# Implication (6A)

Young Won Lim
3/17/18

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

# Semantics of Implication Rule

(1) Suppose **A** → **B** is **true**
If **A** is **true** then **B** must be **true**

(2) Suppose **A** → **B** is false
If **A** is **true** then **B** must be **false**

(3) Suppose **A** → **B** is true
If **A** is **false** then we **do not know** whether **B** is **true** or **false**

| | A | B | A→B |
|---|---|---|---|
| (1) | T | T | T |
| (2) | T | F | F |
| (3) | F | T | T |
| | F | F | T |

# Material Implication & Logical Implication

Given two propositions **A** and **B**,

If **A** ⇒ **B** is a **tautology** ← in every interpretation

It is said that **A** **logically implies** **B** (**A** ⇛ **B**)

**Material Implication** **A ⇒ B** (not a tautology) **A → B**

**Logical Implication** **A ⇛ B** (a **tautology**) **A ⇛ B**

| A | B | A⇒B |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

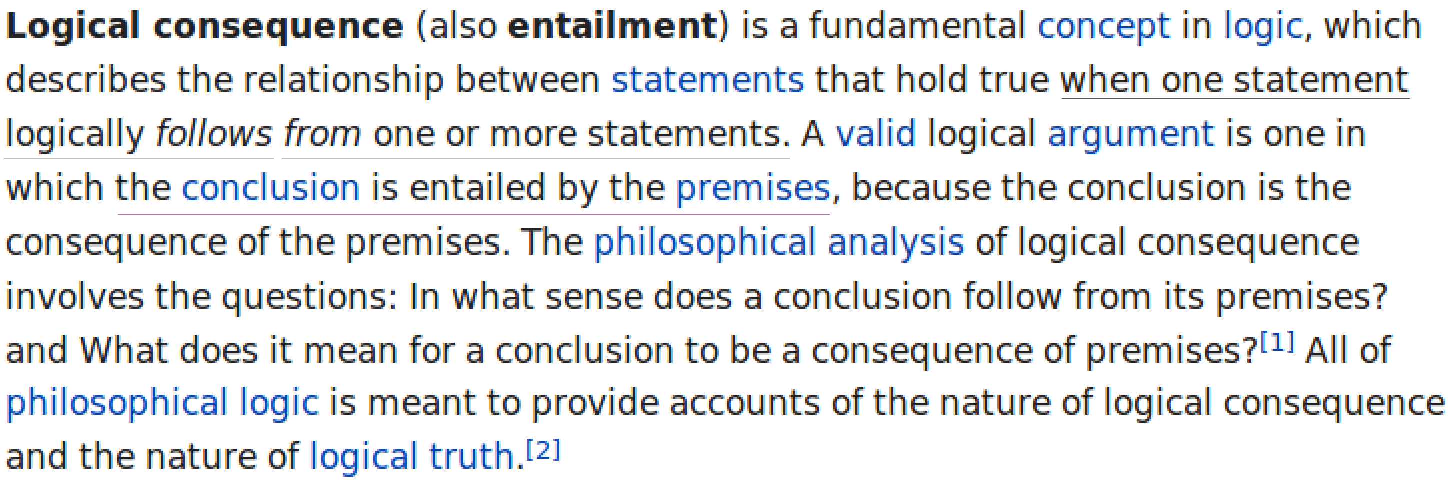
| A | B | A∧B | A∧B ⇒ A |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

tautology

↓

**A∧B ⇛ A**

# Logical consequences



**Logical consequence** (also **entailment**) is a fundamental concept in logic, which describes the relationship between statements that hold true when one statement logically *follows from* one or more statements. A valid logical argument is one in which the conclusion is entailed by the premises, because the conclusion is the consequence of the premises. The philosophical analysis of logical consequence involves the questions: In what sense does a conclusion follow from its premises? and What does it mean for a conclusion to be a consequence of premises?[1] All of philosophical logic is meant to provide accounts of the nature of logical consequence and the nature of logical truth.[2]

https://en.wikipedia.org/wiki/Logical_consequence

# Proofs and Models

The two prevailing techniques for providing accounts of logical consequence involve expressing the concept via

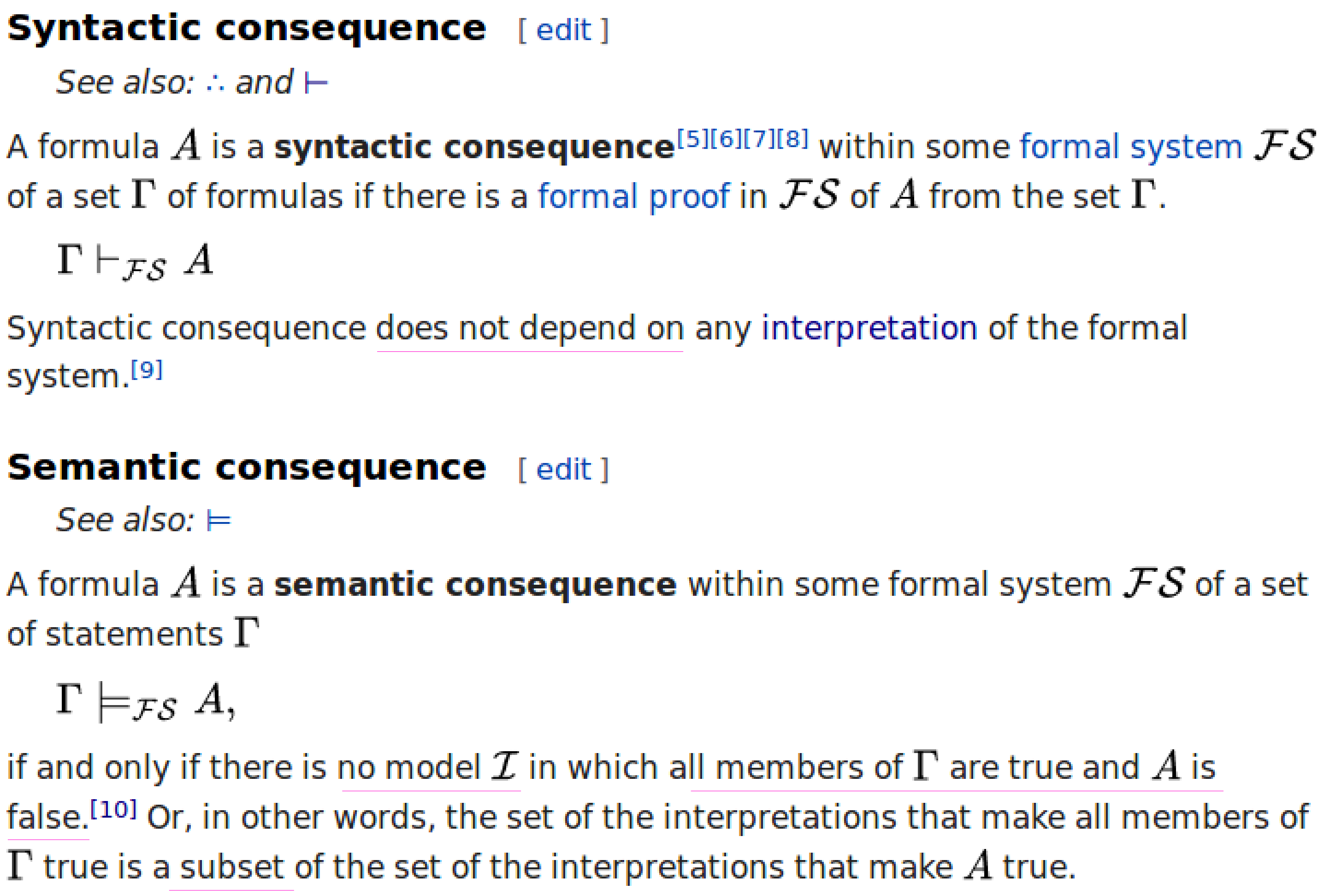
- **proofs**
- **models**

**Logical Consequences**

- **syntactic consequence – proof theory** $\vdash$
- **semantic consequence – model theory** $\models$

https://en.wikipedia.org/wiki/Logical_consequence

# Syntactic vs Semantic Consequences



## Syntactic consequence [ edit ]

*See also: ∴ and ⊢*

A formula $A$ is a **syntactic consequence**[5][6][7][8] within some formal system $\mathcal{FS}$ of a set $\Gamma$ of formulas if there is a formal proof in $\mathcal{FS}$ of $A$ from the set $\Gamma$.

$$\Gamma \vdash_{\mathcal{FS}} A$$

Syntactic consequence does not depend on any interpretation of the formal system.[9]

## Semantic consequence [ edit ]

*See also: ⊨*

A formula $A$ is a **semantic consequence** within some formal system $\mathcal{FS}$ of a set of statements $\Gamma$

$$\Gamma \models_{\mathcal{FS}} A,$$

if and only if there is no model $\mathcal{I}$ in which all members of $\Gamma$ are true and $A$ is false.[10] Or, in other words, the set of the interpretations that make all members of $\Gamma$ true is a subset of the set of the interpretations that make $A$ true.

https://en.wikipedia.org/wiki/Logical_consequence

# Syntactic Consequence Example

| A | B | A⇒B |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

No need
- models
- interpretations

| A | B | A∧B | A∧B ⇒ A |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

Entailment **A∧B** ⊢ **A**, or **A∧B** ⇛ **A**

syntactic consequence

logical consequence

# Semantic Consequence Example (1)

| | A | B | A⇒B |
|---|---|---|---|
| Interpretation **I1** | T | T | T |
| Interpretation **I2** | T | F | F |
| Interpretation **I3** | F | T | T |
| Interpretation **I4** | F | F | T |

But if we force the use of

- implications
- models

| | A | B | A∧B | A∧B ⇒ A |
|---|---|---|---|---|
| Interpretation **I1** | T | T | T | T |
| Interpretation **I2** | T | F | F | T |
| Interpretation **I3** | F | T | F | T |
| Interpretation **I4** | F | F | F | T |

# Semantic Consequence Example (2)

| | A | B | A⇒B | |
|---|---|---|---|---|
| a **Model** of A | T | T | T | a **Model** of A⇒B |
| a **Model** of A | T | F | F | |
| | F | T | T | a **Model** of A⇒B |
| | F | F | T | a **Model** of A⇒B |

$\mathbf{A} \cancel{\Rrightarrow} \mathbf{B}$

| | A | B | A∧B | A∧B ⇒ A | |
|---|---|---|---|---|---|
| a **Model** of A∧B | T | T | T | T | a **Model** of A∧B ⇒ A |
| | T | F | F | T | a **Model** of A∧B ⇒ A |
| | F | T | F | T | a **Model** of A∧B ⇒ A |
| | F | F | F | T | a **Model** of A∧B ⇒ A |

Entailment **A∧B ⊨ A**, or **A∧B ⇛ A**

syntactic consequence

logical consequence

# Entailment

if **A → B** holds in every model then **A ⊨ B**,
and conversely
if **A ⊨ B** then **A → B** is true in every model

any model that makes **A ∧ B** true

also makes **A** true : **A ∧ B ⊨ A**

No case : True ⇒ False

| A | B | A⇒B |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| A | B | A∧B | A∧B ⇒ A |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

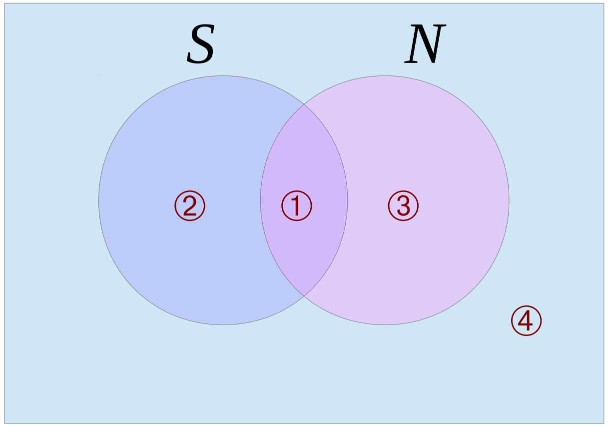
Entailment **A ∧ B ⊨ A**, or **A ∧ B ⇛ A**

semantic
consequence

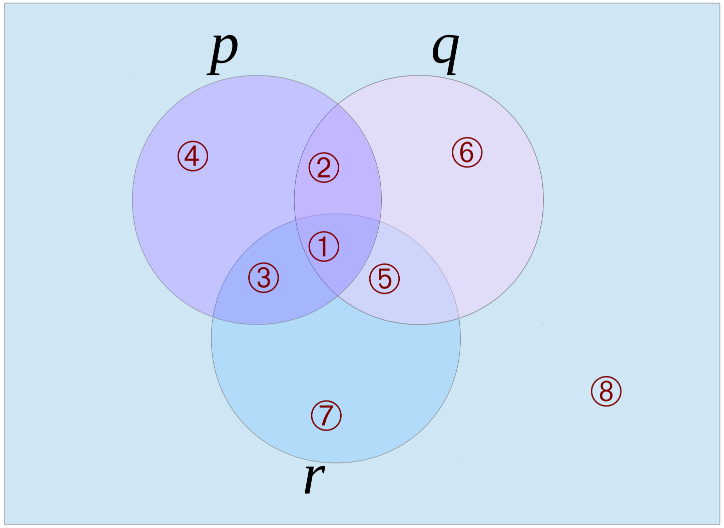
logical
consequence

# Logic and Venn diagram  (1)

| | S | N | S⇒N |
|---|---|---|---|
| case ① | T | T | T |
| case ② | T | F | F |
| case ③ | F | T | T |
| case ④ | F | F | T |



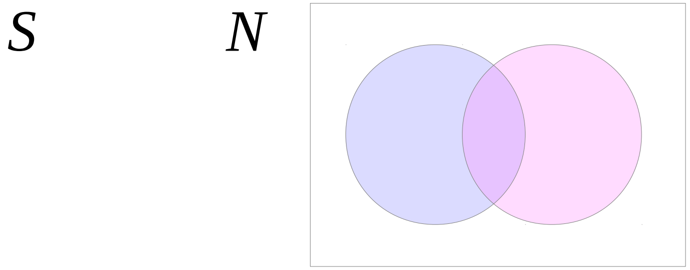
# Logic and Venn diagram (2)



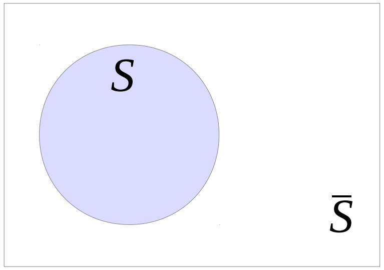
|  | $p$ | $q$ | $r$ | $p \to q$ | $\cdots$ |
|---|---|---|---|---|---|
| case ① | $T$ | $T$ | $T$ | $T$ |  |
| case ② | $T$ | $T$ | $F$ | $T$ |  |
| case ③ | $T$ | $F$ | $T$ | $F$ |  |
| case ④ | $T$ | $F$ | $F$ | $F$ |  |
| case ⑤ | $F$ | $T$ | $T$ | $T$ |  |
| case ⑥ | $F$ | $T$ | $F$ | $T$ |  |
| case ⑦ | $F$ | $F$ | $T$ | $T$ |  |
| case ⑧ | $F$ | $F$ | $F$ | $T$ |  |

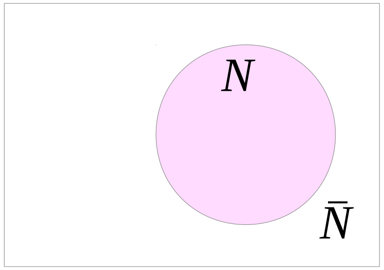
# Material Implication and Venn Diagram

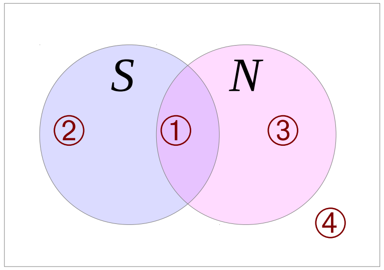
|  | S | N | S⇒N |
|---|---|---|---|
| case ① | T | T | T |
| case ② | T | F | F |
| case ③ | F | T | T |
| case ④ | F | F | T |

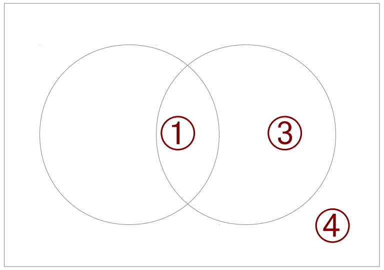


When S⇒N is True



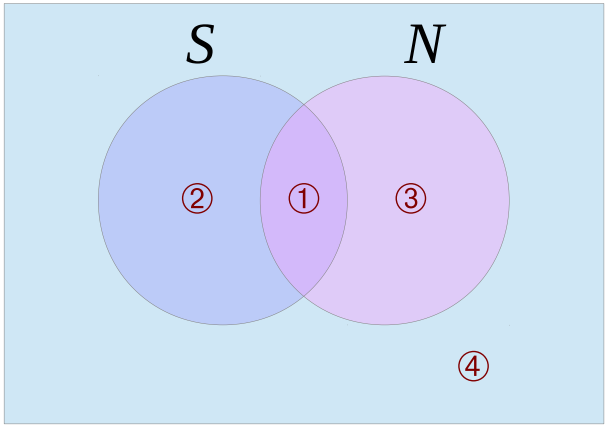






# When S⇒N is a true statement

| | S | N | S⇒N | |
|---|---|---|---|---|
| case ① | T | T | T | (1) |
| case ② | T | F | F | |
| case ③ | F | T | T | |
| case ④ | F | F | T | (2) |



S⇒N is True

cases ①+③+④

if the conditional statement (**S⇒N**) is a **true** statement,

(1) then the consequent **N** must be **true** if **S** is **true**

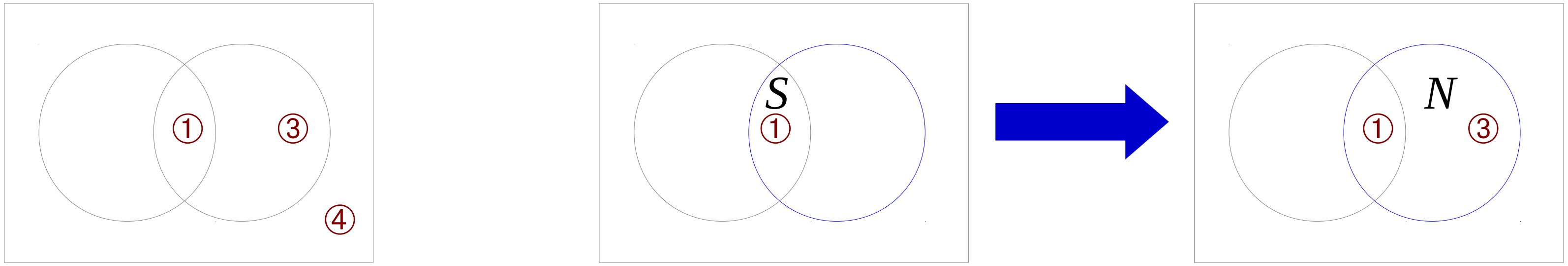
(2) the antecedent **S** can not be **true** **without** **N** being **true**

# **S ⊆ N**

if the conditional statement **(S⇒N**) is a **true** statement,

then the consequent **N** must be **true** if **S** is **true**

**S⇒N**



case ①

cases ①+③

**S⊆N**

x∈**S**

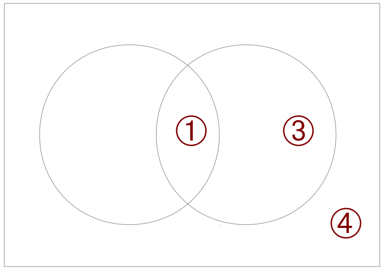
x∈**N**

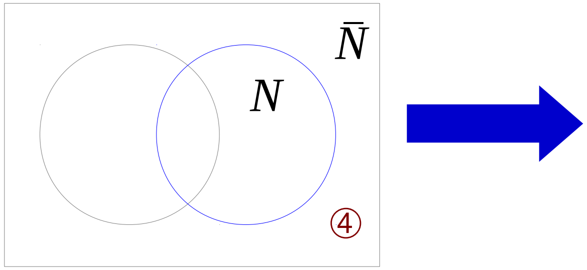
# ~N ⊆ ~S

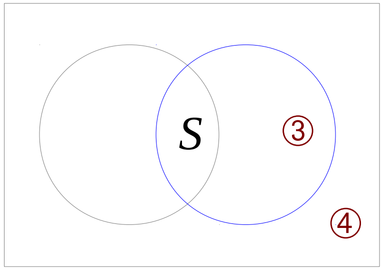
if the conditional statement (S⇒N) is a **true** statement,

the antecedent **S** can not be **true** without **N** being **true**

¬N⇒¬S







case ④

cases ③+④

¬N⊆¬S

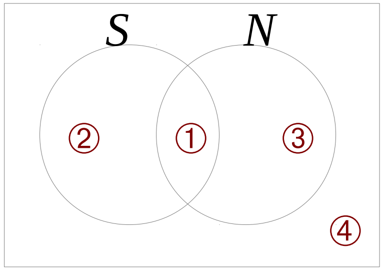
x∈¬N

x∈¬S

# Material Implication vs. Logical Consequence

## Material Implication

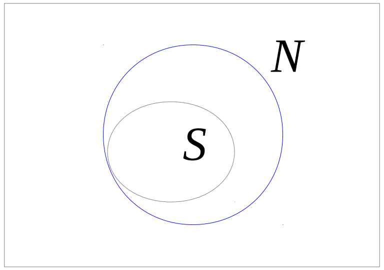
**S⇒N**



T⇒F exists ②

## Logical Consequences

**S⇛N**



Always True
(Tautology)

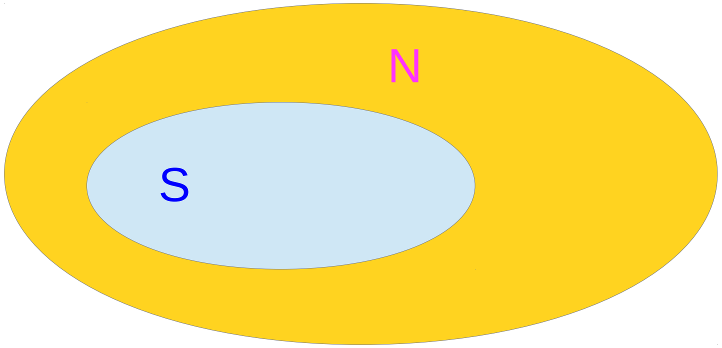
**entailment**

**S ⊢ N syntactic (proof)**

**S ⊨ N semantic (model)**

# Implication

$$S \longrightarrow N$$



If **S**, then N.

**S** implies N.

N whenever **S**.

**S** is sufficient for N.

S only if **N**.

not S if not **N**.

not S without **N**.

**N** is necessary S.

http://en.wikipedia.org/wiki/

# Necessity and Sufficiency

condition that guarantees N

sufficiency for N

**S** satisfies at least N

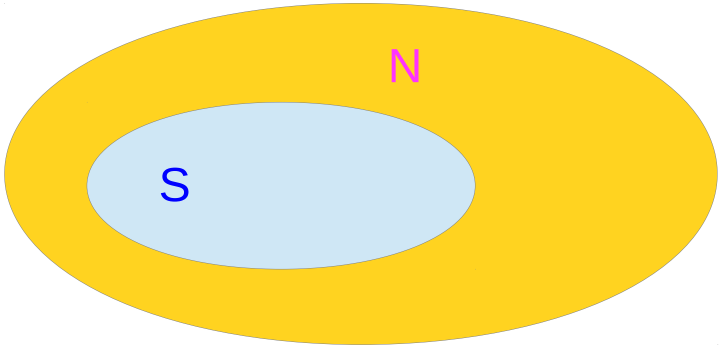
N if **S**

condition that must be satisfied for S

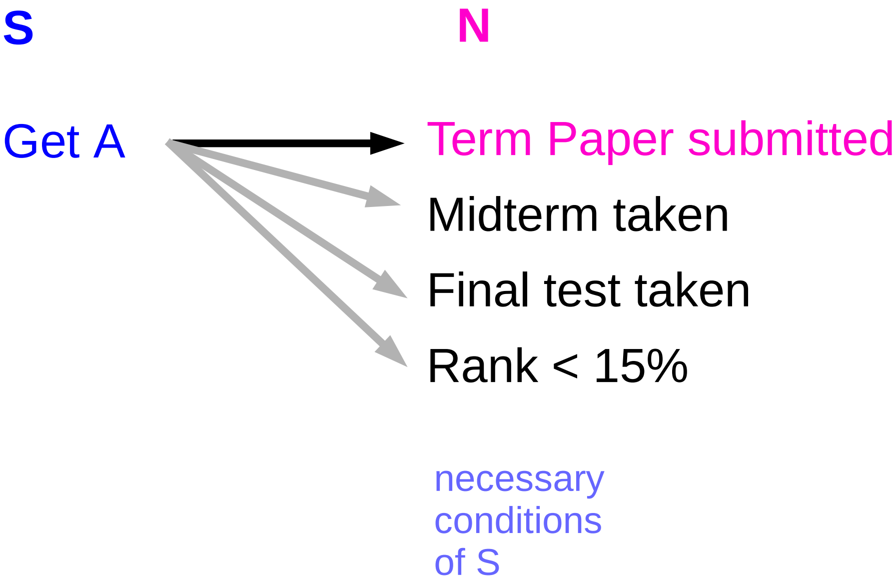
necessity for S

without **N**, it can’t be S

**S** only if **N**



# Other Necessary Conditions



**S**

**N**

Get A

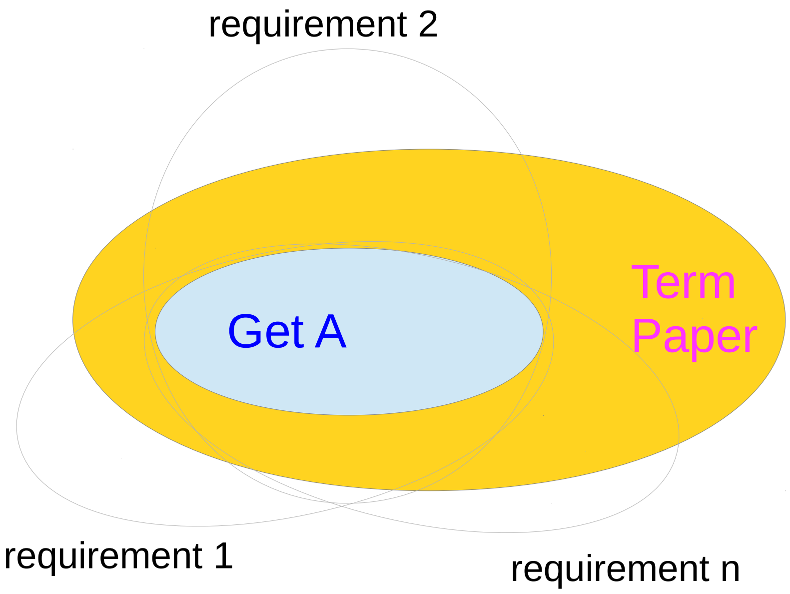
Term Paper submitted

Midterm taken

Final test taken

Rank < 15%

necessary
conditions
of S

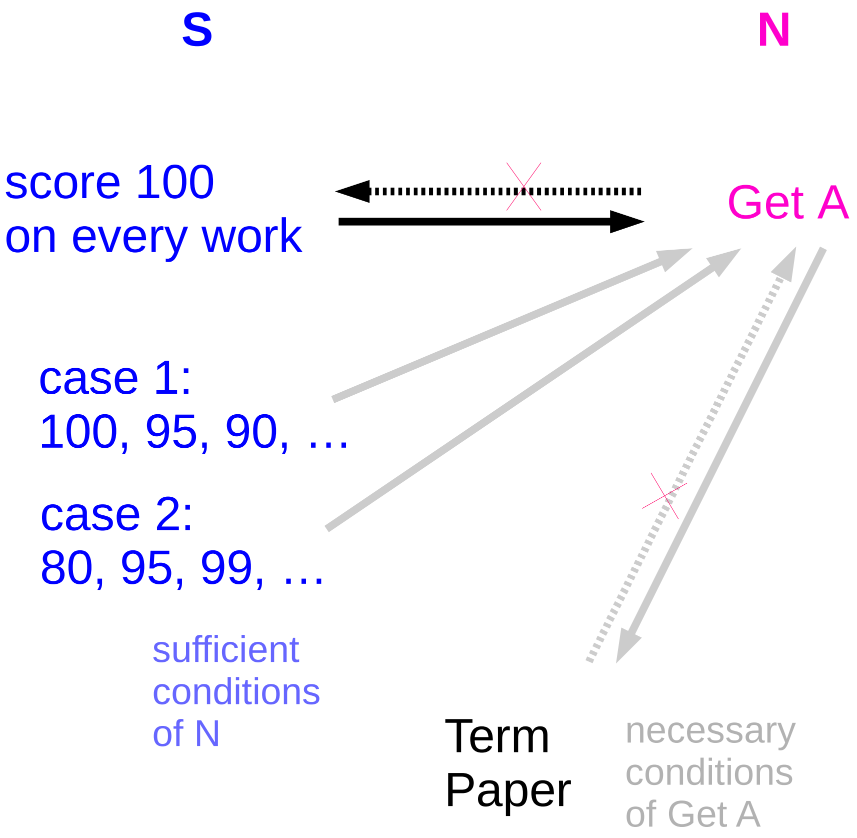


S only if N

not S if not N

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm`

# Other Sufficient Conditions



**S**

**N**

score 100
on every work

Get A

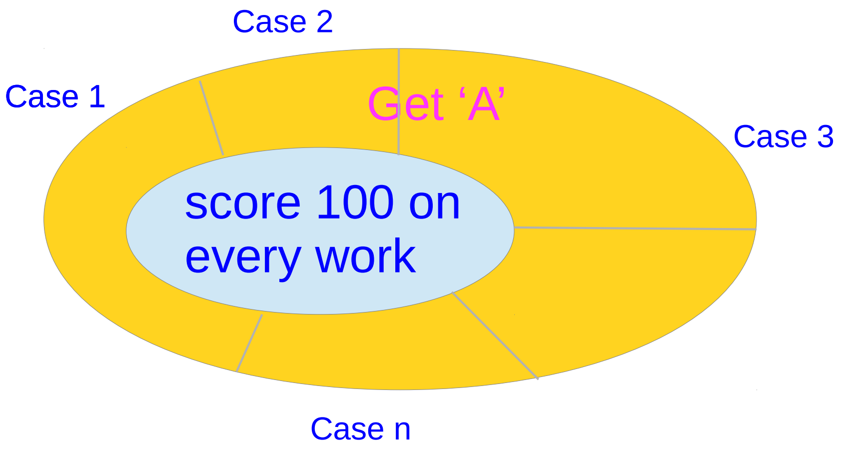
case 1:
100, 95, 90, …

case 2:
80, 95, 99, …

sufficient
conditions
of N

Term
Paper

necessary
conditions
of Get A



Case 1

Case 2

Case 3

Case n

Get ‘A’

score 100 on
every work

N if S

N whenever S

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm`

# Necessity Definition

**Definition**: A **necessary condition** for some state of affairs **S** is a condition that must be satisfied in order for **S** to obtain.

a necessary condition for getting an A in 341 is that a student hand in a term paper.

This means that if a student does not hand in a term paper, then a student will not get an A,

or, equivalently, if a student gets an A, then a student hands in a term paper.

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm

# Sufficiency Definition

**Definition**: A **sufficient condition** for some state of affairs **N** is a condition that, if satisfied, guarantees that **N** obtains.

a sufficient condition for getting an A in 341 is
getting an A on every piece of graded work in the course.

This means that if a student gets an A on every piece of graded work in the course,
then the student gets an A.

Handing in a term paper is not a sufficient condition
for getting an A in the course.

It is possible to hand in a term paper and not to get an A in the course.

Getting an A on every piece of graded work is not a necessary condition
for getting an A in the course.

It is possible to get an A in the course
even though one fails to get an A on some piece of graded work.

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm`

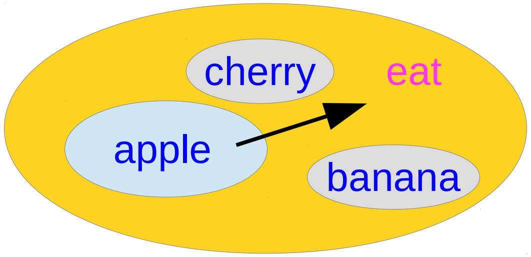
# IF / Only IF

"Madison will eat the fruit if it is an apple."

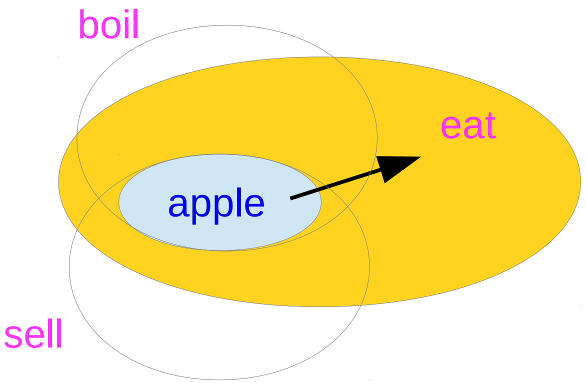
"Only if Madison will eat the fruit, is it an apple;"

"Madison will eat the fruit ← fruit is an apple"

- This states simply that Madison will eat fruits that are apples.
- It does not, however, *exclude* the possibility that Madison might also eat bananas or other types of fruit.
- All that is known for certain is that she will eat any and all apples that she happens upon.
- That the fruit is an apple is **a sufficient condition** for Madison to eat the fruit.



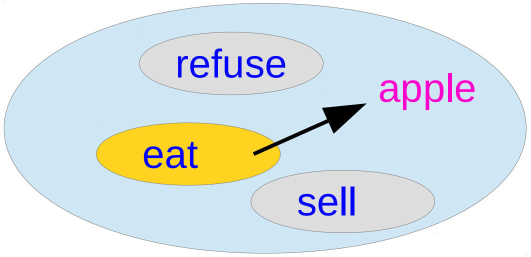
**sufficient conditions**



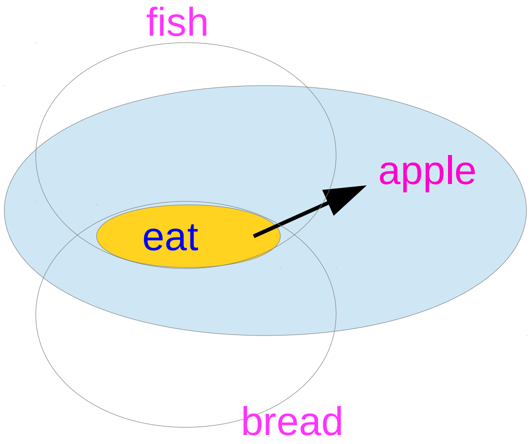
**necessary conditions**

http://en.wikipedia.org/wiki/Derivative

# IF / Only IF



**apple : a necessary condition**



**eat : a sufficient condition**

"Madison will eat the fruit only if it is an apple."

"If Madison will eat the fruit, then it is an apple"

"Madison will eat the fruit → fruit is an apple"

- This states that the only fruit Madison will eat is an apple.
- It does not, however, *exclude* the possibility that Madison will refuse an apple if it is made available
- in contrast with (1), which requires Madison to eat any available apple.
- In this case, that a given fruit is an apple is **a necessary condition** for Madison eating it.
- It is not a sufficient condition since Madison might not eat all the apples she is given.

http://en.wikipedia.org/wiki/Derivative

# IF / Only IF

"Madison will eat the fruit if it is an apple."

"Only if Madison will eat the fruit, is it an apple;"

"Madison will eat the fruit ← fruit is an apple"

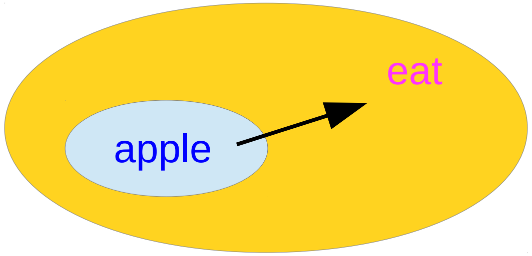
- This states simply that Madison will eat fruits that are apples.
- It does not, however, *exclude* the possibility that Madison might also eat bananas or other types of fruit.
- All that is known for certain is that she will eat any and all apples that she happens upon.
- That the fruit is an apple is **a sufficient condition** for Madison to eat the fruit.

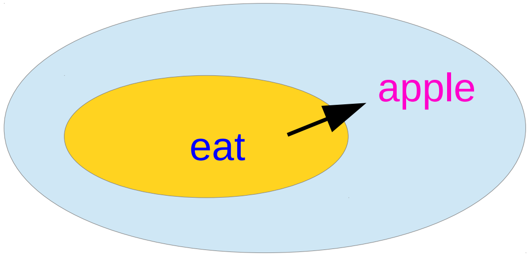
"Madison will eat the fruit only if it is an apple."

"If Madison will eat the fruit, then it is an apple"

"Madison will eat the fruit → fruit is an apple"

- This states that the only fruit Madison will eat is an apple.
- It does not, however, *exclude* the possibility that Madison will refuse an apple if it is made available
- in contrast with (1), which requires Madison to eat any available apple.
- In this case, that a given fruit is an apple is **a necessary condition** for Madison eating it.
- It is not a sufficient condition since Madison might not eat all the apples she is given.





http://en.wikipedia.org/wiki/Derivative

apple

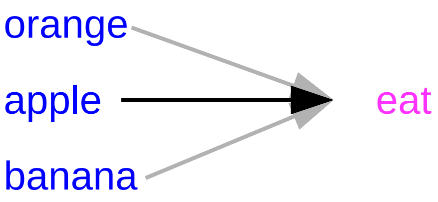
# IF / Only IF

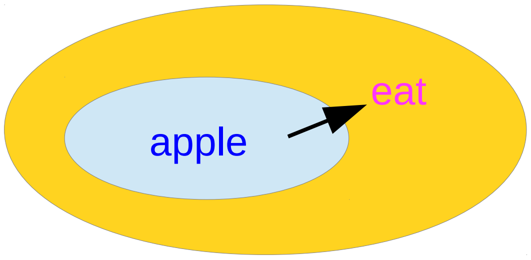
Madison will eat the fruit if it is an apple.

Only if Madison will eat the fruit, is it an apple.
If Madison will not eat the fruit, it is not an apple.

fruit is an apple → Madison will eat the fruit ←

A → E





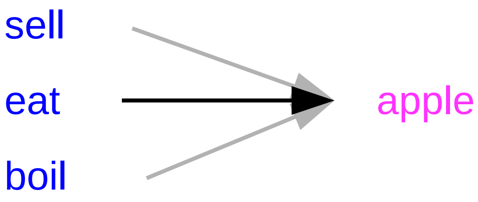
apple: a sufficient condition

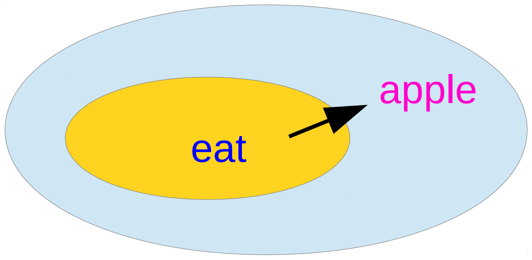
Madison will eat the fruit only if it is an apple.
Madison will not eat the fruit, if it is not an apple.

If Madison will eat the fruit, then it is an apple.

Madison will eat the fruit → fruit is an apple

E → A





apple: a necessary condition

http://en.wikipedia.org/wiki/Derivative

# IFF

"Madison will eat the fruit if and only if it is an apple"

"Madison will eat the fruit ↔ fruit is an apple"

- This statement makes it clear that Madison will eat all and only those fruits that are apples.
- She will not leave any apple uneaten, and
- she will not eat any other type of fruit.
- That a given fruit is an apple is both a necessary and a sufficient condition for Madison to eat the fruit.

http://en.wikipedia.org/wiki/Derivative

# Necessary Condition (1)

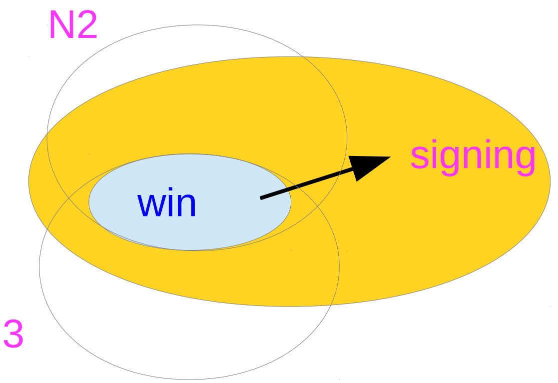
**Definition**: A condition that is necessary for a particular outcome to be achieved.

The condition does **not** guarantee the outcome; but

if the condition does **not** hold, the outcome will **not** achieved

if the Cubs **win** the World Series, we can be sure that they **signed** a right-handed relief pitcher,

since, **without** such a **signing**, they would **not** have **won** the World Series.



Discrete Mathematics, Johnsonbaugh

# Necessary Condition (2)

if the Cubs **win** the World Series, we can be sure that they **signed** a right-handed relief pitcher,

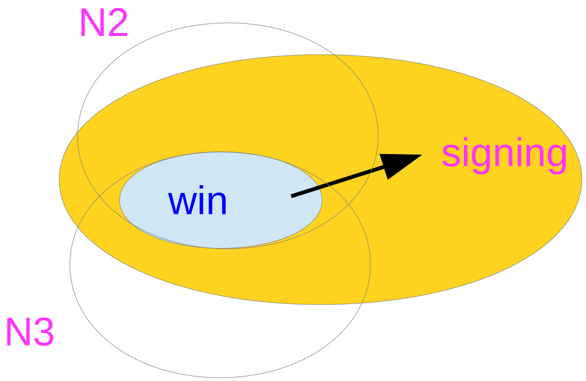
since, **without** such a **signing**, they would **not** have **won** the World Series.

*The equivalent statement:*

if the Cubs **win** the World Series,

then they **signed** a right-handed relief pitcher

The conclusion expresses a necessary condition



Discrete Mathematics, Johnsonbaugh

# Necessary Condition (3)

if the Cubs **win** the World Series, we can be sure that they **signed** a right-handed relief pitcher,

since, **without** such a **signing**, they would **not** have **won** the World Series.

*Not equivalent statement:*

if the Cubs **sign** a right-handed relief pitcher,

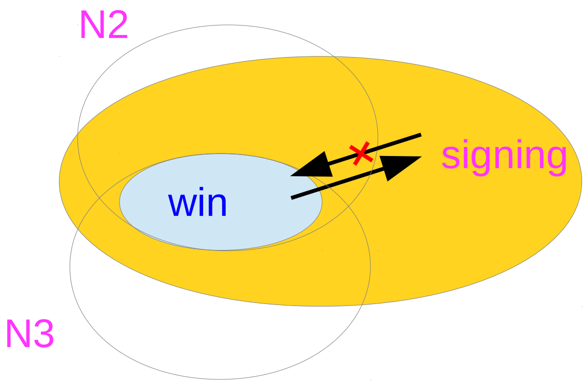
then they **win** the World Series

**Signing** a right-handed relief pitcher

does **not** guarantee a World Series **win**.

However, **not signing** a right-handed relief pitcher

guarantees that they will **not win** the World Series



Discrete Mathematics, Johnsonbaugh

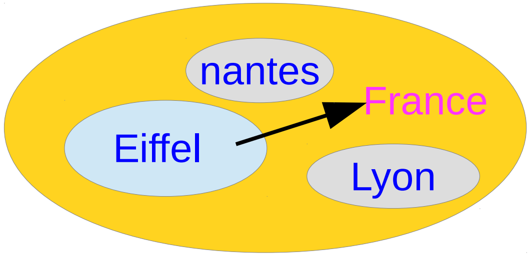
# Sufficient Condition (1)

**Definition**: a condition that suffices to guarantee a particular outcome.

If the condition does **not** hold,
the outcome might be achieved **in other ways**
or it might **not** be achieved at all; but
If the condition does hold, the outcome guaranteed.

To be sure that Maria visits **France**, it suffices for her to go to the **Eiffel Tower**.

There are surely **other ways** to ensure that Maria visits **France**; for example, she could go to **Lyon**.



Discrete Mathematics, Johnsonbaugh

**sufficient conditions**

# Sufficient Condition (2)

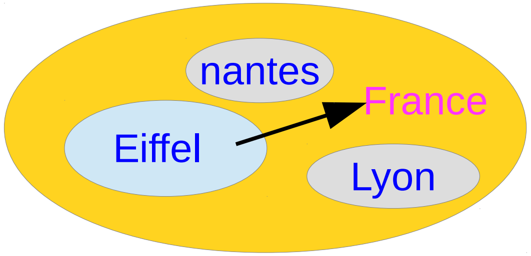
To be sure that Maria visits **France**, it suffices for her to go to the **Eiffel Tower**.

There are surely **other ways** to ensure that Maria visits **France**; for example, she could go to **Lyon**.

*The equivalent statement:*

If Maria goes to the **Eiffel Tower**, then she visits **France**

The hypothesis expresses a sufficient condition



Discrete Mathematics, Johnsonbaugh

**sufficient conditions**

# Sufficient Condition (3)

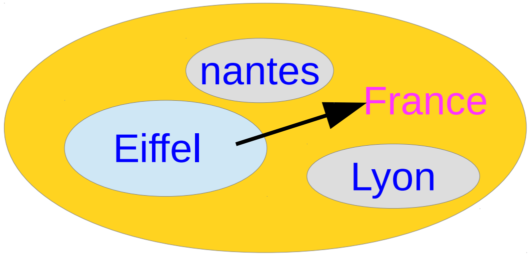
To be sure that Maria visits **France**, it suffices for her to go to the **Eiffel Tower**.

There are surely **other ways** to ensure that Maria visits **France**; for example, she could go to **Lyon**.

*Not equivalent statement:*

If Maria visits **France**, then she goes to the **Eiffel Tower**.

There are **ways other than** going to the **Eiffel Tower** to ensure that Maria visits **France**

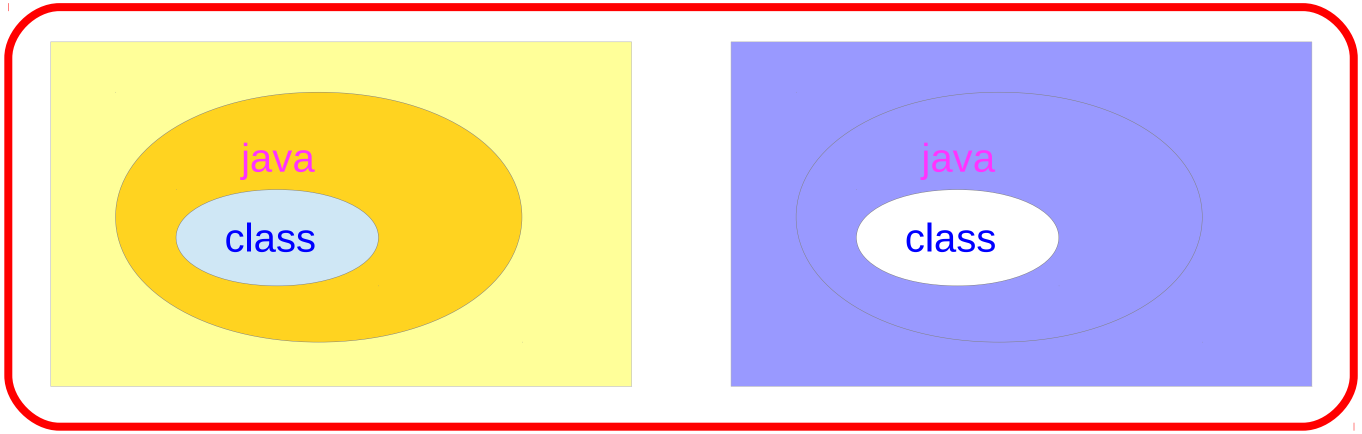


Discrete Mathematics, Johnsonbaugh

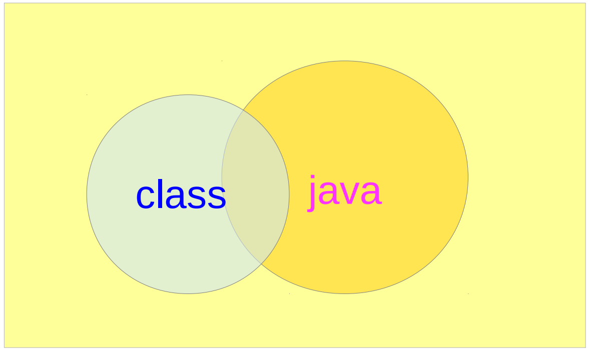
**sufficient conditions**

# Implication in First Order Logic

**Every** student in this **class** has studied **Java**.



$$\forall x\,(C(x) \rightarrow J(x))$$

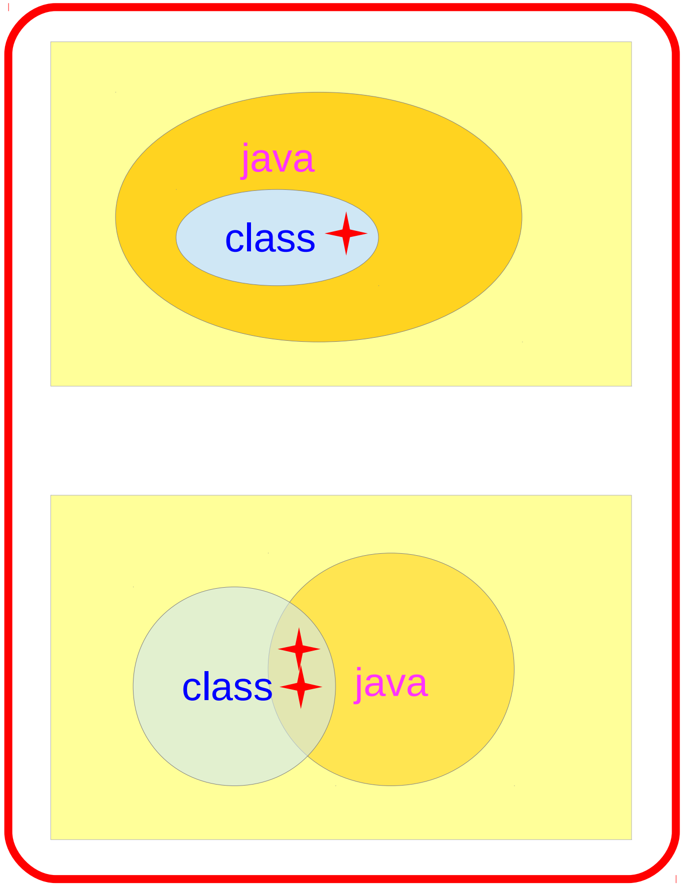


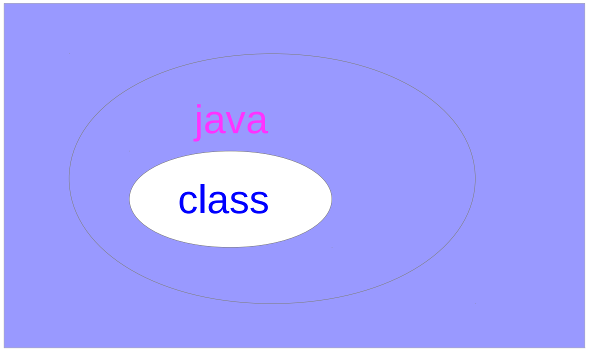
Discrete Mathematics, Johnsonbaugh

**sufficient conditions**

# Implication in First Order Logic

**Some** student in this **class** has studied **Java**.





$$\exists x \left( C(x) \wedge J(x) \right)$$

Discrete Mathematics and its Applications, Rosen

**sufficient conditions**

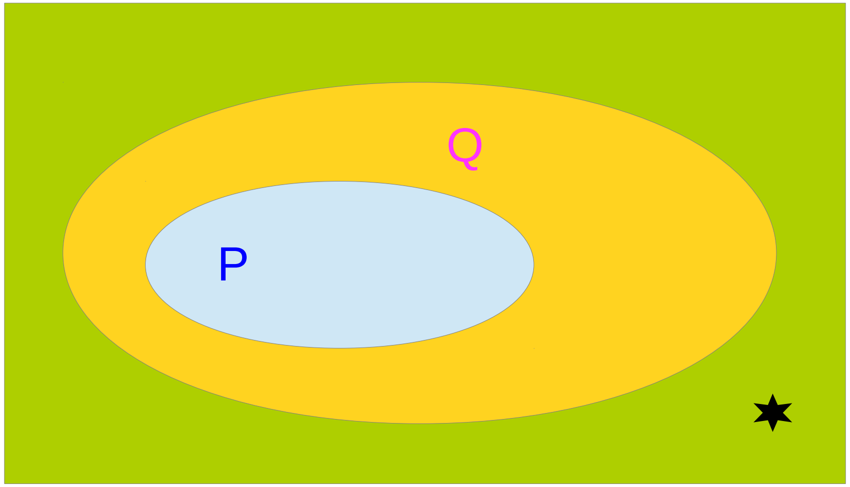
# To prove implications by contradiction

P ⟶ Q

Assume this is false

$$\neg(\neg p \vee q)$$

$$p \wedge \neg q$$



Assume P is true and Q is false

Derive contradiction

Discrete Mathematics and its Applications, Rosen

# Indirect Proof

contradiction

| $\neg p$ | $p$ | $q$ | $r$ | $p \rightarrow q$ | $p \wedge \neg q$ | $r \wedge \neg r$ | $(p \wedge \neg q) \rightarrow (r \wedge \neg r)$ |
|---|---|---|---|---|---|---|---|
| F | T | T | T | T | F | F | T |
| F | T | T | F | T | F | F | T |
| F | T | F | T | F | T | F | F |
| F | T | F | F | F | T | F | F |
| T | F | T | T | T | F | F | T |
| T | F | T | F | T | F | F | T |
| T | F | F | T | T | F | F | T |
| T | F | F | F | T | F | F | T |

## References

[1] http://en.wikipedia.org/
[2] http://web.stanford.edu/class/archive/cs/cs103/cs103.1132/lectures/02/Small02.pdf