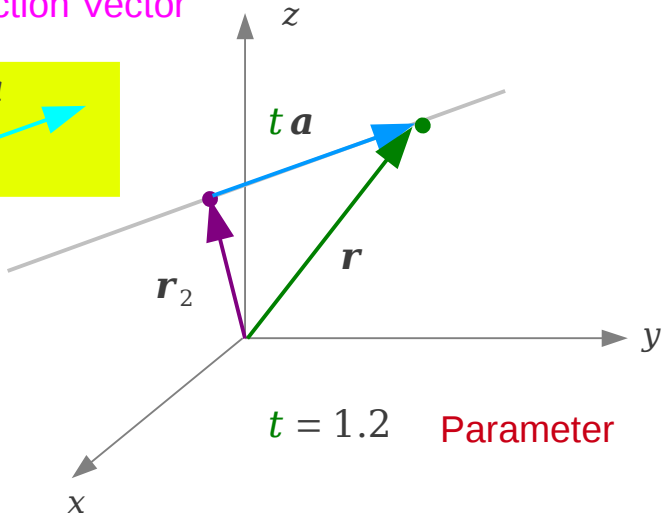
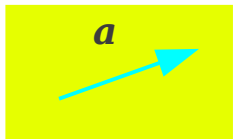
$$t = \frac{x - x_2}{a_1} = \frac{y - y_2}{a_2} = \frac{z - z_2}{a_3}$$

Elimination of parameter

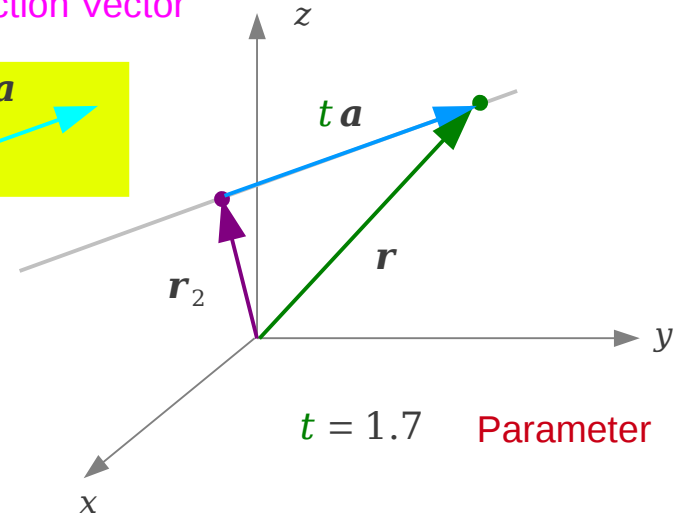
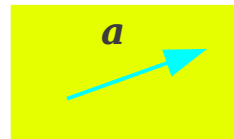
Line Equations (2)



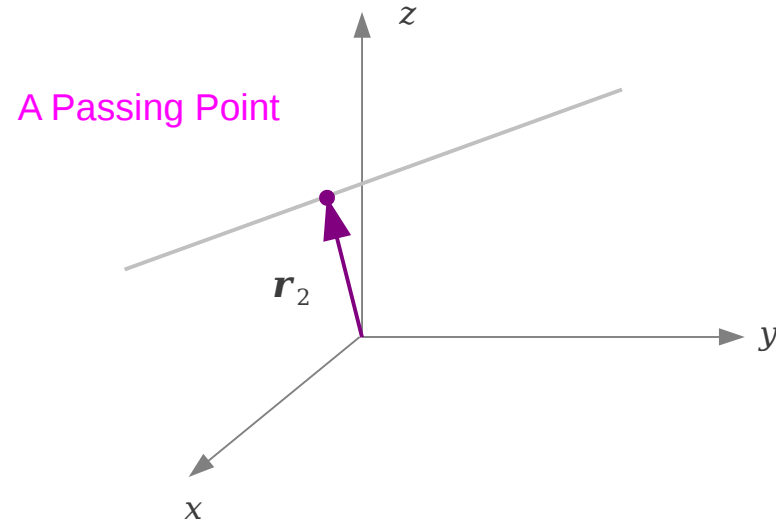
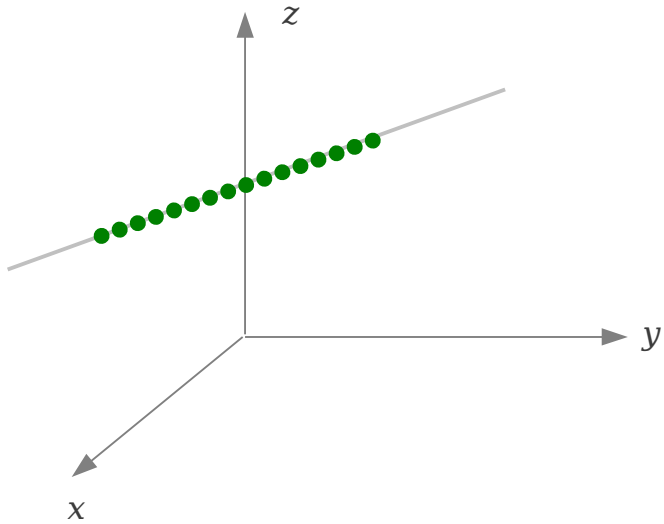
Direction Vector



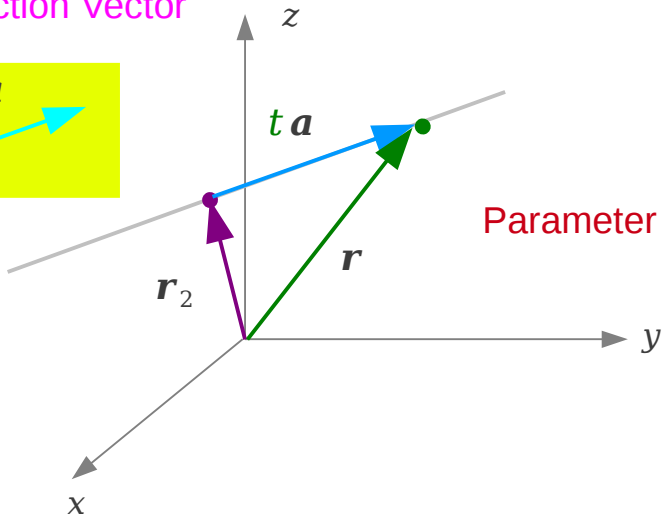
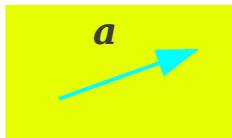
Direction Vector



Line Equations (3)



Direction Vector



$$\mathbf{r} = \mathbf{r}_2 + t\mathbf{a}$$

$$\mathbf{r}_2 = \langle x_2, y_2, z_2 \rangle$$

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Plane Equations (1)

Vector equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

Normal Vector

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

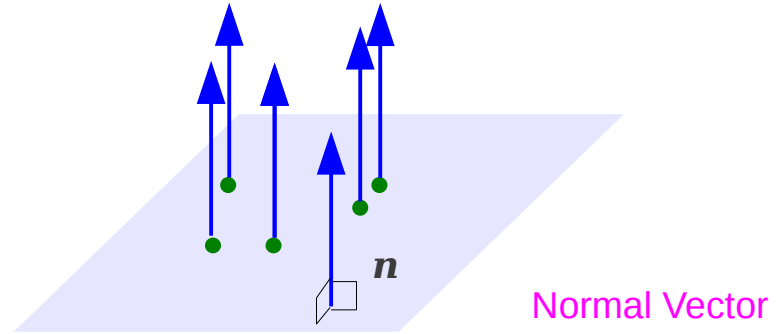
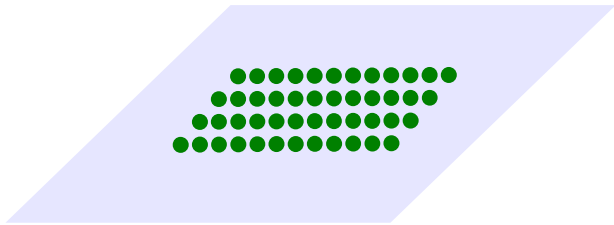
$$\mathbf{r} - \mathbf{r}_1 = \langle x - x_1, y - y_1, z - z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

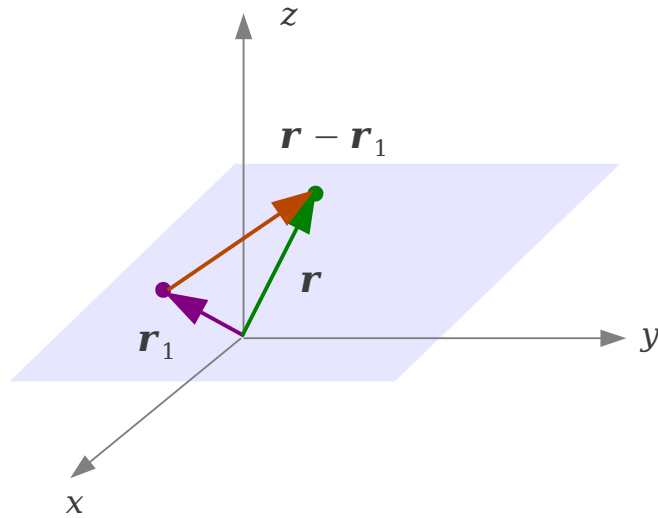
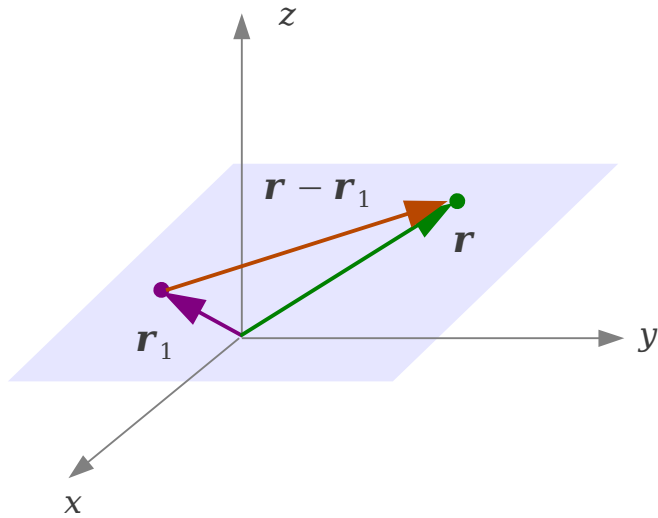
Cartesian equation

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

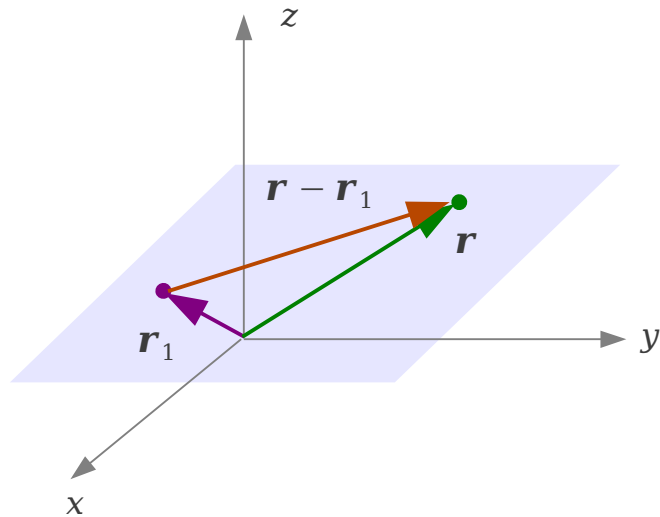
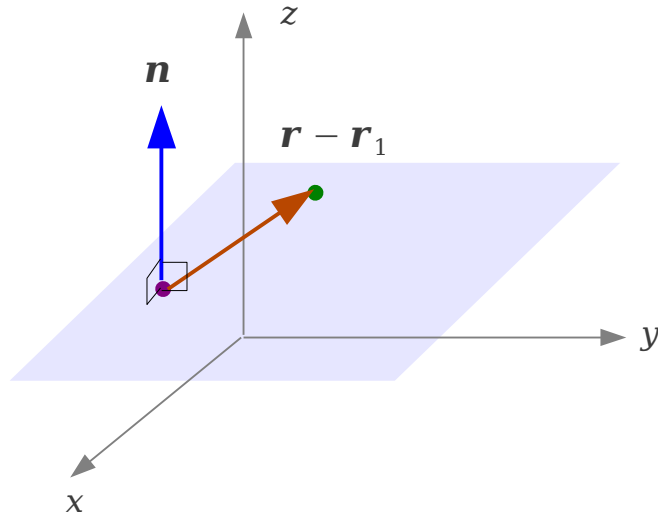
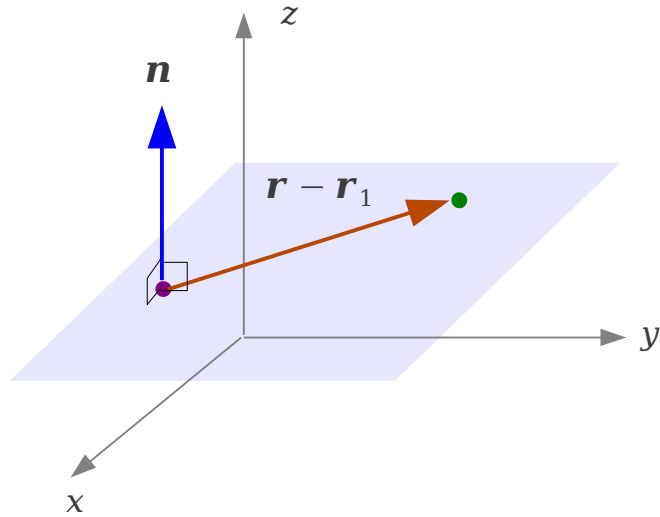
Plane Equations (2)



No Parameter



Plane Equations (3)



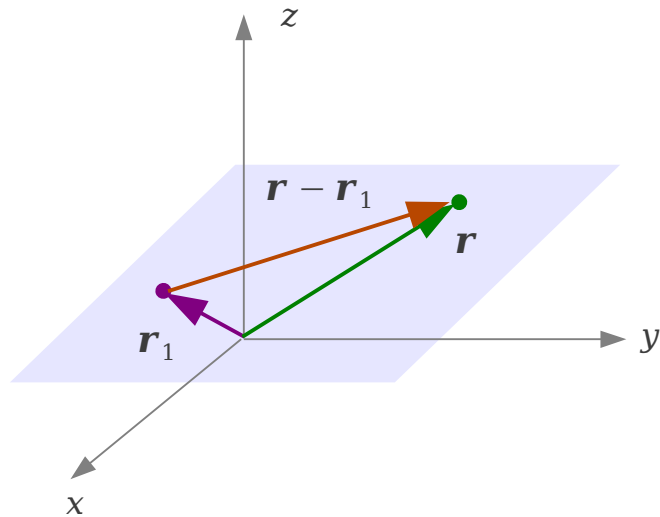
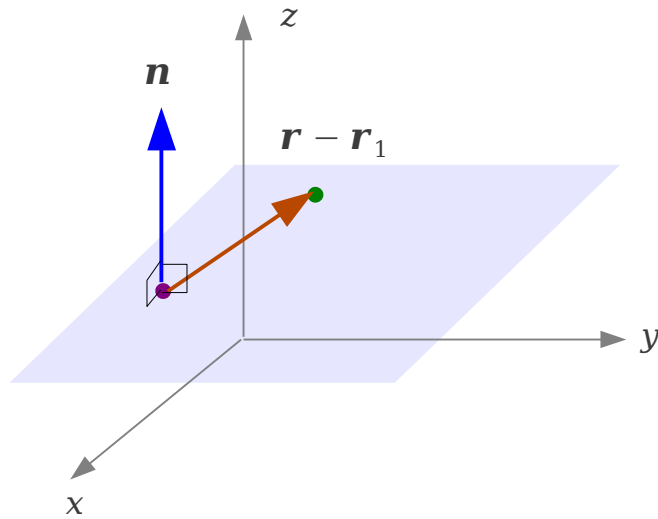
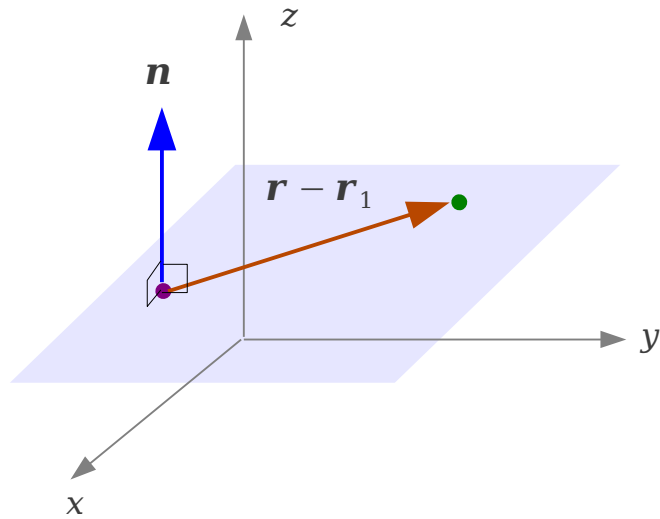
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

Plane Equations (3)



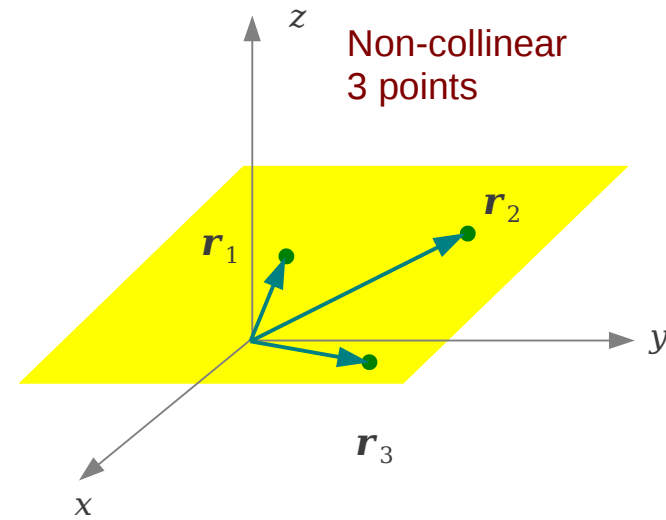
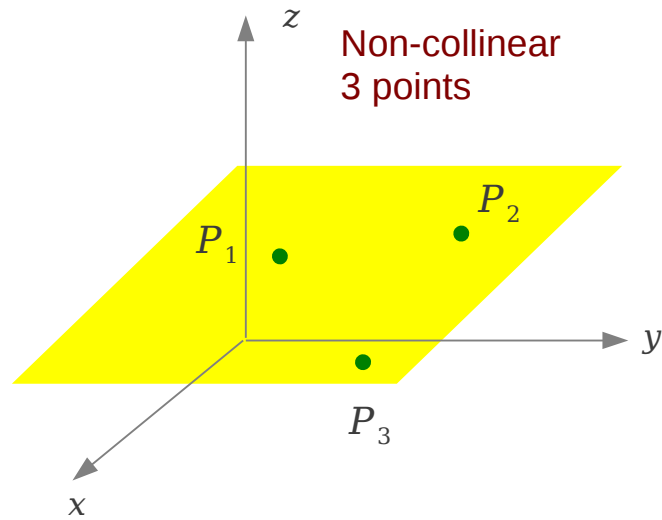
$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_1) = 0$$

$$\mathbf{r} = \langle x, y, z \rangle$$

$$\mathbf{r}_1 = \langle x_1, y_1, z_1 \rangle$$

$$\mathbf{n} = \langle a, b, c \rangle$$

Normal Vector & 3 Points

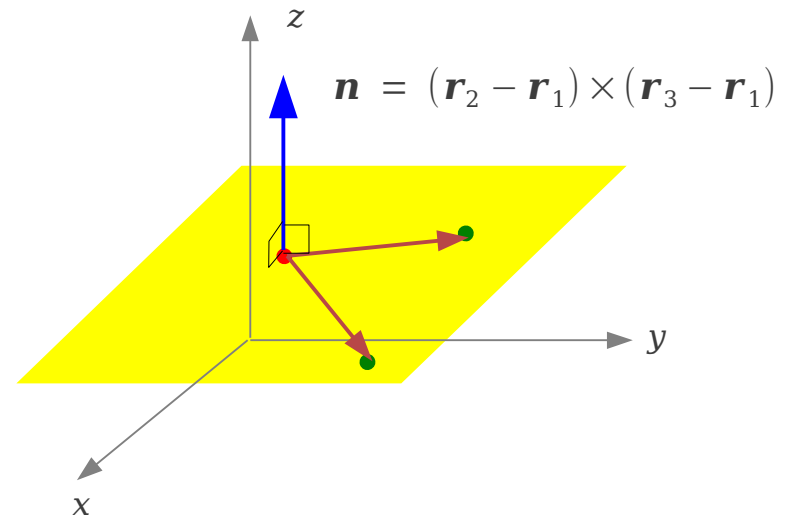
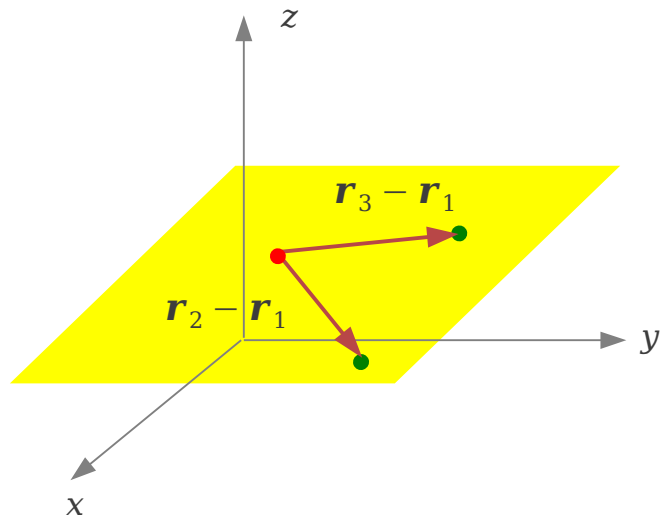
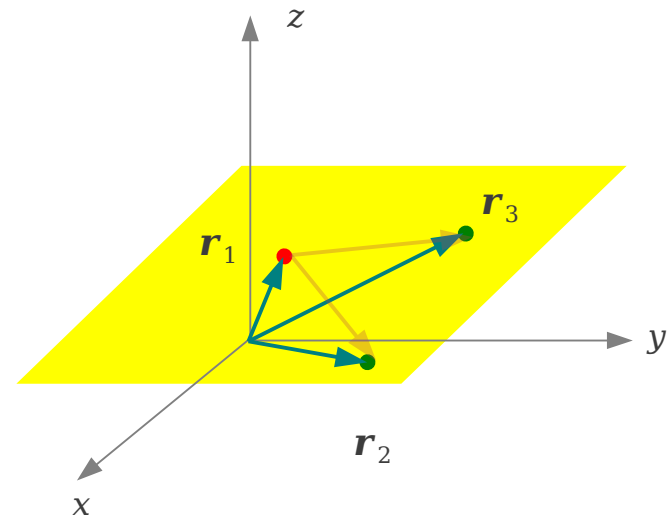
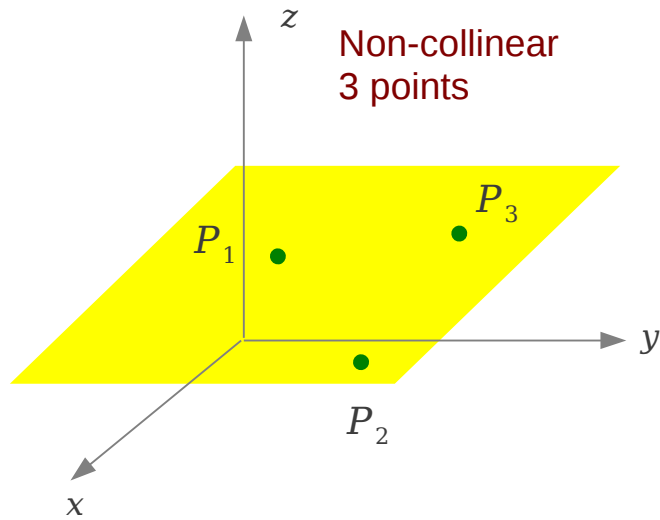


Graph of a plane

Line intersection of two planes

Point of intersection of a line and plane

Normal Vector & 3 Points



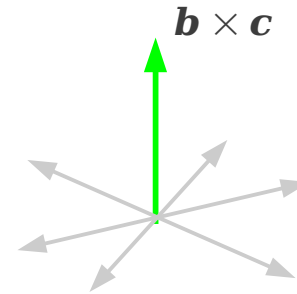
Vector Triple Product (1)

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

⇒ perpendicular to $\mathbf{b} \times \mathbf{c}$

⇒ any vector perpendicular to $\mathbf{b} \times \mathbf{c}$
lies in the plane perpendicular to $\mathbf{b} \times \mathbf{c}$

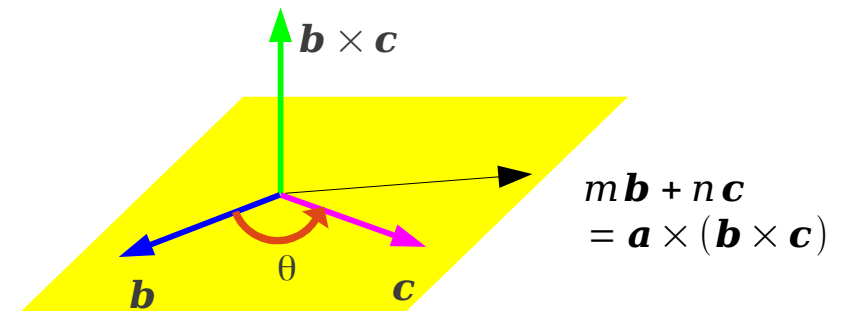
⇒ lies in the plane of \mathbf{b} and \mathbf{c}



$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$$

⇒ $m\mathbf{b} + n\mathbf{c}$

$$(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$



Vector Triple Product (2)

$$\mathbf{b} = b_1 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k}$$

$$\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + 0 \mathbf{k}$$

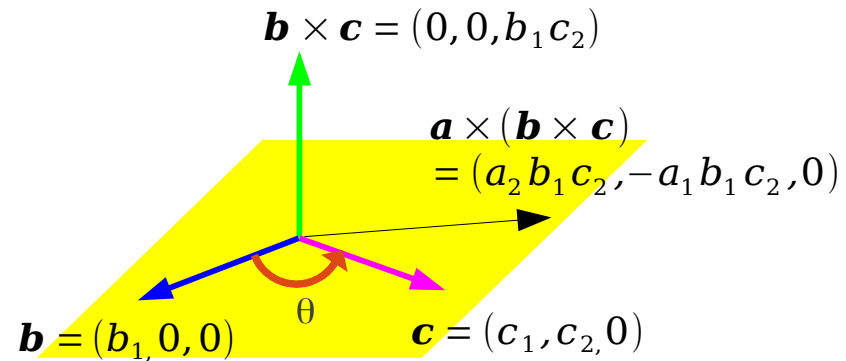
$$\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1$$

$$\mathbf{a} \cdot \mathbf{c} = a_1 c_1 + a_2 c_2$$

$$\mathbf{b} \times \mathbf{c} = b_1 \mathbf{i} \times (c_1 \mathbf{i} + c_2 \mathbf{j} + 0 \mathbf{k}) = b_1 c_2 \mathbf{k}$$

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times b_1 c_2 \mathbf{k} \\ &= -a_1 b_1 c_2 \mathbf{j} + a_2 b_1 c_2 \mathbf{i} \\ &= + (a_2 c_2 + a_1 c_1) b_1 \mathbf{i} - a_1 b_1 (c_1 \mathbf{i} + c_2 \mathbf{j}) \\ &= + (\mathbf{a} \cdot \mathbf{c}) b_1 \mathbf{i} - (\mathbf{a} \cdot \mathbf{b}) (c_1 \mathbf{i} + c_2 \mathbf{j}) \\ &= + (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \end{aligned}$$



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

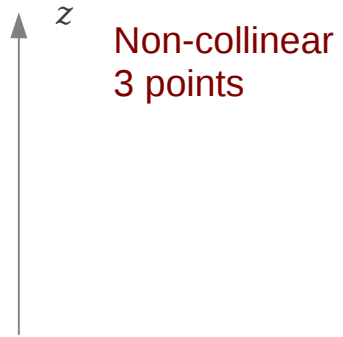
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

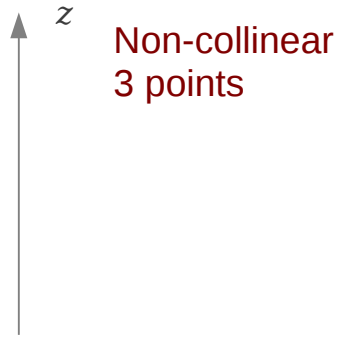
$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot \left(\mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) \\ &= \left(a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Normal Vector & 3 Points



$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Normal Vector & 3 Points



References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [4] D.G. Zill, “Advanced Engineering Mathematics”