Second Order ODEs (H.1)

20151228

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = \boxed{g(x,y)} \qquad y' = \boxed{g(x,y)}$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y) \qquad y' = g_1(x) g_2(y) \qquad y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x) y = g(x) \qquad a_1(x) y' + a_0(x) y = g(x) \qquad y = f(x)$$

Exact Equations

$$M(x, y) dx + N(x, y) dy = 0 \qquad \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0 \qquad z = f(x, y)$$

Second Order ODEs

First Order Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x) y = g(x) \qquad a_1(x) y' + a_0(x) y = g(x)$$

Second Order Linear Equations

$$a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x) y = g(x) \qquad a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

Second Order Linear Equations with Constant Coefficients

$$a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x) \qquad a_2 y'' + a_1 y' + a_0 y = g(x)$$

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c\, y = g(x) \qquad a\, y'' + b\, y' + c\, y = g(x)$$

Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$

$$a\frac{d^2}{dx^2}\{e^{mx}\} + b\frac{d}{dx}\{e^{mx}\} + c\{e^{mx}\} = 0 \qquad a\{e^{mx}\}'' + b\{e^{mx}\}' + c\{e^{mx}\} = 0$$

$$a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0 \qquad a\{m^2e^{mx}\} + b\{me^{mx}\} + c\{e^{mx}\} = 0$$

$$(am^2 + bm + c)\cdot e^{mx} = 0 \qquad (am^2 + bm + c)\cdot e^{mx} = 0$$

auxiliary equation

$$\boxed{(am^2 + bm + c) = 0} \qquad \boxed{(am^2 + bm + c) = 0}$$

Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

auxiliary equation

try a solution $y = e^{mx}$ $\Rightarrow$ $\boxed{(am^2 + bm + c) = 0}$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

$y_1 = e^{m_1x}$, $y_2 = e^{m_2x}$ $\Leftarrow$	(A)	$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$y_1 = e^{m_1x} = y_2 = e^{m_2x}$ $\Leftarrow$	(B)	$b^2 - 4ac = 0$	Real, equal m_1, m_2
$y_1 = e^{m_1x}$, $y_2 = e^{m_2x}$ $\Leftarrow$	(C)	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

Linear Combination of Solutions

DEQ

$$\boxed{a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0} \leftarrow y_1, \; y_2$$

$$\boxed{C_1\, y_1 + C_2\, y_2}$$

$$a\,y_1'' + b\,y_1' + c\,y_1 = 0$$
$$a\,y_2'' + b\,y_2' + c\,y_2 = 0$$

$$a(y_1'' + y_2'') + b(y_1' + y_2') + c(y_1 + y_2) = 0$$
$$a(y_1 + y_2)'' + b(y_1 + y_2)' + c(y_1 + y_2) = 0$$

$$y_3 = y_1 + y_2$$
$$y_4 = y_1 - y_2$$
$$y_5 = y_3 + 2\,y_4$$
$$y_6 = y_3 - 2\,y_4$$

$$a(C_1 y_1'' + C_2 y_2'') + b(C_1 y_1' + C_2 y_2') + c(C_1 y_1 + C_2 y_2) = 0$$
$$a(C_1 y_1 + C_2 y_2)'' + b(C_1 y_1 + C_2 y_2)' + c(C_1 y_1 + C_2 y_2) = 0$$

Solutions of 2nd Order ODEs

DEQ

$$\boxed{a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = 0} \leftarrow y_1, \; y_2$$

$$\boxed{C_1\, y_1 + C_2\, y_2}$$

$$\begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \; (D > 0) \qquad \begin{cases} y_1 = e^{m_1 x} \\ \end{cases} \; (D = 0) \qquad \begin{cases} y_1 = e^{m_1 x} \\ y_2 = e^{m_2 x} \end{cases} \; (D < 0)$$

$$\begin{cases} y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D > 0) \\ y = C_1 e^{m_1 x} \quad ? & (D = 0) \\ y = C_1 e^{m_1 x} + C_2 e^{m_2 x} & (D < 0) \end{cases}$$

auxiliary equation

$$\boxed{(a\,m^2 + b\,m + c) = 0}$$

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2\,a$$
$$m_2 = (-b - \sqrt{b^2 - 4ac})/2\,a$$

Linear Independent Functions Example (2)

linearly dependent functions

$$y_1 = e^{2x} \qquad y_2 = 3e^{2x}$$

$$y = c_1 e^{2x} + c_2 \cdot 3e^{2x}$$
$$y = (c_1 + 3c_2) e^{2x} = C e^{2x}$$

~~general solution~~

linearly independent functions

$$y_1 = e^{2x} \qquad y_2 = x e^{2x}$$

$$y = c_1 e^{2x} + c_2 x e^{2x}$$

general solution

$$y'' - 4y' + 4y = 0 \qquad m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$y_1 = e^{2x}$ | $4y_1 = 4e^{2x}$
$y_1' = 2e^{2x}$ | $-4y_1' = -8e^{2x}$
$y_1'' = 4e^{2x}$ | $y_1'' = 4e^{2x}$
| 0

$y_2 = x e^{2x}$ | $4y_2 = 4xe^{2x}$
$y_2' = e^{2x} + 2xe^{2x}$ | $-4y_2' = -4e^{2x} - 8xe^{2x}$
$y_2'' = 2e^{2x} + 2e^{2x} + 4xe^{2x}$ | $y_2'' = 4e^{2x} + 4xe^{2x}$
| 0

Fundamental Set of Solutions

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \leftarrow y_1, \; y_2$$

$$C_1 y_1 + C_2 y_2$$

Functions y_1 and y_2 are either
- linearly independent functions or
- linearly dependent functions

$$\{y_1, \; y_2\}$$

Second Order there can be at most two linearly independent functions

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$
$$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$$
$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

any *n* linearly independent solutions of the homogeneous linear *n-th* order differential equation

Fundamental Set of Solutions

(A) Real Distinct Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ ➡ $(am^2 + bm + c) = 0$ auxiliary equation

$$m_1 = (-b + \sqrt{b^2-4ac})/2a \qquad y_1 = e^{m_1 x}$$
$$m_2 = (-b - \sqrt{b^2-4ac})/2a \qquad y_2 = e^{m_2 x}$$

$b^2-4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2-4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2-4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$

(B) Repeated Real Roots Case

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ ➡ $(am^2 + bm + c) = 0$ auxiliary equation

$$\begin{aligned} m_1 &= (-b + \sqrt{b^2-4ac})/2a \\ m_2 &= (-b - \sqrt{b^2-4ac})/2a \end{aligned} \;\Rightarrow\; b^2-4ac = 0 \;\Rightarrow\; \begin{aligned} m_1 &= -b/2a \\ m_2 &= -b/2a \end{aligned}$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$b^2-4ac > 0$	Real, distinct m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$
$b^2-4ac = 0$	Real, equal m_1, m_2	$y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$
$b^2-4ac < 0$	Conjugate complex m_1, m_2	$y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$

(C) Complex Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

try a solution $y = e^{mx}$ $\Rightarrow$ $(am^2 + bm + c) = 0$ auxiliary equation

$m_1 = (-b + \sqrt{b^2 - 4ac})/2a \Rightarrow m_1 = (-b + \sqrt{4ac - b^2}\, i)/2a \qquad y_1 = e^{m_1 x}$

$m_2 = (-b - \sqrt{b^2 - 4ac})/2a \Rightarrow m_2 = (-b - \sqrt{4ac - b^2}\, i)/2a \qquad y_2 = e^{m_2 x}$

$b^2 - 4ac > 0$ Real, distinct m_1, m_2 $\qquad y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$b^2 - 4ac = 0$ Real, equal m_1, m_2 $\qquad y = C_1 e^{m_1 x} + C_2 x e^{m_1 x}$

$b^2 - 4ac < 0$ Conjugate complex m_1, m_2 $\qquad y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

Complex Exponential Conversion

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad ay'' + by' + cy = 0$$

$m_1 = (-b + \sqrt{b^2 - 4ac})/2a \Rightarrow m_1 = (-b + \sqrt{4ac - b^2}\, i)/2a = \alpha + i\beta \qquad y_1 = e^{m_1 x}$

$m_2 = (-b - \sqrt{b^2 - 4ac})/2a \Rightarrow m_2 = (-b - \sqrt{4ac - b^2}\, i)/2a = \alpha - i\beta \qquad y_2 = e^{m_2 x}$

$b^2 - 4ac < 0$ Conjugate complex m_1, m_2 $\qquad y = C_1 e^{m_1 x} + C_2 e^{m_2 x} = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$

Pick **two** homogeneous solution

$y_1 = \{e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x}\}/2 = e^{\alpha x}\cos(\beta x) \qquad (C_1 = +1/2, \quad C_2 = +1/2)$

$y_2 = \{e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}\}/2i = e^{\alpha x}\sin(\beta x) \qquad (C_1 = +1/2i, \quad C_2 = -1/2i)$

$$y = C_3 e^{\alpha x}\cos(\beta x) + C_4 e^{\alpha x}\sin(\beta x) = e^{\alpha x}(C_3\cos(\beta x) + C_4\sin(\beta x))$$

Advanced Eng Math in plain view

ODE

2nd Order ODE : 2P.pdf → Complex numbers
: Linear Equation (1A.pdf)
: Reduction of Order (....)

Zill & Wright Sec 3.3, 3.2

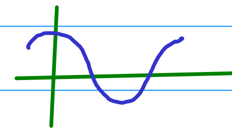
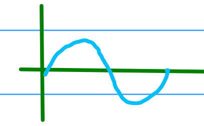
Complex Number Background

Taylor Series

Euler Formula

Taylor Series

$f(x) = \cos x$

$f'(x) = -\sin x$ $\qquad f'(0) = -\sin 0 = 0$

$f^{(2)}(x) = -\cos x$ $\qquad f^{(2)}(0) = -\cos 0 = -1$

$f^{(3)}(x) = +\sin x$ $\qquad f^{(3)}(0) = +\sin 0 = 0$

$f^{(4)}(x) = +\cos x$ $\qquad f^{(4)}(0) = +\cos 0 = +1$

$f^{(5)}(x) = -\sin x$ $\qquad f^{(5)}(0) = -\sin 0 = 0$

$f^{(6)}(x) = -\cos x$ $\qquad f^{(6)}(0) = -\cos 0 = -1$

$f^{(7)}(x) = +\sin x$ $\qquad f^{(7)}(0) = +\sin 0 = 0$

$f^{(8)}(x) = +\cos x$ $\qquad f^{(8)}(0) = +\cos 0 = +1$

$\vdots \qquad\qquad \vdots$

$x=0$ 근방에서 Taylor series

$$\boxed{f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x^1 + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4}$$

$$\cos x = +1 + \frac{0}{1!}x^1 + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{-1}{4!}x^4$$

$f'(0) = -\sin 0 = 0$

$f^{(2)}(0) = -\cos 0 = -1$

$f^{(3)}(0) = +\sin 0 = 0$

$f^{(4)}(0) = +\cos 0 = +1$

$$\cos x = \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}}_{\text{Series (급수)}}$$

0의근방에서

```
(%i1) f1(x) := 1;
```
(%o1) $f1(x):=1$

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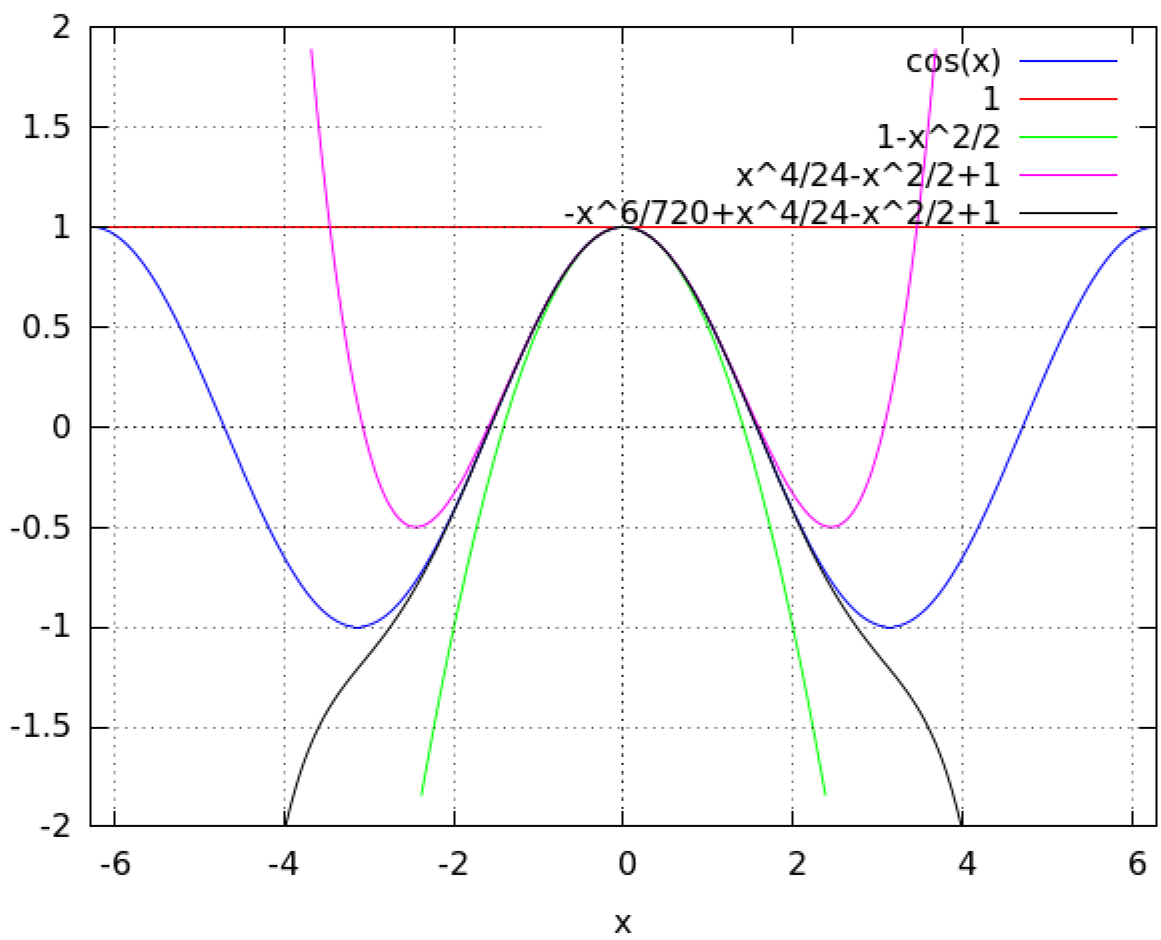
(%i2) f2(x) := f1(x) - x^2/factorial(2);
```
(%o2) $f2(x):=f1(x)-\frac{x^2}{2!}$

```
(%i3) f2(x);
```
(%o3) $1-\frac{x^2}{2}$

```
(%i4) f3(x) := f2(x) + x^4 / factorial(4);
```
(%o4) $f3(x):=f2(x)+\frac{x^4}{4!}$



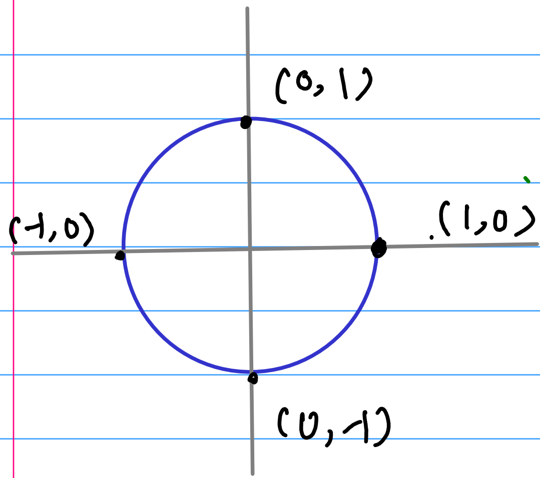
Taylor Series Example

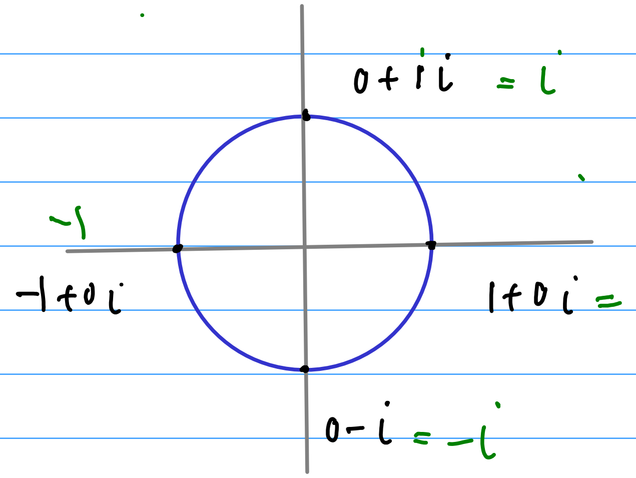
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

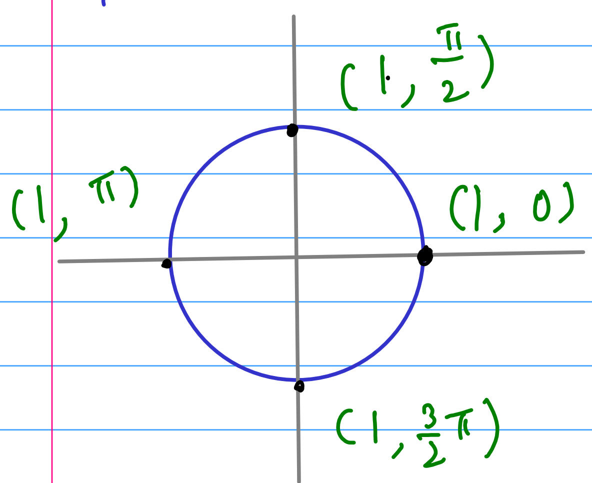
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

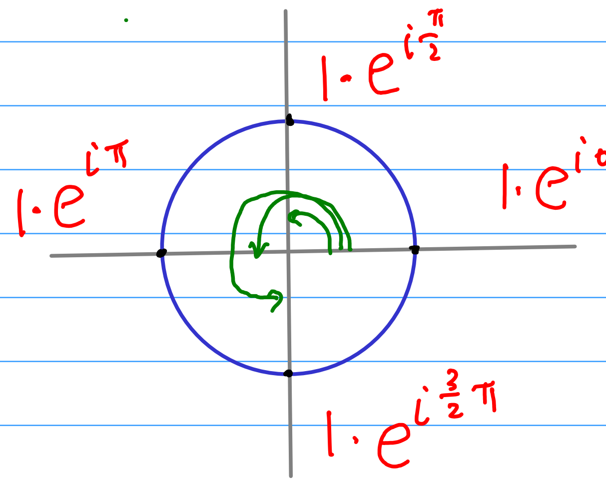
직교좌표





극좌표





$$x + yi = re^{i\theta}$$

$$e^{i0} = \cos 0 + i\cancel{\sin 0}$$

$$e^{i\frac{\pi}{2}} = \cancel{\cos\frac{\pi}{2}} + i\sin\frac{\pi}{2}$$

$$e^{i\pi} = \cos\pi + \cancel{i\sin\pi}$$

$$e^{i\frac{3}{2}\pi} = \cancel{\cos\frac{3}{2}\pi} + i\sin\frac{3}{2}\pi$$

Euler Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

hw 20150626

Paul's online math note

Differential Equation
Second Order :
- Two distinct real roots
- Complex conjugate roots
- repeated roots
- Reduction of Orders
- Undetermined Coefficients

$$a y'' + b y' + c y = 0$$

homogeneous eq

y_h : homogeneous solution

$$a y'' + b y' + c y = \underbrace{g(x)}_{x\text{의 식}}$$

y_p : particular solution

① 3.4 Undetermined Coefficients
② 3.5 Variation of Parameters
③ Green's function.

finding y_p : undetermined coefficients

$(e^x)' = e^x$

$(e^x)'' = e^x$

$(e^x)'' - 2(e^x)' + (e^x) = 0$ for all x

$$\boxed{y'' - 2y' + y = 0}$$

$m^2 - 2m + 1 = 0$ $\quad (m-1)^2 = 0$

$m = 1$ $\quad y_h = c_1 e^x + c_2 x e^x$

repeated root

$$\boxed{y'' - 2y' + y = e^{3x}}$$

$y = Ae^{3x}$ $\quad y' = 3Ae^{3x}$ $\quad y'' = 9Ae^{3x}$

$9Ae^{3x} - 2 \cdot 3Ae^{3x} + Ae^{3x} = e^{3x}$

$(\underline{9A - 6A + A})e^{3x} = e^{3x}$

$4A = 1$ $\quad A = \frac{1}{4}$

$y_p = \frac{1}{4}e^{3x}$

When the multiplication rule must be considered

$$ay'' + by' + cy = \boxed{e^{kt}}$$

$$am^2 + bm + c = 0$$

real distinct m_1, m_2

$$e^{m_1 t} \qquad e^{m_2 t} \qquad A e^{kt}$$

$k = m_1$, $k = m_2$

if e^{kt} is already in y_h

When the multiplication rule must be considered

$$ay'' + by' + cy = \boxed{\begin{matrix} e^{\alpha t}\cos(\beta t) \\ e^{\alpha t}\sin(\beta t) \end{matrix}}$$

complex conjugate roots

$$\{e^{m_1 t}, \quad e^{m_2 t}\}$$

$$\{e^{\alpha t + i\beta t}, \quad e^{\alpha t - i\beta t}\}$$

$$\{e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t)\}$$

already in y_h

$\alpha = 0$

$$ay'' + by' + cy = \boxed{\begin{matrix} \cos(\beta t) \\ \sin(\beta t) \end{matrix}}$$

complex conjugate roots

$$\{e^{m_1 t}, \quad e^{m_2 t}\}$$

$$\{e^{+i\beta t}, \quad e^{-i\beta t}\}$$

$$\{\cos(\beta t), \quad \sin(\beta t)\}$$

already in y_h

When the multiplication rule must be considered

Zill & Wright 3.4 Ex 2.13, 2.15

$y'' + 4y = 3\sin 2x$

$A\sin 2x + B\cos 2x$ ✗

$m^2 + 4 = 0$

$m = \pm 2i \quad (\alpha = 0)$

$y_p = x(A\sin 2x + B\cos 2x)$

$c_1 e^{2ix} + c_2 e^{-2ix}$

$c_3\cos(2x) + c_4\sin(2x)$

$x\sin x$, $\cancel{\sin x}$

$y'' + y = 3x\sin x$ → $(Ax+B)\sin x + (Cx+D)\cos x$

$m^2 + 1 = 0$

$m = +i, -i$

already in y_h

$c_1\cos x + c_2\sin x$

$x(Ax+B)\sin x + x(Cx+D)\cos x$

$$4y'' + 16y' + 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$$

$$4y'' + 16y' + 17y = 0$$

$$4m^2 + 2\cdot 8\, m + 17 = 0 \qquad m = \frac{-8 \pm \sqrt{64 - 68}}{4}$$

$$m = -2 \pm \frac{1}{2} i$$

$$y_c = c_1 e^{(-2+\frac{1}{2}i)t} + c_2 e^{(-2-\frac{1}{2}i)t}$$

$$= c_3 e^{-2t}\cos\left(\frac{1}{2}t\right) + c_4 e^{-2t}\sin\left(\frac{1}{2}t\right)$$

$$4y'' + 16y' + 17y = \underline{e^{-2t}\sin\left(\frac{t}{2}\right)} + \underline{6t\cos\left(\frac{t}{2}\right)}$$

$$y_p = \underline{e^{-2t}\left[A\sin\left(\frac{t}{2}\right) + B\cos\left(\frac{t}{2}\right)\right]} + (Ct + D)\sin\left(\frac{t}{2}\right) + (Et + F)\cos\left(\frac{t}{2}\right)$$

already in y_h

$$y_p = \underline{te^{-2t}\left[A\sin\left(\frac{t}{2}\right) + B\cos\left(\frac{t}{2}\right)\right]} + (Ct + D)\sin\left(\frac{t}{2}\right) + (Et + F)\cos\left(\frac{t}{2}\right)$$

$$y_p = t e^{-2t}\left[A\sin\left(\tfrac{t}{2}\right) + B\cos\left(\tfrac{t}{2}\right)\right] + (Ct + D)\sin\left(\tfrac{t}{2}\right) + (Et + F)\cos\left(\tfrac{t}{2}\right)$$

$$y_p: \left\{ \begin{array}{lll} te^{-2t}\sin\left(\tfrac{t}{2}\right), & t\sin\left(\tfrac{t}{2}\right), & \sin\left(\tfrac{t}{2}\right), \\ te^{-2t}\cos\left(\tfrac{t}{2}\right), & t\cos\left(\tfrac{t}{2}\right), & \cos\left(\tfrac{t}{2}\right) \end{array} \right\}$$

$$y_h: \left\{ \begin{array}{l} e^{-2t}\sin\left(\tfrac{t}{2}\right), \\ e^{-2t}\cos\left(\tfrac{t}{2}\right) \end{array} \right\}$$

$$4y'' + 16y' + 17y = e^{-2t}\sin\left(\frac{t}{2}\right) + 6t\cos\left(\frac{t}{2}\right)$$

$$e^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) \quad (Et+F)\sin\left(\frac{t}{2}\right)$$

already in y_h $(= y_c)$

$$y_c = c_3 e^{-2t}\cos\left(\frac{1}{2}t\right) + c_4 e^{-2t}\sin\left(\frac{1}{2}t\right)$$

$$t\,e^{-2t}\left(A\cos\left(\frac{t}{2}\right) + B\sin\left(\frac{t}{2}\right)\right) + (Ct+D)\cos\left(\frac{t}{2}\right) \quad (Et+F)\sin\left(\frac{t}{2}\right)$$

$$y_c = c_1 \underbrace{\boxed{e^{(-2+\frac{1}{2}i)t}}}_{y_1} + c_2 \underbrace{\boxed{e^{(-2-\frac{1}{2}i)t}}}_{y_2}$$

$$\boxed{4y'' + 16y' + 17y = 0} \leftarrow y_1,\ \leftarrow y_2$$

$$y_3 = \frac{1}{2}y_1 + \frac{1}{2}y_2$$

$$y_4 = \frac{1}{2i}y_1 + \frac{1}{2i}y_2$$

if y_1 & y_2 are solutions, then so are y_3 & y_4

$$y_3 = \frac{1}{2}e^{(-2+\frac{1}{2}i)t} + \frac{1}{2}e^{(-2-\frac{1}{2}i)t}$$ a solution

$$= e^{-2t}\left(\frac{e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}}{2}\right) = \underline{e^{-2t}\cos\left(\frac{t}{2}\right)}$$

$$y_4 = \frac{1}{2i}e^{(-2+\frac{1}{2}i)t} - \frac{1}{2i}e^{(-2-\frac{1}{2}i)t}$$ a solution

$$= e^{-2t}\left(\frac{e^{i\frac{t}{2}} - e^{-i\frac{t}{2}}}{2i}\right) = \underline{e^{-2t}\sin\left(\frac{t}{2}\right)}$$

$$y_c = c_3 \overbrace{\boxed{e^{-2t}\cos\left(\frac{t}{2}\right)}}^{y_3} + c_4 \overbrace{\boxed{e^{-2t}\sin\left(\frac{t}{2}\right)}}^{y_4}$$

$$ay'' + by' + cy = g(x)$$

general solution

$$y_h + y_p$$

Finding y_h

$$ay'' + by' + cy = 0$$

$$am^2 + bm^1 + cm^0 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D > 0$

$m_1, \quad m_2$

$\downarrow \quad \downarrow$

$$\underbrace{e^{m_1 x}, \quad e^{m_2 x}}$$

$\Downarrow$

$$c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

linear combination

$D = 0$

$m_1 = m_2$

$\downarrow$

$$\underbrace{e^{m_1 x}, \quad x e^{m_2 x}}$$

$\Downarrow$

$$c_1 e^{m_1 x} + c_2 x e^{m_2 x}$$

linear combination

Finding y_h

$$a y'' + b y' + c y = 0$$

$$a m^2 + b m^1 + c m^0 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$D < 0$

$m_1, \quad m_2$

$\downarrow \qquad \downarrow$

$$\underbrace{e^{m_1 x}, \quad e^{m_2 x}}$$

$\Downarrow$

$$c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

linear combination

$m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

① $c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

② $e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right]$

③ $e^{\alpha x}\left[c_3 \cos(\beta x) + c_4 \sin(\beta x)\right]$

Finding y_p Example

(a) $\underline{\underline{y'' + 3y' - 28y = 0}}$

$m^2 + 3m - 28 = (m+7)(m-4) = 0$

$m = -7 \quad , \quad 4$

$\downarrow$

$c_1 e^{-7t} + c_2 e^{4t}$

(b) $\underline{\underline{y'' - 100y = 0}}$

$m^2 - 100 m^0 = 0 \qquad\qquad m^2 - 100 = 0$

$m = 10 \qquad -10$

$c_1 e^{+10t} + c_2 e^{-10t}$

(c) $\underline{\underline{4y'' + y = 0}}$

$4m^2 + m^0 = 0 \qquad\qquad 4m^2 + 1 = 0$

$m^2 = -\frac{1}{4} \qquad m = \frac{i}{2} \qquad -\frac{i}{2}$

$c_1 e^{\frac{i}{2}t} + c_2 e^{-\frac{i}{2}t}$

$c_3 \underline{\cos\left(\frac{t}{2}\right)} + c_4 \underline{\sin\left(\frac{t}{2}\right)}$

(d) $4y'' + 16y' + 17y = 0$

$4m^2 + 16m + 17 = 0$ $\qquad 2 \quad 17$

$\qquad\qquad\qquad\qquad 2 \quad 2$

$4(m^2 + 4m + 4) + 1 = 0$

$4(m+2)^2 + 1 = 0$

$(m+2)^2 = -\frac{1}{4}$

$(m+2) = \pm \frac{i}{2}$

$m = -2 + \frac{i}{2} \quad . \quad -2 - \frac{i}{2}$

$$C_1 e^{(-2+\frac{i}{2})t} + C_2 e^{(-2-\frac{i}{2})t}$$

$$C_1 e^{-2t} e^{+\frac{i}{2}t} + C_2 e^{-2t} e^{-\frac{i}{2}t}$$

$$e^{-2t}\left(C_1 e^{+\frac{i}{2}t} + C_2 e^{-\frac{i}{2}t} \right)$$

$$e^{-2t}\left(C_3 \cos\left(\frac{t}{2}\right) + C_4 \sin\left(\frac{t}{2}\right) \right)$$

(e) $\underline{\underline{y'' + 8y' + 16y = 0}}$

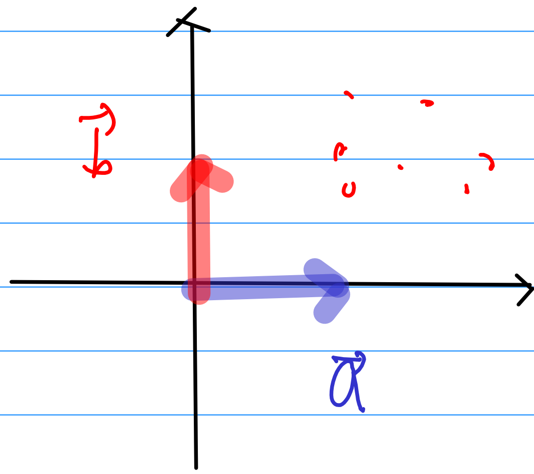
$m^2 + 8m + 16 = (m+4)^2 = 0$

$m = -2$

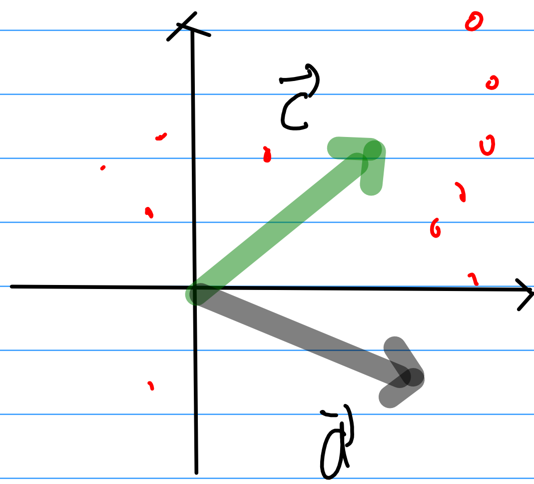
$\downarrow$

$c_1 e^{-2t} + c_2 t \cdot e^{-2t}$

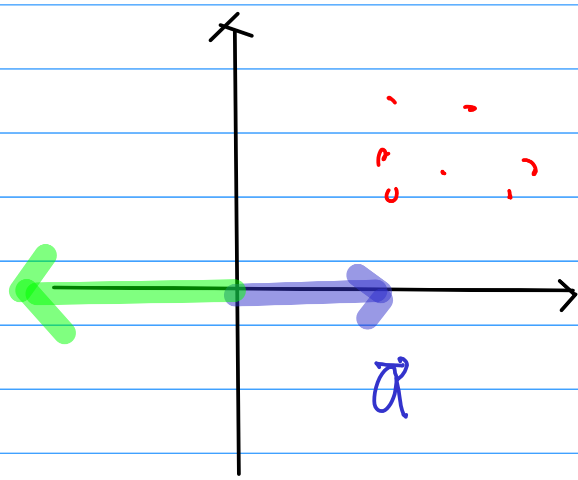
Linear Independent Vectors



$c_1 \vec{a} + c_2 \vec{b}$



$c_3 \vec{c} + c_4 \vec{d}$



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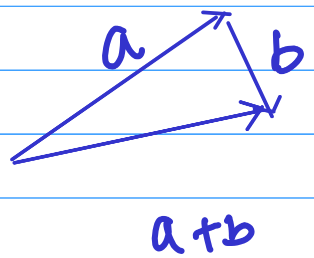
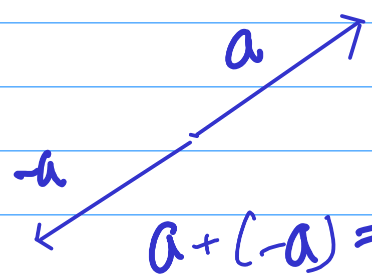
Second Order Differential Equation (2.P.pdf)

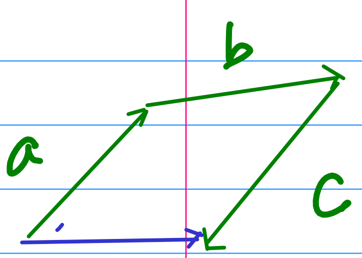
- Linear Independence
- Wronskian

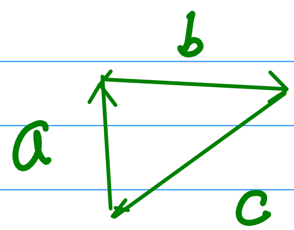




$a + (-a) = 0$

한개 vec = 나머지 vect들 linear comb.





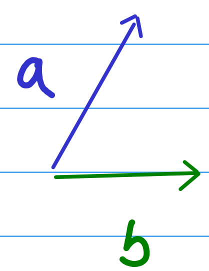
$a = -b - c$

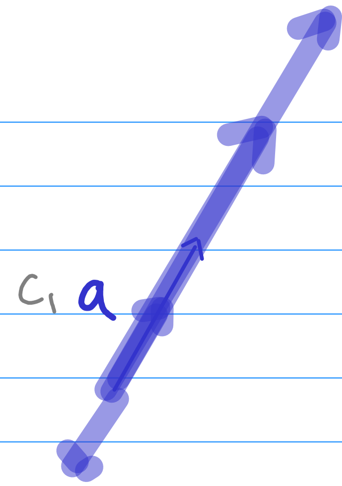
$b = -a - c$

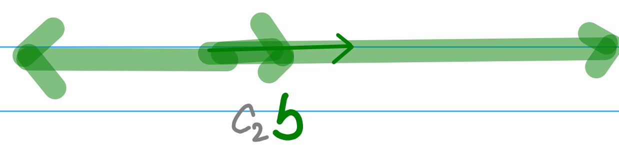
$c = -a - b$

~~linearly indep~~

$1a + 1b + 1c = 0$







$c_1\vec{a} + c_2\vec{b}$

a, b의 선형결합

linear combination

$\vec{a}$는 $\vec{b}$로 만들수 없고

$\vec{b}$도 $\vec{a}$로 만들수 없다

a와 b는 서로 독립

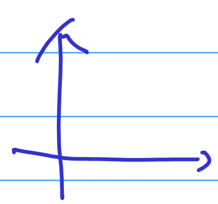
다른 종류의 벡터들이다

$c_1\vec{a} + c_2\vec{b} = 0$ 을

만족시키는 c_1, c_2는 $c_1 = 0, c_2 = 0$ all zero case

⇔ $\vec{a}$와 $\vec{b}$는 서로 linear independent

2차원 평면의 다른 모든 vector들은 $c_1\vec{a} + c_2\vec{b}$로 표현가능



n 개 벡터의 linear independent

n 개 ⟨함수, x의 수식⟩ linear independent → 미분방정식

2 차 미분 방정식 $y'' = g(x, y, y')$

→ max 2 개 linear independent solution을 찾아야 함

3 차 미분 방정식 $y^{(3)} = g(x, y, y', y'')$

→ max 3 개 linear independent solution을 찾아야 함

4 차 미분 방정식 $y^{(4)} = g(x, y, y', y'', y^{(3)})$

→ max 4 개 linear independent solution을 찾아야 함

* Wronskian : 함수가 linearly independent ?

$y_1 = e^{3x}$ $\quad$ $y_2 = e^{-3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ (e^{3x})' & (e^{-3x})' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}$$

$$= -3 - 3 = -6 \neq 0$$

linearly independent

$y_1 = e^{x}, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}$

$$W = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} \neq 0$$

$y_1 = e^{x}, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}$ 서로 linearly independent

$$y_h = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$$

함수 y_1, y_2, y_3 linearly independent?

$y_1(x), \quad y_2(x), \quad y_3(x)$

Form a square Matrix $\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$W \neq 0$: y_1, y_2, y_3 linearly independent!

$W = 0$: y_1, y_2, y_3 linearly ~~independent~~!

P 4n 3

determinant

matrix

(11) $\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$ (12) $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{bmatrix}$

(13) $\begin{bmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{bmatrix}$ (14) $\begin{bmatrix} -3-\lambda & -4 \\ -2 & 5-\lambda \end{bmatrix}$

determinant.

(11) $\begin{vmatrix} 3 & 5 \\ -1 & 4 \end{vmatrix}$ (12) $\begin{vmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{3} & -\frac{4}{3} \end{vmatrix}$

(13) $\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix}$ (14) $\begin{vmatrix} -3-\lambda & -4 \\ -2 & 5-\lambda \end{vmatrix}$

(15) $\begin{vmatrix} 0 & 2 & 0 \\ 3 & 0 & 1 \\ 0 & 5 & 8 \end{vmatrix}$ (16) $\begin{vmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

p 495 예제 1

$$3x_1 + 2x_2 + x_3 = 7$$
$$x_1 - x_2 + 3x_3 = 3$$
$$5x_1 + 4x_2 - 2x_3 = 1$$

1469 6113 21

$$\begin{pmatrix} 3x_1 + 2x_2 + x_3 = 7 \\ 1x_1 - 1\cdot x_2 + 3x_3 = 3 \\ 5x_1 + 4\cdot x_2 - 2x_3 = 1 \end{pmatrix}$$ a unique solution

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

① $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$ using inverse matrix

② Cramer's rule

$$\Delta = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -1 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 13$$

$$\frac{1}{\Delta} \begin{vmatrix} 7 & 2 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & -2 \end{vmatrix} = \frac{1}{13}(-39) = x_1$$

$$\frac{1}{\Delta} \begin{vmatrix} 3 & 7 & 1 \\ 1 & 3 & 3 \\ 5 & 1 & -2 \end{vmatrix} = \frac{1}{13}(78) = x_2$$

$$\frac{1}{\Delta} \begin{vmatrix} 3 & 2 & 7 \\ 1 & -1 & 3 \\ 5 & 4 & 1. \end{vmatrix} = \frac{1}{13}(52) = x_3$$

2nd Order, Linear, Homogeneous, constant coefficients

$$a\boxed{y''} + b\boxed{y'} + c\boxed{y} = 0$$

$y = e^{mx}$

$$am^2 + bm + c = 0$$

$D < 0 \qquad m_1 = \alpha + i\beta \qquad m_2 = \alpha - i\beta$

$$\boxed{c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}}$$

$$W\left(e^{(\alpha+i\beta)t}, e^{(\alpha-i\beta)t}\right) \neq 0$$

linearly independent. → Fundamental set of solution

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ 일 때 solution이 된다

$$\frac{1}{2}\left(e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\cos(\beta t)$$

$c_1 = \frac{1}{2i}$, $c_2 = \frac{-1}{2i}$ 일 때도 solution이 된다

$$\frac{1}{2i}\left(e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\sin(\beta t)$$

$$\boxed{c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}}$$ General Solution

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$ 일 때 solution이 된다

$$\frac{1}{2}\left(e^{(\alpha+i\beta)t} + e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\cos(\beta t)$$

$c_1 = \frac{1}{2i}$, $c_2 = \frac{-1}{2i}$ 일 때도 solution이 된다

$$\frac{1}{2i}\left(e^{(\alpha+i\beta)t} - e^{(\alpha-i\beta)t}\right) = e^{\alpha t}\sin(\beta t)$$

$e^{\alpha t}\cos(\beta t)$ 은 solution이고

$e^{\alpha t}\sin(\beta t)$ 도 solution이다

$c_3 e^{\alpha t}\cos(\beta t) + c_4 e^{\alpha t}\sin(\beta t)$ 도 solution이다.

$$W\left(e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)\right) \neq 0$$

linearly independent. → Fundamental set of solution

$$\boxed{c_3 e^{\alpha t}\cos(\beta t) + c_4 e^{\alpha t}\sin(\beta t)}$$ General Solution

$$\begin{vmatrix} e^{(\alpha+i\beta)x} & e^{(\alpha-i\beta)x} \\ (e^{(\alpha+i\beta)x})' & (e^{(\alpha-i\beta)x})' \end{vmatrix}$$

$$= \begin{vmatrix} e^{(\alpha+i\beta)x} & e^{(\alpha-i\beta)x} \\ (\alpha+i\beta)e^{(\alpha+i\beta)x}, & (\alpha-i\beta)e^{(\alpha-i\beta)x} \end{vmatrix}$$

$$= (\alpha-i\beta)e^{2\alpha x} - (\alpha+i\beta)e^{2\alpha x}$$

$$= (-i2\beta)e^{2\alpha x} \neq 0$$

these are fundamental solution

$W \neq 0$

these also fundamental solution

$W \neq 0$

$$\begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ (e^{\alpha x}\cos(\beta x))' & (e^{\alpha x}\sin(\beta x))' \end{vmatrix} = \begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ (\alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x)) & (\alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x)) \end{vmatrix}$$

$$= e^{\alpha x}\cos(\beta x)(\cancel{\alpha e^{\alpha x}\sin(\beta x)} + \beta e^{\alpha x}\cos(\beta x)) - e^{\alpha x}\sin(\beta x)(\cancel{\alpha e^{\alpha x}\cos(\beta x)} - \beta e^{\alpha x}\sin(\beta x))$$

$$= e^{\alpha x}\cos(\beta x)\beta e^{\alpha x}\cos(\beta x) + e^{\alpha x}\sin(\beta x)\beta e^{\alpha x}\sin(\beta x) = \beta e^{2\alpha x}(\cos^2(\beta x) + \sin^2(\beta x)) = \beta e^{2\alpha x} \neq 0$$

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ch4

Second Order Differential Equation
Variation of Parameters (4.A.pdf)

– ch1

Linear Algebra
Determinants (A.pdf)

3x3 matrix invert

Paul's online math note

158p - 160p examples

→ variation of parameters

Reduction of Orders

Undertermined Coefficients [7]

Finding a second solution from y_1

$$a y'' + b y' + c y = 0$$

2nd Order $\rightarrow$ there are 2 fundamental solutions

(i.e. linearly independent)

* If we know one solution y_1,

then the other solution can be found

by assuming $y_2 = u(x) \cdot y_1$

* Repeated root ($D = 0$)

we know $y_1 = e^{-\frac{b}{2a}x}$ solution.

then the other solution can be found

by assuming $y_2 = u(x) \cdot y_1$

$$\Downarrow$$

$$y_2 = x \cdot y_1$$

Finding y_p from y_h

particular solution — homogeneous solution

$$a\boxed{y''} + b\boxed{y'} + c\boxed{y} = g(x)$$

* y_1, y_2 ..... homogeneous solutions known

Finding a particular solution y_p

$$\left.\begin{aligned} a y_1'' + b y_1' + c y_1 &= 0 \leftarrow y_1 \\ a y_2'' + b y_2' + c y_2 &= 0 \leftarrow y_2 \end{aligned}\right\} \quad y_h = c_1 y_1 + c_2 y_2$$

then the particular solution y_p can be found

by assuming $y_p = u_1 y_1 + u_2 y_2$

* Finding a second solution : Reduction of Orders

given $\boxed{y_1 = e^x}$ $\Rightarrow$ $y_2 = u(x) \cdot e^x$ suppose

$\boxed{y'' - y = 0}$ $\quad$ $u(x)$?

$m^2 - 1 = 0$

$m = \pm 1$

e^{+x}, e^{-x}

$$y_2' = u' e^x + u \cdot e^x$$

$$y_2'' = u'' e^x + u' e^x + u' e^x + u e^x$$

$$= u'' e^x + 2u' e^x + u e^x$$

$$y_2'' - y_2 = \left(u'' e^x + 2u' e^x + \cancel{u e^x}\right) - \left(\cancel{u e^x}\right)$$

$$= u'' e^x + 2u' e^x = 0$$

$$\underset{\substack{\neq \\ 0}}{e^x} \underset{\substack{\parallel \\ 0}}{\underline{(u'' + 2u')}} = 0$$

y_2가 solution이 되기 위한 condition → $u(x)$.

Reduction of Orders

2차 $\quad u'' + 2u' = 0$

$\Downarrow$

1차 $\quad (w)' + 2(w) = 0$

Reduction of Orders

※ $\quad u'' + 2u' + u = 0$

$\Downarrow$

$(w)' + 2(w) + ? = 0$

~~Reduction of Orders~~

given $\boxed{y_1 = e^x}$ $\Rightarrow$ $y_2 = u(x) \cdot e^x$ suppose

$\boxed{y'' - y = 0}$

$u(x)?$

2차 $u'' + 2u' = 0$

1차 $(w)' + 2(w) = 0$

$w = u'$

$y_1 = e^x$

$y_2 = e^{-x}$

$w' = -2w$

$\frac{w'}{w} = -2$

$\int \frac{1}{w} \frac{dw}{dx} dx = \int -2\, dx$

$\int \frac{1}{w} dw = \int -2\, dx$

$\boxed{\ln|w| = -2x + c}$

$|w| = e^{-2x+c}$

$= c e^{-2x}$

$w = \pm c e^{-2x}$

$\boxed{w = c e^{-2x}}$

$\boxed{u' = c e^{-2x}}$

$u = \int c e^{-2x} dx$

$= -\frac{c_1}{2} e^{-2x} + C_2$

$\boxed{u = -\frac{c_1}{2} e^{-2x} + c_2}$

$y_2 = u \cdot y_1$

$= \left(\frac{c_1}{2} e^{-2x} + c_2\right) e^x$

$= \frac{c_1}{2} e^{-x} + \cancel{c_2 e^x}$

$y = c_1 e^x + c_2 e^{-x}$

$= 0$ $\quad a, b, c$

homogeneous, constant coefficients

※ Finding a second solution

$y_1 \longrightarrow y_2$?

y_1과 다른 solution!

$y_2 = u\, y_1$ $\quad u(x)$ x's formula

$= \left(\boxed{\boxed{\cancel{x^4}} + C}\right) y_1$

$y_2 = \boxed{\boxed{\cancel{x^4}}\, y_1} + C\, y_1$

$\neq y_1$ $\quad = y_1$

exclude!

$D < 0$ repeating solution

Finding another solution y_2 from the known y_1

Second Order EQ

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$y_1 = f(x)$ *known solution*

$y_2 = u(x)f(x)$ *another solution to be found*

$b^2 - 4ac = 0$ $\qquad m_1 = m_2 = -\frac{b}{2a}$

We know one solution $\qquad y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$

Suppose the other solution $\qquad y_2(x) = u(x)\, y_1(x) = u(x)e^{m_1 x}$

Condition for $y_2(t)$ to be a solution ⇨ Find u(x)

$$e^{-\frac{b}{2a}x} \longrightarrow u(x)\cdot e^{-\frac{b}{2a}x}$$

$u(x) = ?$ ---> x

$$\boxed{y''} + 4\boxed{y'} + 4\boxed{y} = 0$$

$$m^2 + 4m + 4 = (m+2)^2 = 0 \qquad m = -2$$

$$y_1 = e^{-2x}$$

$$y_2 = u \cdot e^{-2x}$$

$$y_2' = u' \cdot e^{-2x} - 2u \cdot e^{-2x}$$

$$y_2'' = u'' \cdot e^{-2x} - 2u' \cdot e^{-2x} - 2u' \cdot e^{-2x} + 4u \cdot e^{-2x}$$

$$= u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u \cdot e^{-2x}$$

$$\boxed{y''} + 4\boxed{y'} + 4\boxed{y}$$

$$= \left(u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u \cdot e^{-2x}\right)$$

$$+ 4\left(u' \cdot e^{-2x} - 2u \cdot e^{-2x}\right)$$

$$+ 4\left(u \cdot e^{-2x}\right)$$

$$4u \cdot e^{-2x}(1 - 2 + 1) = 0$$

$$\boxed{y''} + 4\boxed{y'} + 4\boxed{y} = u'' \cdot e^{-2x} - 4u' \cdot e^{-2x} + 4u' \cdot e^{-2x}$$

$$= u'' \cdot e^{-2x} = 0$$

$$u'' = 0 \Rightarrow u' = c_1 \Rightarrow u = c_1 x + c_2$$

$$y_1 = e^{-2x}$$

$$y_2 = u \cdot e^{-2x} = (c_1 x + c_2)\, e^{-2x}$$

$$= \underbrace{c_1 x e^{-2x}}_{y_2} + \underbrace{c_2 e^{-2x}}_{y_1}$$

$$y_1 = e^{-2x}$$

$$y_2 = x \cdot e^{-2x}$$

* y_1, y_2 ..... homogeneous solution given

Finding a particular solution

$$y_p = \underbrace{u_1 y_1 + u_2 y_2}$$

$u_1(x) = ?$

$u_2(x) = ?$

homogeneous solution y_1, y_2 주어 있는 partial

$$y_p = \left(\boxed{\quad} + c_1\right) y_1 + \left(\boxed{\quad} + c_2\right) y_2$$

$$= \boxed{\quad}\, y_1 + \boxed{\quad}\, y_2 + c_1 y_1 + c_2 y_2$$

y_1

y_2

$\neq y_1$ $\neq y_2$

$$a y'' + b y' + c y = 0$$

Linear Eq with ~~constant coefficient~~

$$a m^2 + b m + c = 0 \quad \text{aux}$$

$$m = m_1, \quad m_2$$

$$c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$a(t)\, y'' + b(t)\, y' + c(t)\, y = 0$$

$$a m^2 \cancel{+ bm +} c = 0$$

$$t y'' - (t+1) y' + y = t^2$$

$$\begin{cases} a(t) = t \\ b(t) = -t-1 \\ c(t) = 1 \end{cases}$$

~~Constant~~ coefficients

examples) . Variable coefficients

Cauchy-Euler Eq

$$ax^2 y'' + bx y' + c y = 0$$ (Sec 3.6)

Bessel Eq.

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Legendre Eq

$$(1-x^2) y'' - 2x y' + n(n+1) y = 0$$

$$y'' + p(t)\,y' + q(t)\,y = 0$$

$y_1, \quad y_2 \quad \cdots$ linearly indep.

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$y = C_1 y_1(t) + C_2 y_2(t)$$

$t = t_0$

$$y_3 = C_1 y_1(t) + C_2 y_2(t) \qquad \boxed{y(t_0) = 0 \qquad y'(t_0) = 1}$$

C_1 & C_2 를

구할수 있다

$$y_4 = C_1 y_1(t) + C_2 y_2(t) \qquad \boxed{y(t_0) = 1 \qquad y'(t_0) = 0}$$

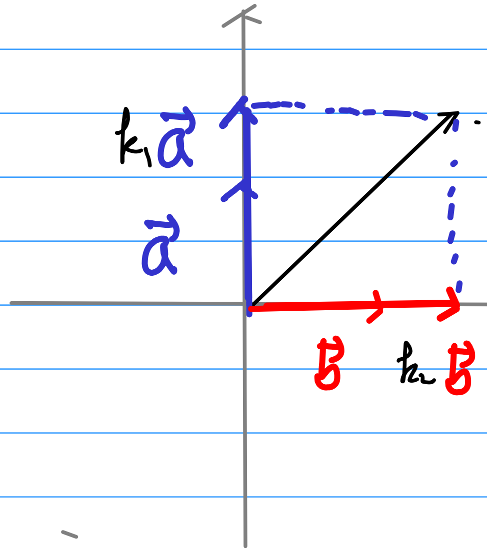
C_1 & C_2 를

구할수 있다

$y_3 \quad y_4$ linear indep

$$\begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} \neq 0$$

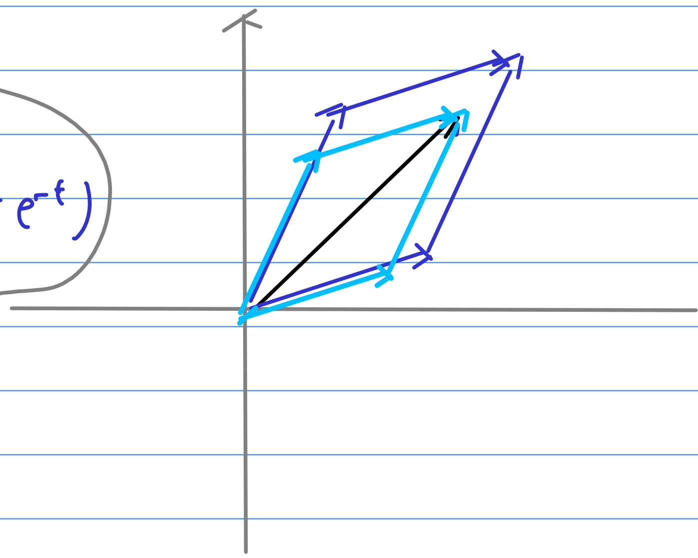
basis vector



$$\vec{x} = k_1\vec{a} + k_2\vec{b}$$

$$c_1 e^{-3t} + c_2 e^{-t}$$

$$c_1\left(-\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}\right) + c_2\left(-\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t}\right)$$



$$y'' - 4y' - 12y = 3e^{5t} + \sin(2t) + te^{4t}$$

$$y'' - 4y' - 12y = 0 \quad \leftarrow \quad y_h$$

$$y'' - 4y' - 12y = 3e^{5t} \quad \leftarrow \quad y_{p1}$$

$$y'' - 4y' - 12y = \sin(2t) \quad \leftarrow \quad y_{p2}$$

$$y'' - 4y' - 12y = te^{4t} \quad \leftarrow \quad y_{p3}$$

$$y = y_h + y_{p1} + y_{p2} + y_{p3}$$

2.2 Separable equation
2.3 Linear equation
2.4 Exact equation
2.5 Substitution

3.2 Reduction of orders
3.3 Homogeneous, Constant coefficients
3.4 Undetermined Coeffcients
3.5 Variation of Parameters