Angle Recording CORDIC
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to reduce the number of CORDIC iterations

by encoding the angle of rotation
as a linear combination of
selected elementary angle of micro-rotations

Signal / Image processing DFT & DCT
— the rotation angle known a priori

greedy algorithms to perform angle recoding

linear combination of
elementary rotation angles (EAS)

FFT, Chirp-z

a circular rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\cos\theta$, $\sin\theta$

CORDIC : a sequence of successive rotation

n elementary rotation angles

$a(i)$, $i = 0, \cdots, n-1$

$\tan[a(i)] = 2^{-i}$

only shifts and adds operations

$$\theta = \sum_{i=0}^{n-1} u(i)\, a(i) + \varepsilon$$

ε: an angle approximation error

$$|\varepsilon| \leq a(n-1)$$

the direction of rotation angle

$$u(i) = +1 \text{ or } -1$$

$$z(0) = \theta$$

$$z(i+1) = z(i) - u(i)\, a(i) \qquad i = 0, \cdots, n-1$$

$$u(i) = \operatorname{sign}(z(i))$$

Initialization $\quad x(0) = x$

$y(0) = y$

For $i = 0$ to $n-1$ do

CORDIC Rotation

$$\begin{bmatrix} x(i+1) \\ y(i+1) \end{bmatrix} = \begin{bmatrix} 1 & \tan u(i)a(i) \\ -\tan u(i)a(i) & 1 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}$$

Scaling operation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \prod_{i=0}^{n-1} \cos u(i)a(i) \cdot \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} \leftarrow \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \leftarrow \quad \cdots \quad \leftarrow \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} \leftarrow \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

shift and add operations

$$\prod_{i=0}^{N-1} \cos u(i)a(i) = \frac{1}{K(n)}$$ norm correction

a known constant

once the set $\{u(i)a(i) : i = 0, \cdots, n-1\}$ is determined

a multiplier recoding method can be applied

Booth's algorithm

For convenience, assume

$$|\theta| < 2a(0) = \frac{\pi}{2}$$

ⓐ if $\theta > 2\pi$, $\quad \theta \leftarrow \theta \bmod 2\pi$

ⓑ if $2\pi > \theta > \pi$, $\quad \begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} -x \\ -y \end{bmatrix}$, $\quad \theta \leftarrow \theta - \pi$

ⓒ if $\pi > \theta > \frac{\pi}{2}$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} \leftarrow \begin{bmatrix} y \\ -x \end{bmatrix}$, $\quad \theta \leftarrow \theta - \frac{\pi}{2}$

CORDIC Angle Recoding Problem

$u(i) = 0$ is allowed. repetition

$$u(i) = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$

desirable to minimize $\boxed{\sum_{i=0}^{n-1} |u(i)|}$

$\rightarrow$ reduce CORDIC iterations

Angle Recoding

given $\boxed{a(i)}$, $i = 0, \cdots, n-1$ $\quad a(i) \in EAS$

θ an angle

find $u(i)$, $i = 0, \cdots, n-1$ $\quad u(i) \in \{-1, 0, +1\}$

such that

(i) $\theta = \sum_{i=0}^{n-1} u(i)\, a(i) + \varepsilon \qquad \varepsilon < a(n-1)$

(ii) $\sum_{i=0}^{n-1} |u(i)|$ is minimized

CORDIC Angle Recoding Algorithm

Greedy Algorithm

Initialization : $\theta(0) = \theta$, $\{ u(i) = 0, \ 0 \leq i \leq n-1 \}$, $k = 0$

Repeat until $|\theta(k)| < a(n-1)$ Do

① choose i_k, $0 \leq i_k \leq n-1$ such that

$$\big| |\theta(k)| - a(i_k) \big| = \min_{0 \leq i \leq n-1} \big| |\theta(k)| - a(i) \big|$$

② $\theta(k+1) = \theta(k) - u(i_k) a(i_k)$

$u(i_k) = \text{sign}(\theta(k))$

greedy

at every step
represent the remaining angle
using a closest elementary CORDIC angle

draw i_k without replacement

Choosing i_k, $\boxed{0 \le i_k \le n-1}$

① with angle replacement

always choose i_k
among n numbers $(0 \dots n-1)$

angle repetition is allowed

$\boxed{\theta(k+1) \le \theta(k)}$ monotonically decreasing

② without angle replacement

at each step choose i_k
among one less numbers
than the previous step's numbers

$\boxed{\theta(k+1) < \theta(k)}$ Strictly decreasing

$$\Big|\, |\theta(k)| - a(i_k) \,\Big| = \min_{0 \le i \le n-1} \Big|\, |\theta(k)| - a(i) \,\Big|$$

$$\theta(k+1) = \theta(k) - u(i_k)\, a(i_k)$$

$$\begin{aligned} |\theta(k+1)| &= |\theta(k) - u(i_k)\, a(i_k)| \\ &= \Big|\, |\theta(k)| - a(i_k) \,\Big| \\ &\le \Big|\, |\theta(k)| - a(n-1) \,\Big| \\ &< |\theta(k)| \end{aligned}$$

loop while

$|\theta(k)| > a(n-1)$

$a(i_k) > a(n-1)$

Strictly decreasing

$g(i) = a(i) - a(i+1) \qquad i = 0, 1, \cdots, n-2$

$a(i) = \tan^{-1}(2^{-i})$

$g(i) > 0 \qquad a(i) > a(i+1)$ strictly decreasing

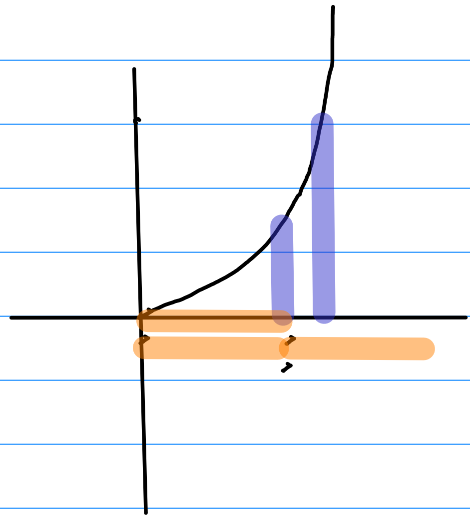
$g(i) = a(i) - a(i+1) = \tan^{-1}(2^{-i}) - \tan^{-1}(2^{-i-1}) > 0$

$g(i+1) = a(i+1) - a(i+2) = \tan^{-1}(2^{-i-1}) - \tan^{-1}(2^{-i-2}) > 0$

$g(i) - g(i+1) = a(i) - 2a(i+1) + a(i+2) > 0$

$\tan(a(i)) = \tan(\tan^{-1}(2^{-i})) = 2^{-i}$

$\tan(a(i+1)) = \tan(\tan^{-1}(2^{-i-1})) = 2^{-i-1}$



$\tan(a(i)) = 2 \cdot \tan(a(i+1))$

$a(i) < 2 \cdot a(i+1) < 2a(i)$

$a(i+2) < g(i) = a(i) - a(i+1) < a(i+1)$

$a(i) < 2 \cdot a(i+1)$

$$|\theta| \leqslant a(0) = \frac{\pi}{4}$$

$$\sum_{i=0}^{n-1} |u(i)| \leqslant \frac{n}{2}$$

$$i_k \geqslant 2k$$

$$a(n-1) < |\theta| < a(0)$$

$$|\theta| \in [a(i_0+1), a(i_0)] \quad \text{or}$$
$$|\theta| \in [a(i_0), a(i_0-1)]$$

$$\theta(1) = \big||\theta(0)| - a(i_0)\big|$$

$$< \max\{g(i_0)/2, \; g(i_0-1)/2\}$$
$$< g(0)/2$$
$$< a(1)/2$$
$$< a(2)$$

$k = 1$

$$|\theta(k)| < a(2k)$$

$$|\theta(k+1)| < g(2k)/2$$
$$< a(2k+1)/2$$
$$< a(2k+2)$$
$$= a(2(k+1))$$

$$2k \geqslant n-1 \qquad k^* \leqslant \frac{(n-1)}{2}$$

$i = 0, 1, 2, \cdots, n-1 \longleftarrow$ n-bit word

$|\theta(k)| < a(n-1)$ termination condition

$k = 0, 1, \cdots\cdots, k'-1$ hopefully less than $n-1$

$k=0 \qquad 0 \leq i_0 \leq n-1$

$k=1 \qquad 0 \leq i_1 \leq n-1$

$\vdots$

$k=k'-1 \qquad 0 \leq i_{k'-1} \leq n-1$

$u(i) = 0$ initialization $i = 0, 1, \cdots, n-1$

$\theta(k) > 0 \qquad u(i_k) = +1 \qquad \theta(k+1) = \theta(k) - a(i_k)$

find i_k s.t. $\min_{0 \leq i \leq n-1} |\theta(k) - a(i)|$

$\theta(k) < 0 \qquad u(i_k) = -1 \qquad \theta(k+1) = -\theta(k) + a(i_k)$

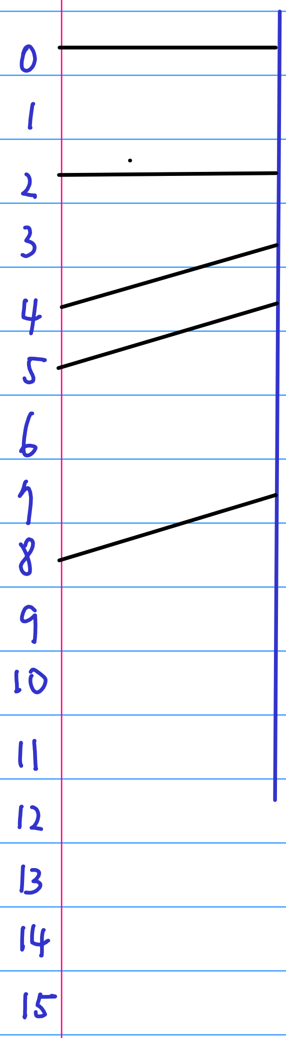
find i_k s.t. $\min_{0 \leq i \leq n-1} |\theta(k) - a(i)|$

$u(j) = 0 \qquad j \in \{0, 1, \cdots, n-1\}$

$j \notin \{i_0, i_1, \cdots, i_{k'-1}\}$

no repeatition

repetition allowed in MVR



0	+
2	−
4	+
5	−
8	−
	0
	0
	0
	0
	0
	0
	0
	0
	0
	0
	0

k'

if the algorithm terminates at $k = k^*$, $k^* < \frac{n}{2}$

$g(i) = a(i) - a(i+1)$ $\quad i = 0, 1, \cdots, n-2$

$a(i) = \tan^{-1} 2^{-i}$

① $g(i) > 0$

② $a(i+2) < g(i) < a(i+1)$

③ $g(i) > g(i+1)$

① $a(i) - a(i+1) > 0$

② $a(i+2) < a(i) - a(i+1) < a(i+1)$

③ $a(i) - a(i+1) > a(i+1) - a(i+2)$

if $|\theta| \leq a(0) = \frac{\pi}{4}$

$$\sum_{i=0}^{n-1} |u(i)| < \frac{n}{2}$$

Elementary Angle Set

$$S = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, -1\},\ r \in \{1, 2, \cdots, n-1\} \}$$

n-bit angle as a linear combination

$$\theta = \sum_{i=0}^{n-1} \sigma_i \cdot \tan^{-1}(2^{-i})$$

AR : $\sigma \in \{-1, 0, +1\}$

EAS (Elementary Angle Set) for AR methods

$$S_{EAS} = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, 0, -1\},\ r \in \{1, 2, \cdots, n-1\} \}$$

Simple angle recording — Hu's greedy algorithm

tries to represent the remaining angle using the closest elementary angle $\pm \tan^{-i}$

$\left\{ \begin{array}{l} \text{rotation mode — Angle Recording} \\ \text{vectoring mode — Backward Angle Recording (BAR)} \end{array} \right.$

initialize $\theta_0 = \theta$

$\sigma_i = 0 \qquad i = 0, 1, \cdots, n-1$

$k = 0$

repeat until $|\theta_k| < \tan^{-1}(2^{-n+1})$ do

1. choose i_k, $i_k = 0, 1, 2, \cdots, n-1$

   such that

   $$\left| |\theta_k| - \tan^{-1}(2^{-i_k}) \right| = \min_{i \in [0:n-1]} \left| |\theta_k| - \tan^{-1}(2^{-i}) \right|$$

2. $$\theta_{k+1} = \theta_k - \sigma_{i_k} \tan^{-1}(2^{-i_k})$$

   $$\sigma_{i_k} = \mathrm{sign}(\theta_k)$$

$$\prod_{i=0}^{n-1} \cos u(i)\, a(i) = 1/K(n)$$

a modified Booth's multiplier recoding
representation for $1/K(n)$

the total number of 1's and $\bar{1}$'s

→ additional iterations needed for CORDIC scaling
(norm correction operation)

the total number of iterations

for CORDIC rotation operations
for norm correction operations