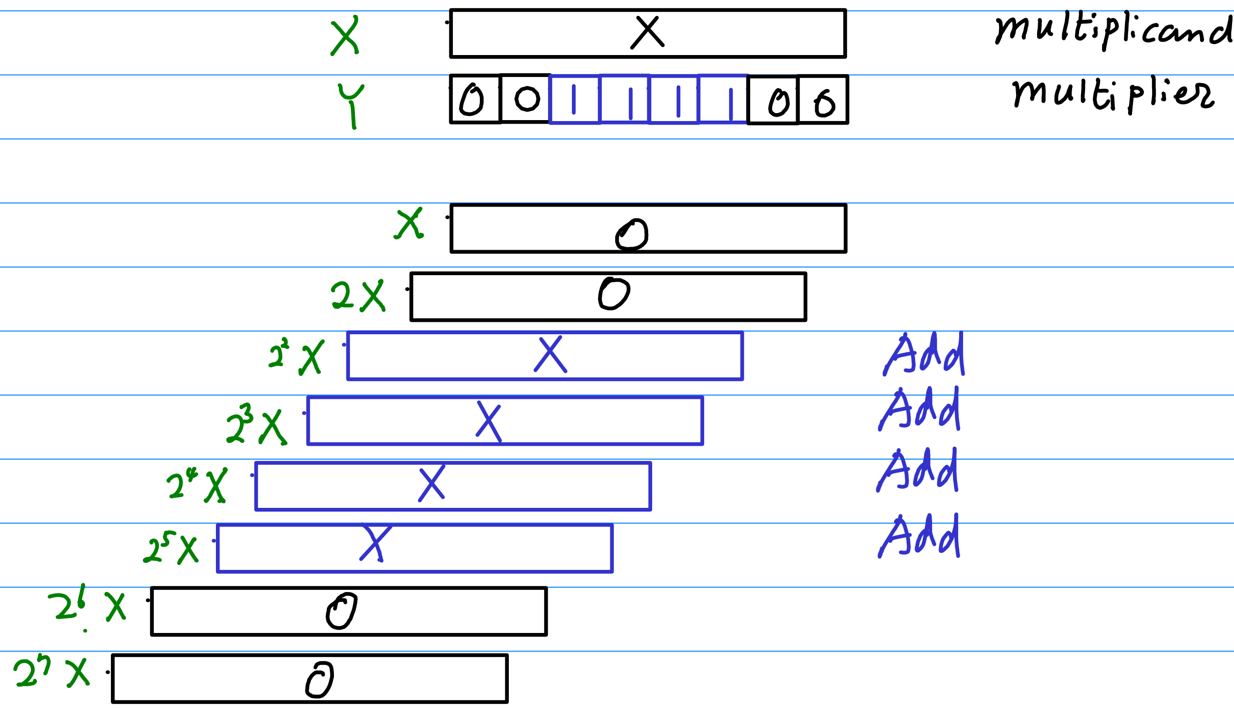
Booth Encoding

20161005

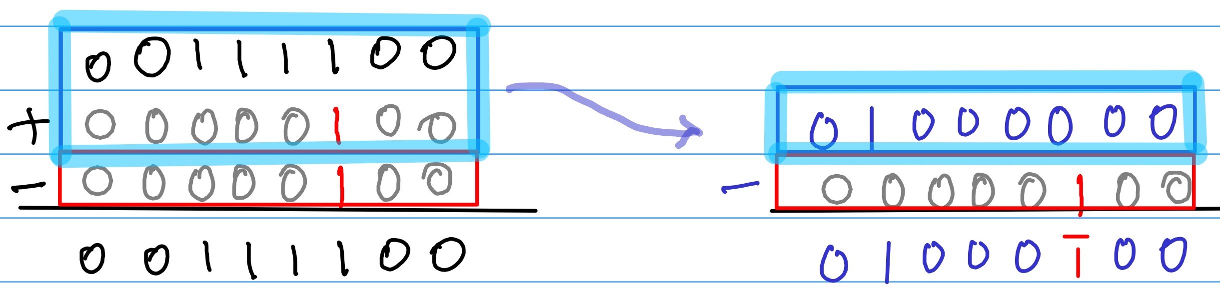
Based on

MJ Flynn's EE 486 Lecture 7 : Integer Multiplication, Stanford University

of Partial Products to be added

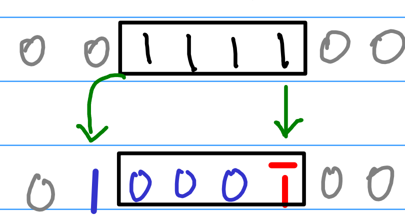


Booth Encoding



+1 −1 = 0

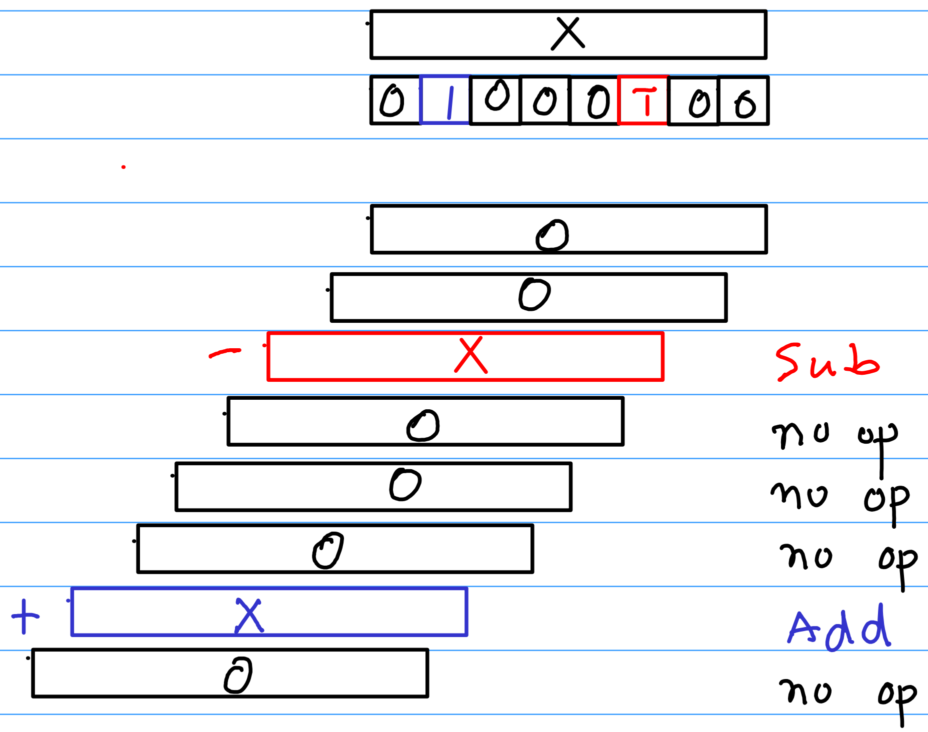
1-run



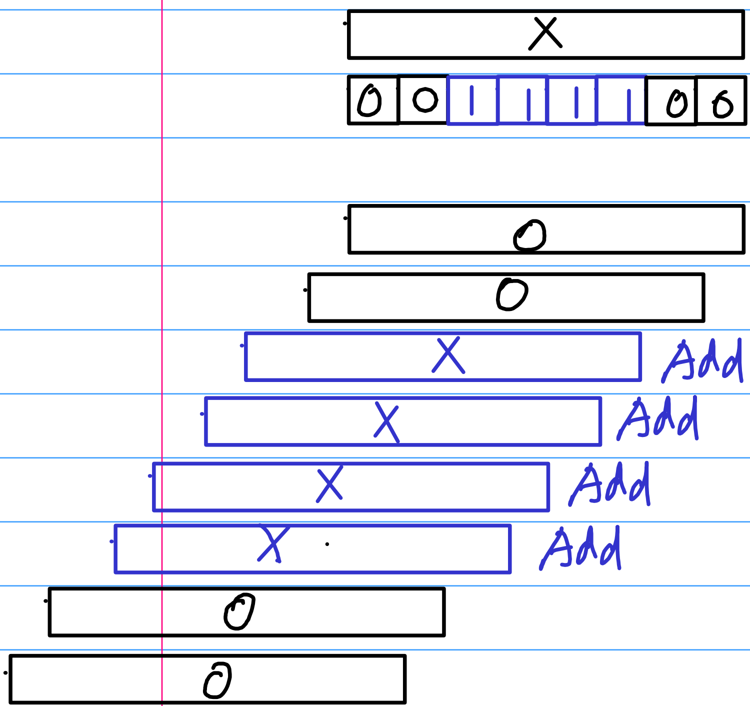
isolated 1 isolated $\bar{1}$

of Partial Products to be added

Original Booth Encoded



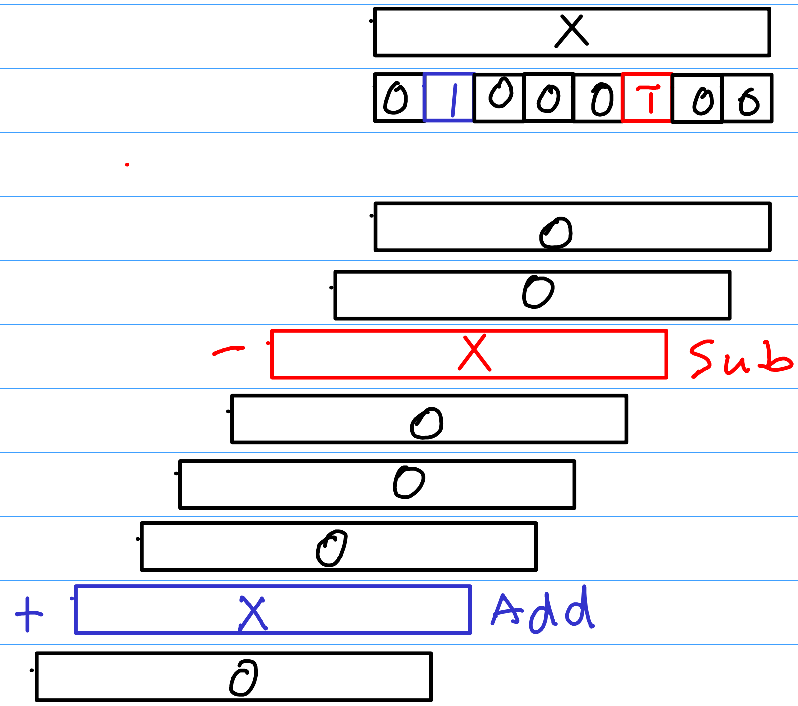
- Best case ✓
- Average case ✓
- Worst case



worst case

n add

=



worst case

n add/sub

booth encoding :

the worst case

the number of partial products $\Rightarrow$ still n

not much gain

reduces the number of partial products to be added

delay reduction

works well for serial multiplication

variable latency

ⓑ worst case : alternating 1's and $\bar{1}$'s

$1\ \bar{1}\ 1\ \bar{1}\ 1\ \bar{1}\ 1\ \bar{1}$

ⓒ Booth encoding does not significantly improve the worst case

⇒ Use **modified** booth encoding

Modified Booth 2

2-bit encoding + 1-bit overlapping

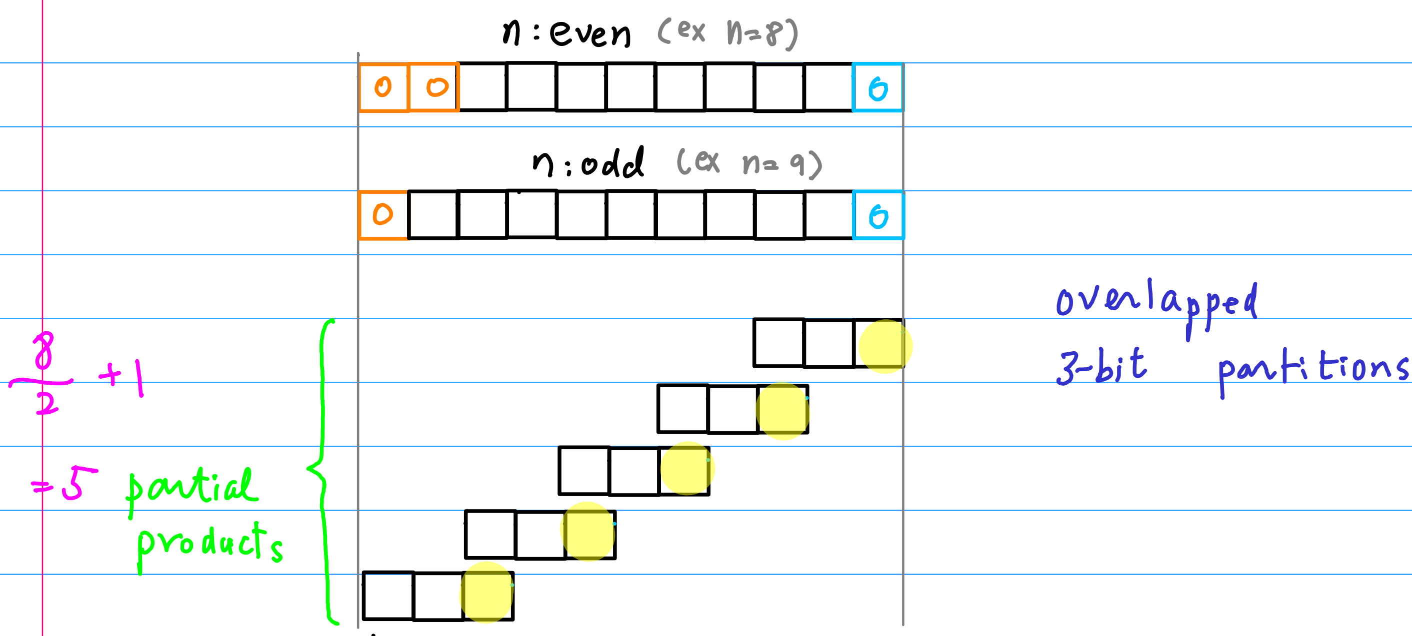
worst case # of p.p's = $\frac{N}{2}+1$

modified Booth 3

3-bit encoding + 1-bit overlapping

worst case # of p.p's = $\frac{N}{3}+1$

Modified Booth 2 (unsigned case)



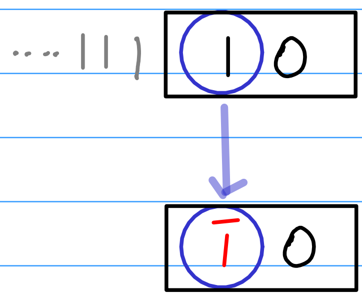
2-bit encoding

		2^1 2^0	scale factor	
all zero's	0 0 0	0 0 0	$0 \cdot 2 + 0 = +0$	+0
end of 1's	0 0 1	0 1 0	$0 \cdot 2 + 1 = +1$	+X
isolated 1	0 1 0	1 T 0	$1 \cdot 2 + T = +1$	+X
end of 1's	0 1 1	1 0 0	$1 \cdot 2 + 0 = +2$	+2X
start of 1's	1 0 0	T 0 0	$T \cdot 2 + 0 = -2$	-2X
isolated 0	1 0 1	T 1 0	$T \cdot 2 + 1 = -1$	-X
start of 1's	1 1 0	0 T 0	$0 \cdot 2 + T = -1$	-X
all 1's	1 1 1	0 0 0	$0 \cdot 2 + 0 = 0$	+0

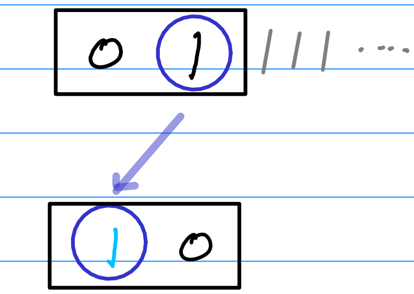
Scale factor $\{0, \pm 1, \pm 2\}$

0-run & 1-run

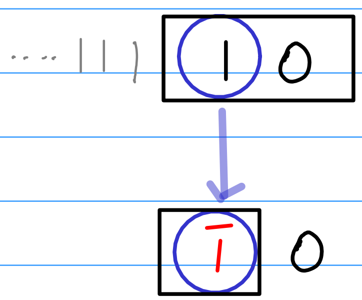
start of a possible 1-run



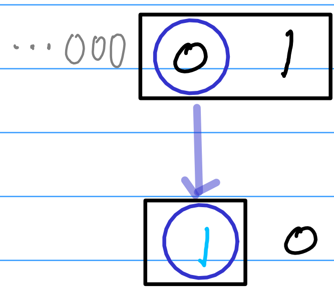
end of a possible 1-run



start of a possible 1's → **start of 1-run**

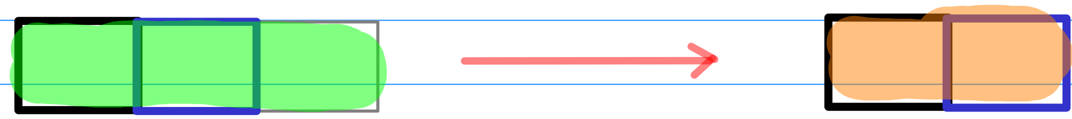


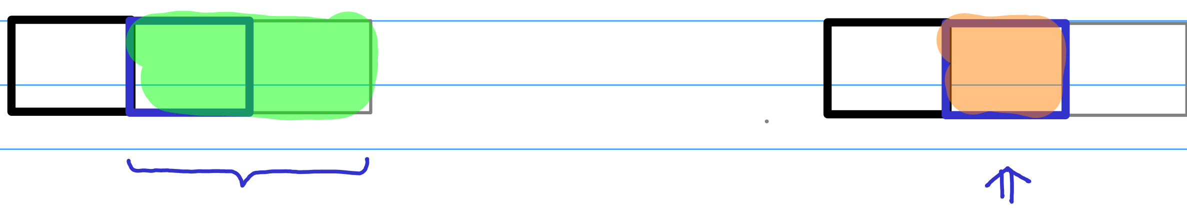
end of a possible 1's → **start of 0-run**



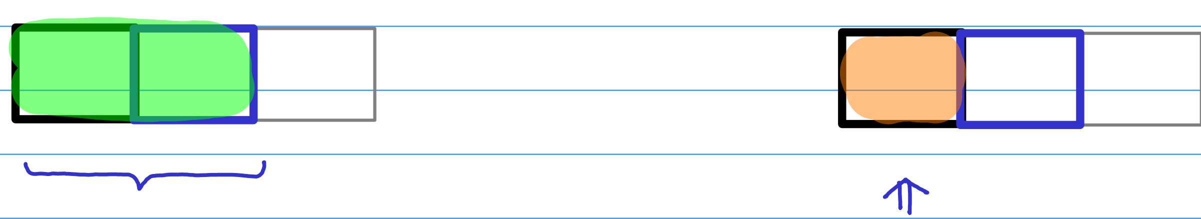
To encode a 1-bit, we need 2-bit info.

Needs 3-bits for encoding a 2-bit



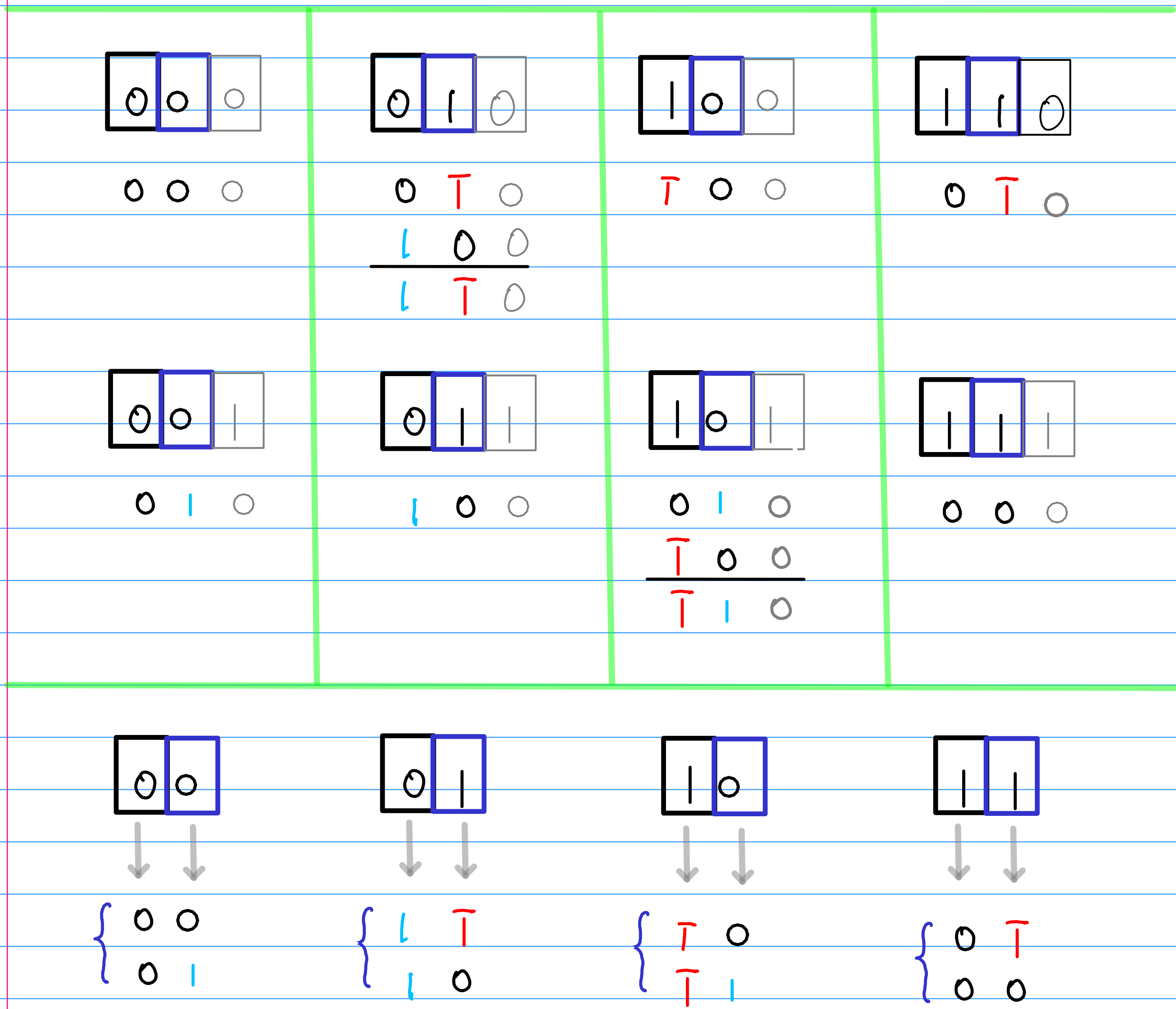


0	0	0-run	0
0	1	0-run starts	1
1	0	1-run starts	1
1	1	1-run	0



0	0	0-run	0
0	1	0-run starts	1
1	0	1-run starts	1
1	1	1-run	0

Encoding 2-bit : 2 choices



0 0 0	→	0 0 0		+0
0 0 1	→	0 1 0	1	+X
0 1 0	→	1 $\bar{1}$ 0	2−1	+X
0 1 1	→	1 0 0	2	+2X
1 0 0	→	$\bar{1}$ 0 0	−2	−2X
1 0 1	→	$\bar{1}$ 1 0	−2+1	−X
1 1 0	→	0 $\bar{1}$ 0	−1	−X
1 1 1	→	0 0 0		+0

Scale Factor for 2-bit encoding

all zero's	middle of 0-run	0 0 0	0 0 0	+0
end of 1's	start of 0-run at (j)	0 0 1	0 1 0	+X
isolated 1	isolated 1 at (j)	0 1 0	1 T 0	+X
end of 1's	start of 0-run at (j+1)	0 1 1	1 0 0	+2X
start of 1's	start of 1-run at (j+1)	1 0 0	T 0 0	-2X
isolated 0	isolated 0 at (j)	1 0 1	T 1 0	-X
start of 1's	start of 1-run at (j)	1 1 0	0 T 0	-X
all 1's	middle of 1-run	1 1 1	0 0 0	+0

X = | X | multiplicand

× Y = | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | multiplier

↓

modified booth encode 2
3

↓

scale factor

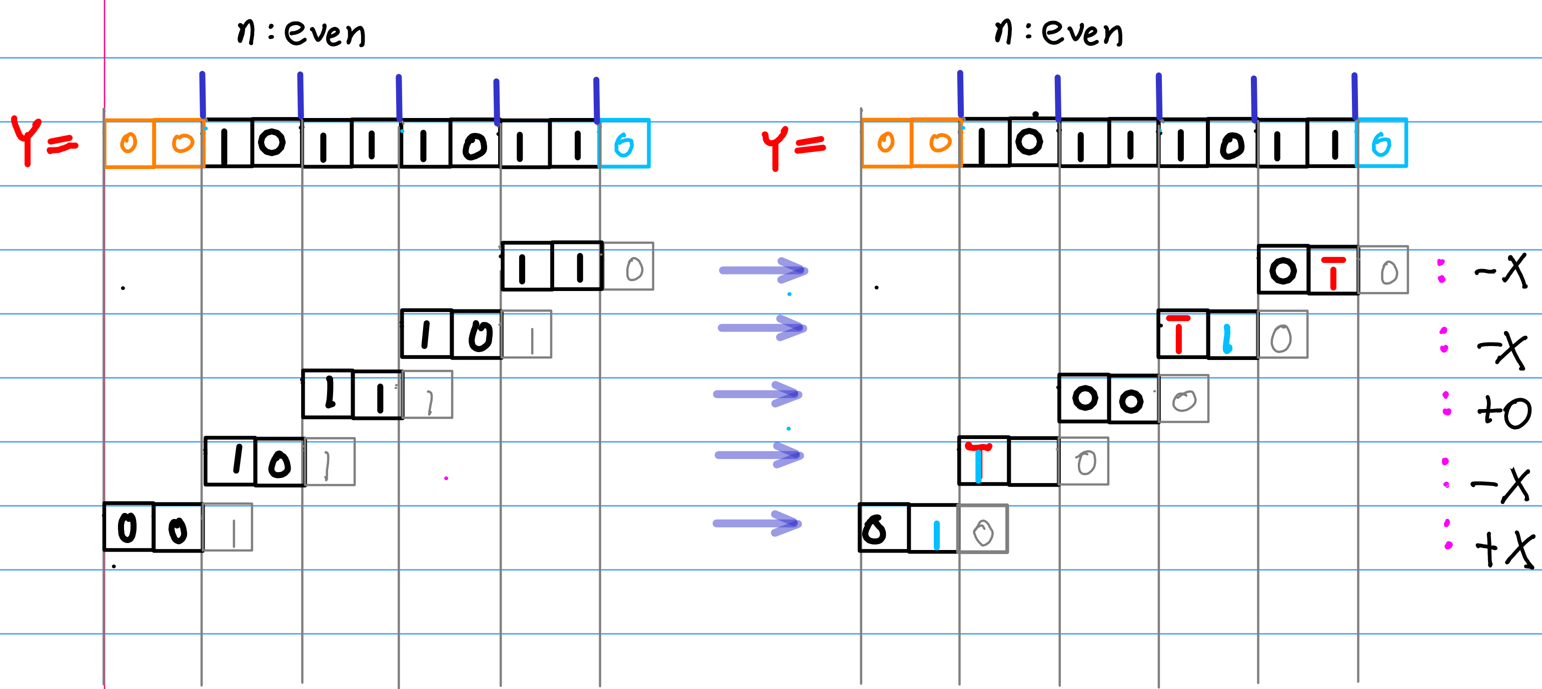
↓

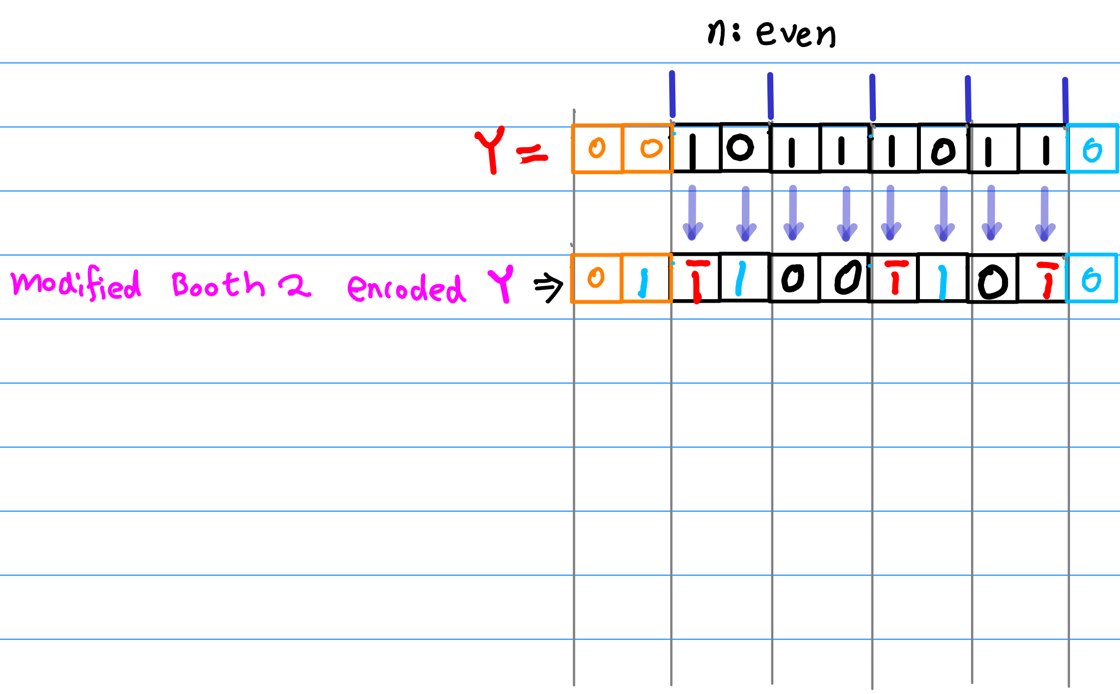
shift multiplicand = X

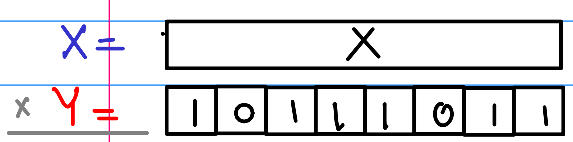
↓

reduced number of partial products for multiplication

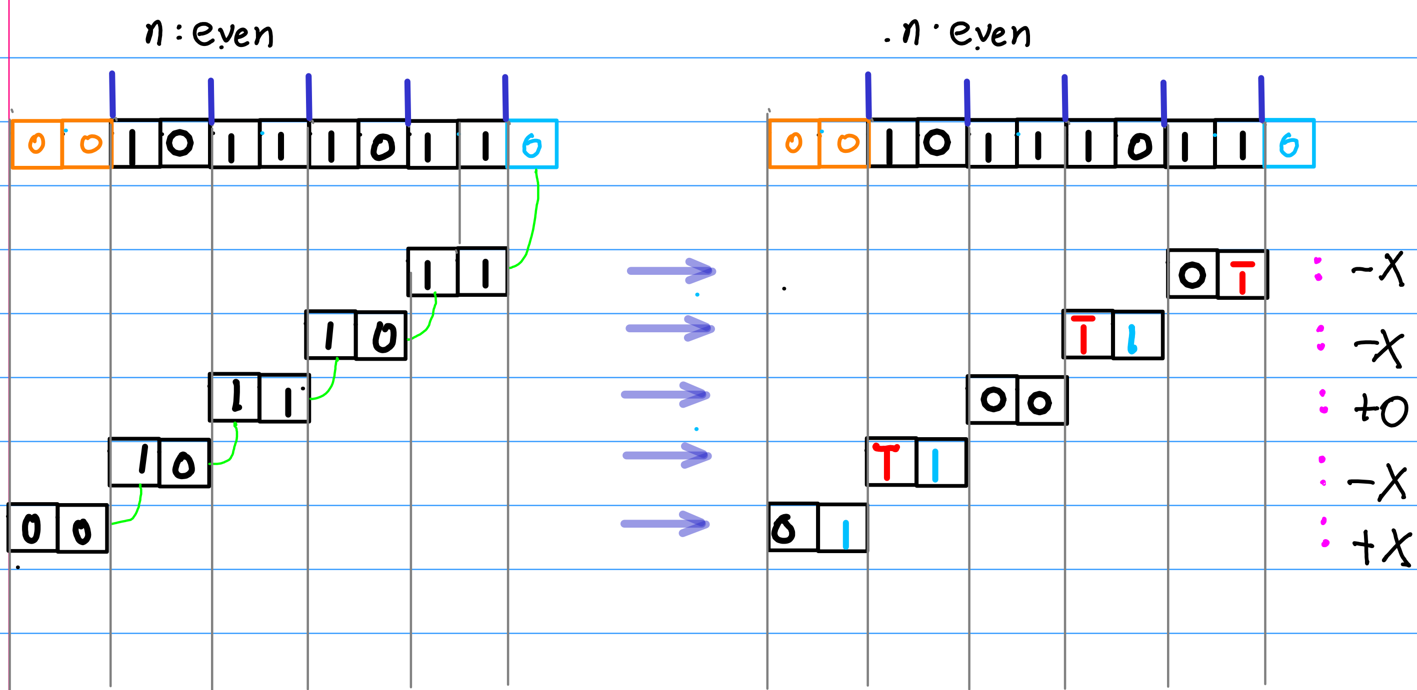
Multiplier Encoding







Basically, $\boxed{\text{2-bit}}$ encoding scheme



context encoding

increasing overlapping bit → no. need

0	0	0	0	middle of 0-run	0	0	0	0
0	0	0	1	middle of 0-run	0	0	0	0
0	0	1	0	start of 0-run at (j)	0	1	0	0
0	0	1	1	start of 0-run at (j)	0	1	0	0
0	1	0	0	isolated 1 at (j)	1	1	0	0
0	1	0	1	isolated 1 at (j)	1	1	0	0
0	1	1	0	start of 0-run at (j+1)	1	0	0	0
0	1	1	1	start of 0-run at (j+1)	1	0	0	0
1	0	0	0	start of 1-run at (j+1)	1	0	0	0
1	0	0	1	start of 1-run at (j+1)	1	0	0	0
1	0	1	0	isolated 0 at (j)	1	1	0	0
1	0	1	1	isolated 0 at (j)	1	1	0	0
1	1	0	0	start of 1-run at (j)	0	1	0	0
1	1	0	1	start of 1-run at (j)	0	1	0	0
1	1	1	0	middle of 1-run	0	0	0	0
1	1	1	1	middle of 1-run	0	0	0	0

Overlapping bit at both ends → no need

0 0 0 0	middle of 0- run	0 0 0 0
0 0 0 1	Start of 0-run at (j)	0 0 1 0
0 0 1 0	isolated 1 at (j)	0 1 T 0
0 0 1 1	start of 0-run at (j+1)	0 1 0 0
0 1 0 0	start of 1-run at (j+1)	0 T 0 0
0 1 0 1	isolated 0 at (j)	0 T 1 0
0 1 1 0	start of 1-run at (j)	0 0 T 0
0 1 1 1	middle of 1- run	0 0 0 0
1 0 0 0	middle of 0- run	0 0 0 0
1 0 0 1	Start of 0-run at (j)	0 0 1 0
1 0 1 0	isolated 1 at (j)	0 1 T 0
1 0 1 1	start of 0-run at (j+1)	0 1 0 0
1 1 0 0	start of 1-run at (j+1)	0 T 0 0
1 1 0 1	isolated 0 at (j)	0 T 1 0
1 1 1 0	start of 1-run at (j)	0 0 T 0
1 1 1 1	middle of 1- run	0 0 0 0

2-bit context encoding

0 0 0	middle of 0- run	0 0 0
1 1 1	middle of 1- run	0 0 0
0 0 1	Start of 0-run at (j)	0 1 0
0 1 1	start of 0-run at (j+1)	1 0 0
1 1 0	start of 1-run at (j)	0 1 0
1 0 0	start of 1-run at (j+1)	1 0 0
1 0 1	isolated 0 at (j)	1 1 0
0 1 0	isolated 1 at (j)	1 1 0

0 0 0	middle of 0- run	0 0 0
0 0 1	Start of 0-run at (j)	0 1 0
0 1 0	isolated 1 at (j)	1 1 0
0 1 1	start of 0-run at (j+1)	1 0 0
1 0 0	start of 1-run at (j+1)	1 0 0
1 0 1	isolated 0 at (j)	1 1 0
1 1 0	start of 1-run at (j)	0 1 0
1 1 1	middle of 1- run	0 0 0

context encoding

5 Types of Partial Products

000 →	000		+0
001 →	010	1	+X
010 →	1T0	2−1	+X
011 →	100	2	+2X
100 →	T00	−2	−2X
101 →	T10	−2+1	−X
110 →	0T0	−1	−X
111 →	000		−0

5 type of scaled multiplicand

+0	0
+X	x_i
+2X ←	x_{i-1}
−X	$\overline{x_i}$
−2X ←	$\overline{x_{i-1}}$

2's complement

i-th bit of the scaled partial products

$1 * X \longrightarrow (i)$

$2 * X \longrightarrow (i-1)$

$(+) \longrightarrow \square$

$(-) \longrightarrow \overline{\square}$ 1's complement

5 types of Scaling X (=multiplier)

$+0$	0
$+X$	x_i
$+2X$ ←	x_{i-1}
$-X$	$\bar{x}_i$
$-2X$ ←	$\bar{x}_{i-1}$

i-th bit of the scaled partial products

$+X$: $x_7\ x_6\ x_5\ x_4\ x_3\ x_2\ x_1\ x_0$ $\quad$ $\boxed{x_i}\ x_{i-1}$

$+2X$: $x_7\ x_6\ x_5\ x_4\ x_3\ x_2\ x_1\ x_0$ (shifted left by one) $\quad$ $x_i\ \boxed{x_{i-1}}$

$\bar{X}+1 \Leftarrow -X$: $\bar{x}_7\ \bar{x}_6\ \bar{x}_5\ \bar{x}_4\ \bar{x}_3\ \bar{x}_2\ \bar{x}_1\ \bar{x}_0$ $+1$ $\quad$ $\boxed{\bar{x}_i}\ x_{i-1}$

$\overline{2X}+1 \Leftarrow -2X$: $\bar{x}_7\ \bar{x}_6\ \bar{x}_5\ \bar{x}_4\ \bar{x}_3\ \bar{x}_2\ \bar{x}_1\ \bar{x}_0$ (shifted left by one) $+1$ $\quad$ $\bar{x}_i\ \boxed{\bar{x}_{i-1}}$

0: $0\ 0\ 0\ 0\ 0\ 0\ 0\ 0$

* the last partial product (sign-bit p.p.)

$y_1\ y_0\ 0 \rightarrow \pm 2X,\ \pm X,\ 0$

$y_3\ y_2\ y_1 \rightarrow \pm 2X,\ \pm X,\ 0$

$y_5\ y_4\ y_3 \rightarrow \pm 2X,\ \pm X,\ 0$

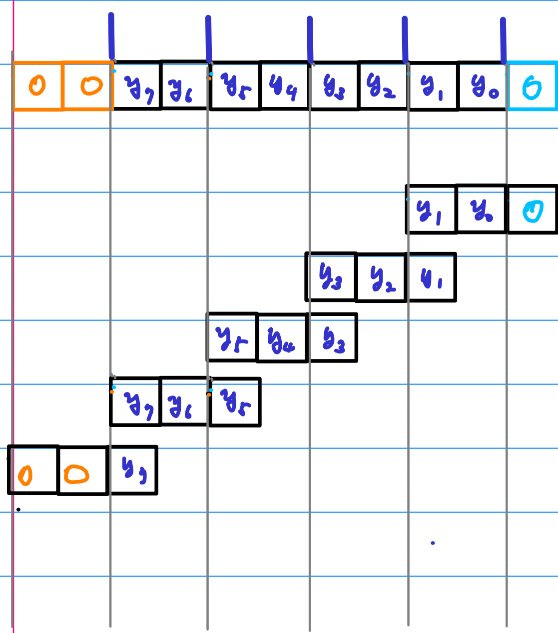
$y_7\ y_6\ y_5 \rightarrow \pm 2X,\ \pm X,\ 0$

$000,\ 001 \rightarrow 0\ 0\ y_7 \rightarrow +X,\ 0$

2 choices

$y_1, y_3, y_5, y_7 = 1 \Rightarrow$ negative scaling

n : even



0 0 0	→	0 0 0	+0
0 0 1	→	0 1 0	+X
0 1 0	→	1 T 0	+X
0 1 1	→	1 0 0	+2X
1 0 0	→	T 0 0	−2X
1 0 1	→	T 1 0	−X
1 1 0	→	0 T 0	−X
1 1 1	→	0 0 0	−0

3-bit partition

$$\boxed{1}\,\boxed{}\,\boxed{} \longrightarrow \begin{cases} -2X & \overline{2X}+1 \\ -X & \overline{X}+1 \\ -0 & \overline{0}+1 \end{cases}$$

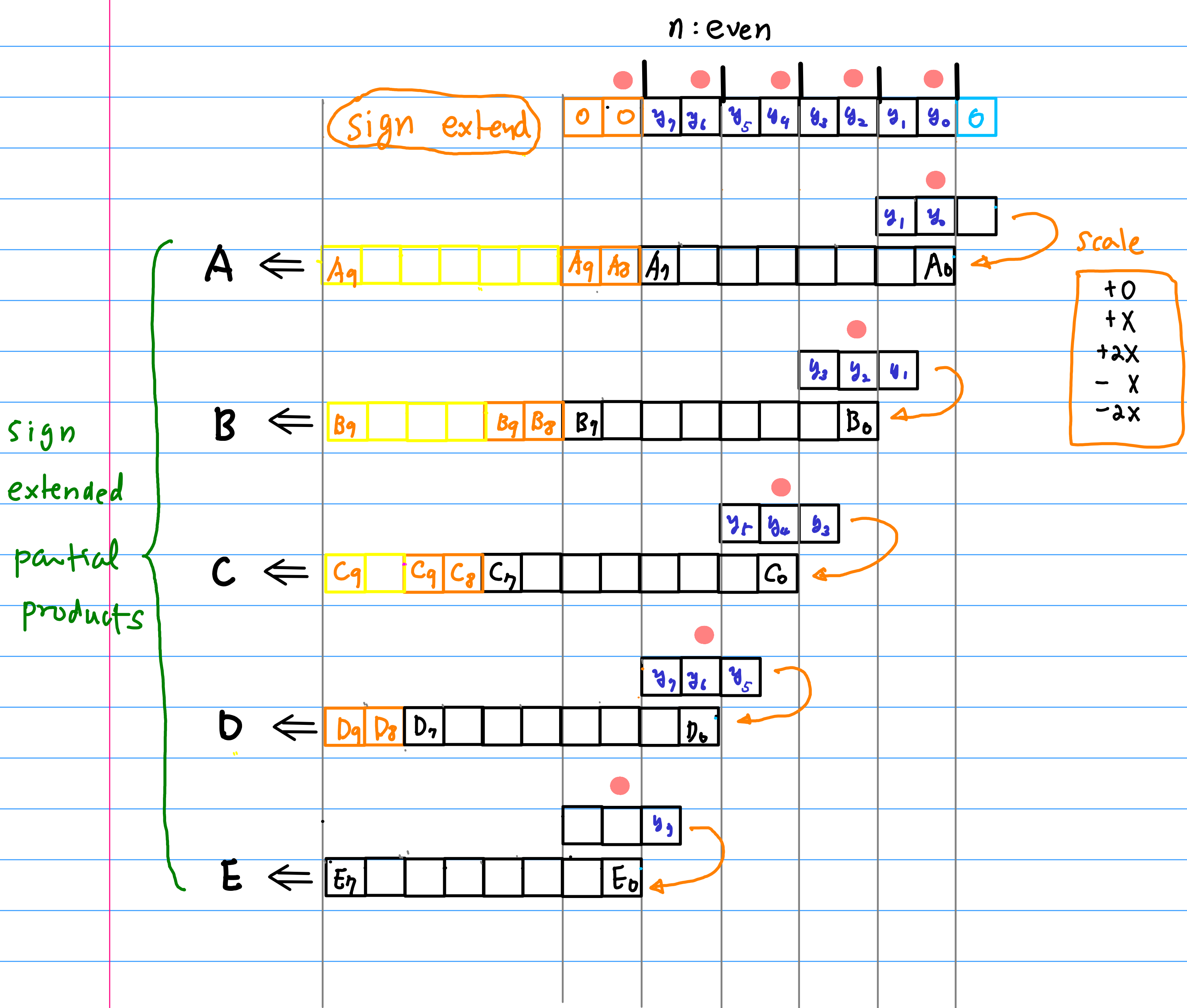
$$\underline{\text{Complement}} \begin{pmatrix} 2X \\ X \\ 0 \end{pmatrix} \underline{+1}$$

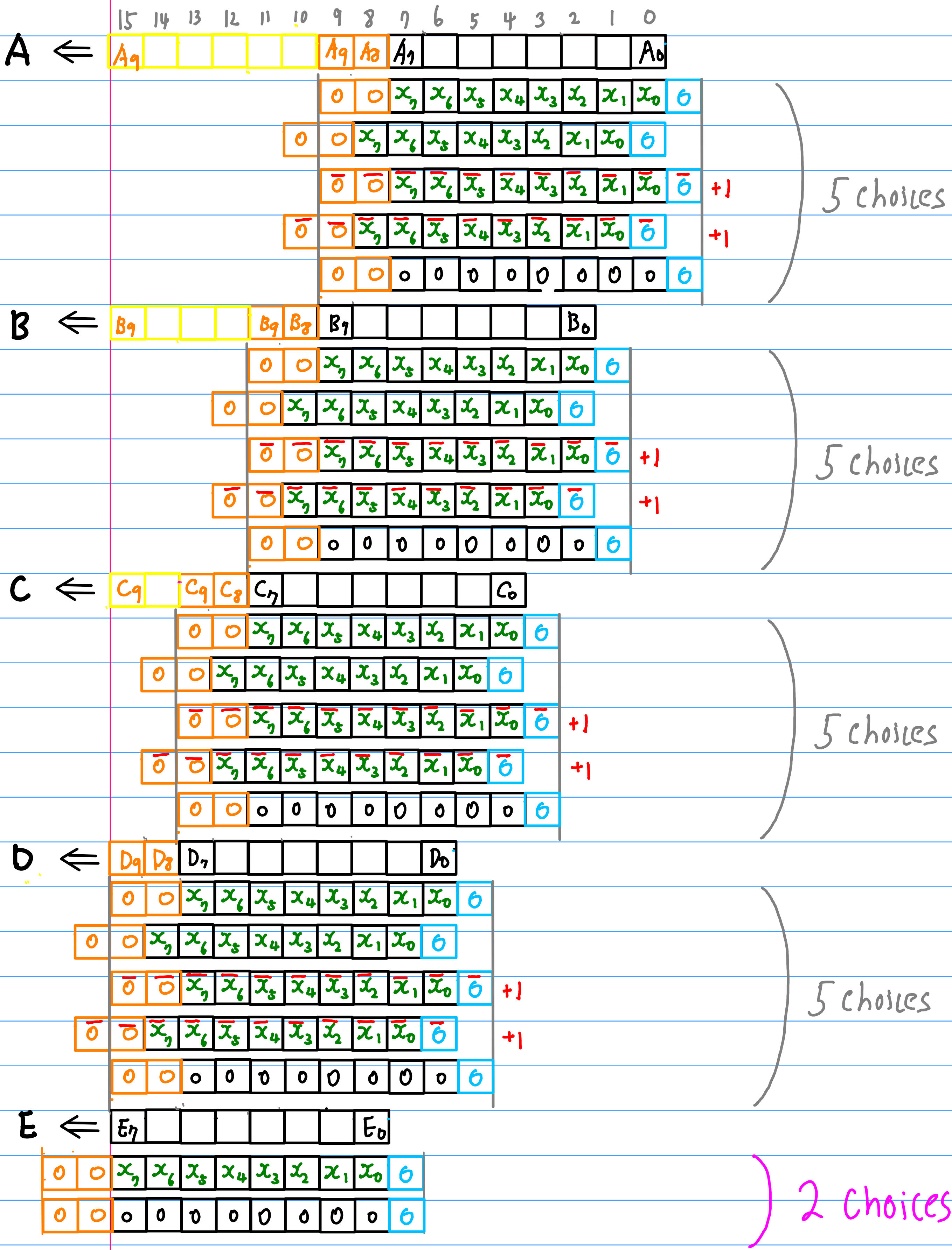
Scaled X's are aligned at y_0, y_2, y_4, y_6, y_8

Sign extended partial product A B C D E

$X = \boxed{x_7}\boxed{x_6}\boxed{x_5}\boxed{x_4}\boxed{x_3}\boxed{x_2}\boxed{x_1}\boxed{x_0}$ multiplicand

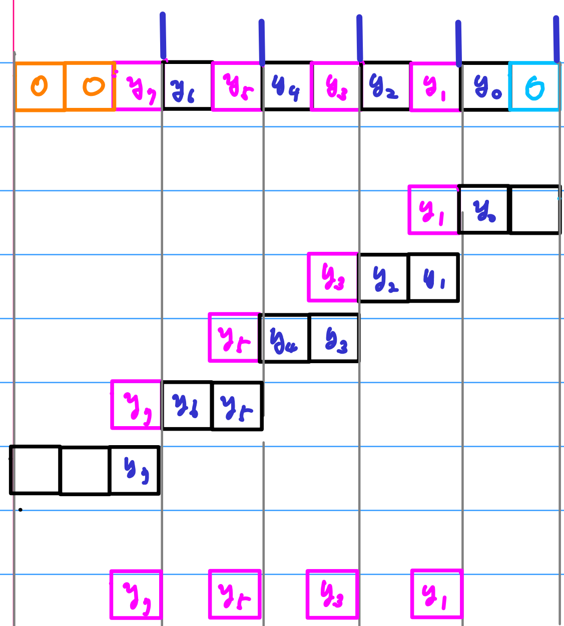
$Y = \boxed{y_7}\boxed{y_6}\boxed{y_5}\boxed{y_4}\boxed{y_3}\boxed{y_2}\boxed{y_1}\boxed{y_0}$ multiplier





15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
A ⇐
B ⇐
C ⇐
D ⇐
E ⇐
+1
5 choices
5 choices
5 choices
5 choices
2 choices

n : even



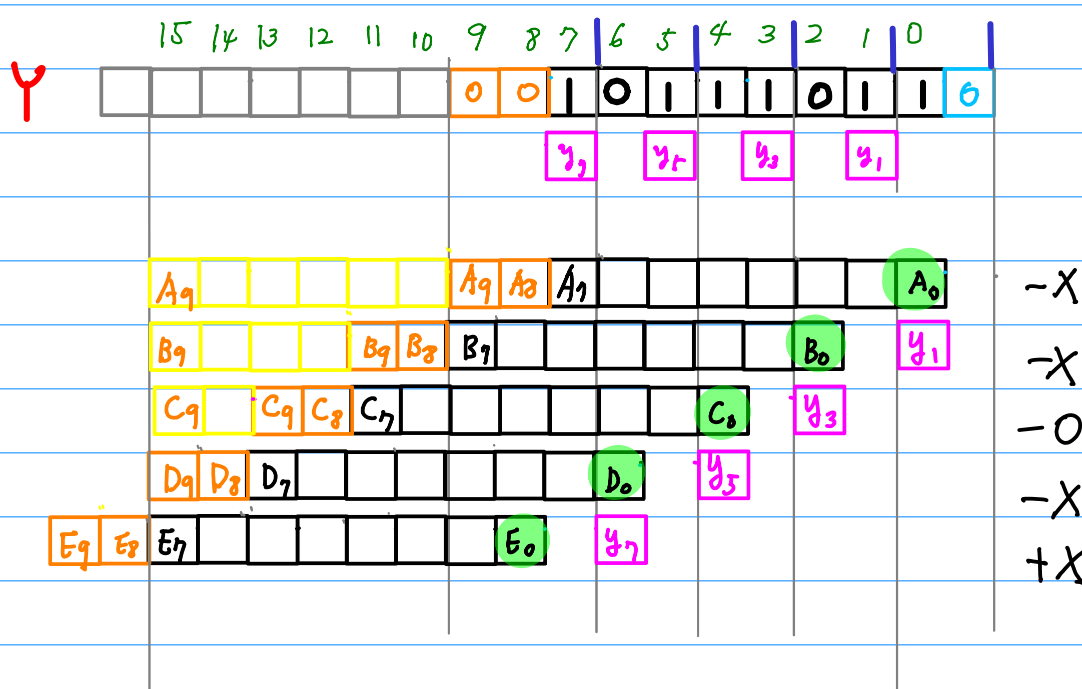
if $y_1 = 1$, complement partial product + y_1

if $y_3 = 1$, complement partial product + y_3

if $y_5 = 1$, complement partial product + y_5

if $y_7 = 1$, complement partial product + y_7

n : even



if $y_1 = 1$, compliment A

if $y_3 = 1$, compliment B

if $y_5 = 1$, compliment C

if $y_7 = 1$, compliment D

E

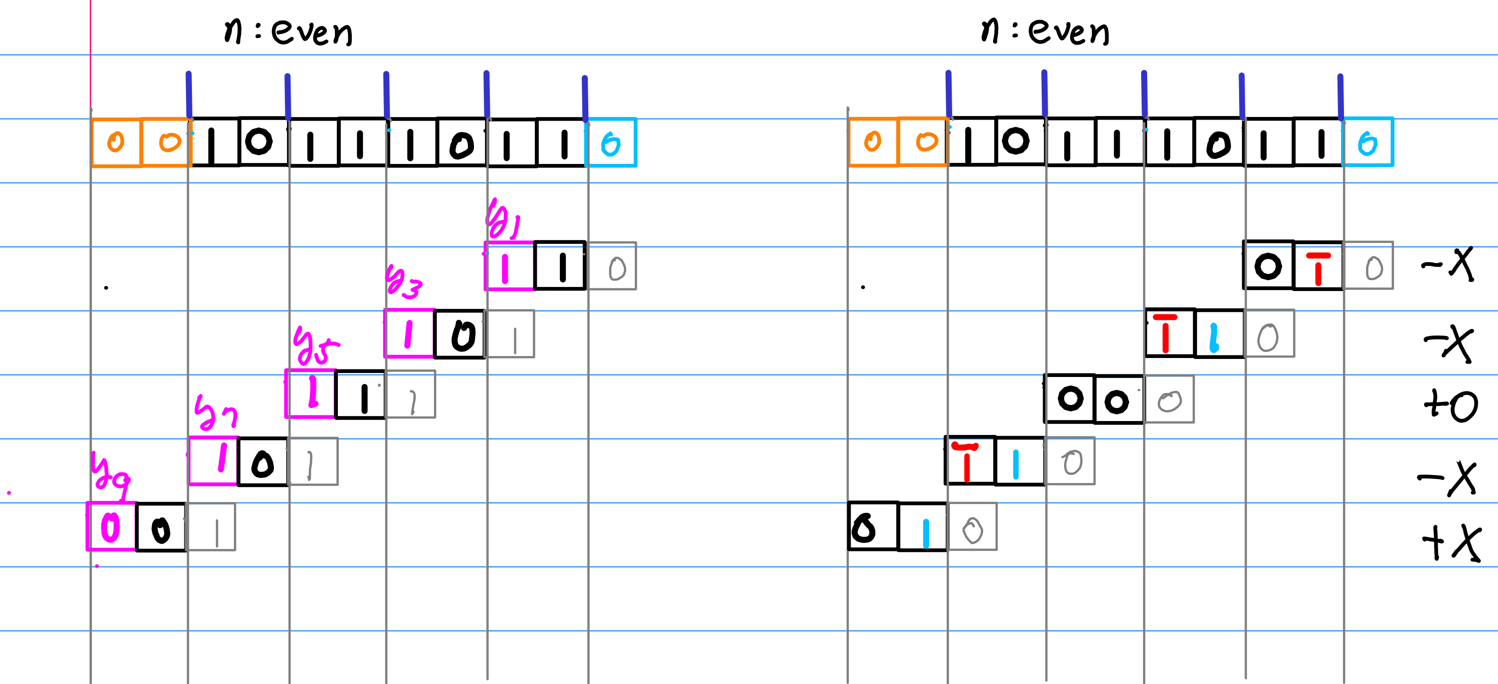
if $y_1 = 1$, add 1 else add 0 ($y_1 = 0$)

if $y_3 = 1$, add 1 else add 0 ($y_3 = 0$)

if $y_5 = 1$, add 1 else add 0 ($y_5 = 0$)

if $y_7 = 1$, add 1 else add 0 ($y_7 = 0$)

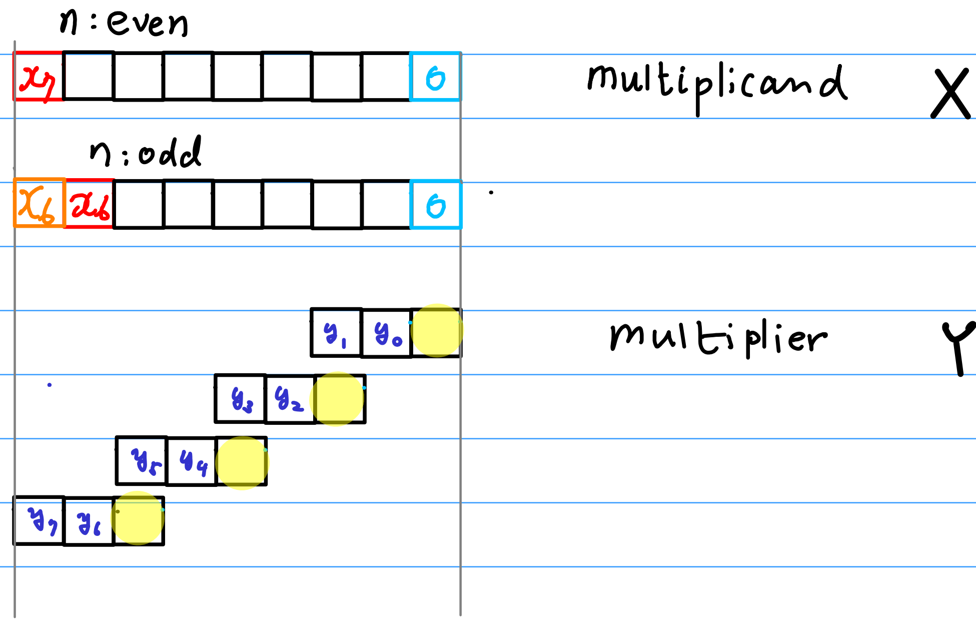
Handling Negative Scales



Modified Booth 2 (signed case)

the worst case $\frac{N}{2}$ partial products

$\frac{8}{2} = 4$



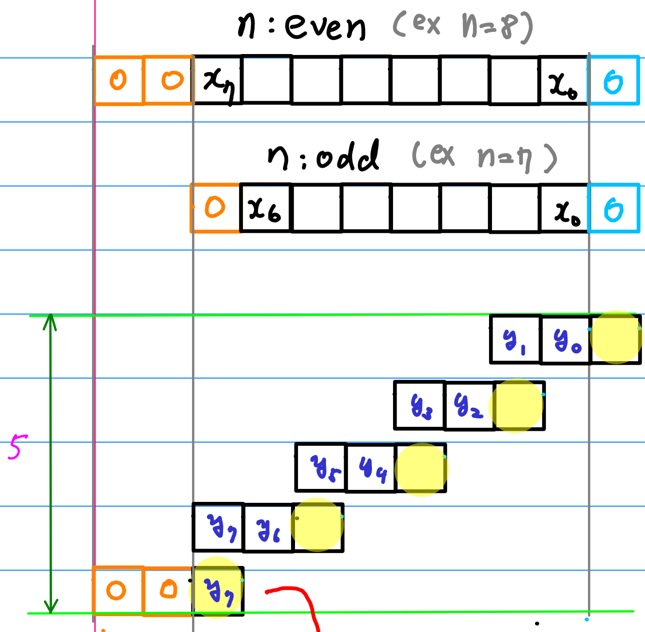
make 4 partial products

x_7, y_7 sign bit

Compared to unsigned multiplication

One less partial products

Booth 2 **Unsigned** Case



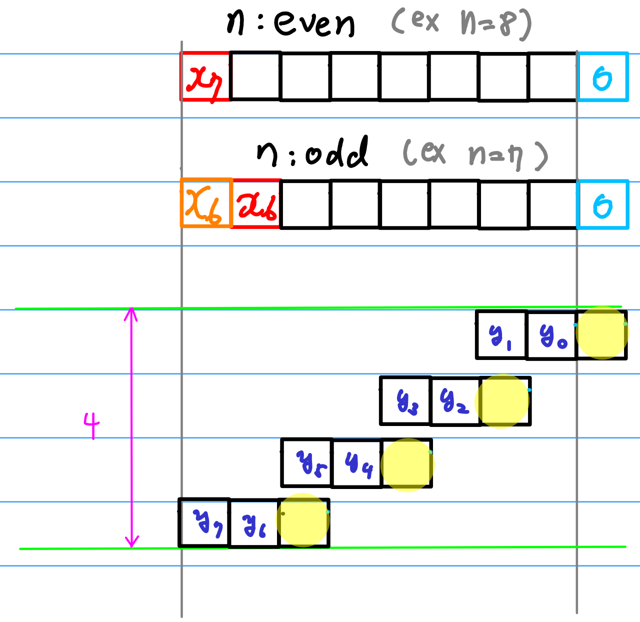
$\frac{8}{2} + 1$

$= 5$

0 0 0 ($y_7 = 0$) +0
0 0 1 ($y_7 = 1$) +X

Necessary to make all positive number (unsigned)

Booth 2 **Signed** case



$\frac{8}{2} = 4$

Booth 2 code :

Signed Digit Number

→ no special treatment for the sign bit y_7

4-Bit unsigned number & Booth 2 Code

0	0 0000	0 0 0 0 0	0
1	0 0001	0 0 0 1 $\bar{1}$	$2-1=1$
2	0 0010	0 0 1 $\bar{1}$ 0	$4-2=2$
3	0 0011	0 0 1 0 $\bar{1}$	$4-1=3$
4	0 0100	0 1 $\bar{1}$ 0 0	$8-4=4$
5	0 0101	0 1 $\bar{1}$ 1 $\bar{1}$	$8-4+2-1=5$
6	0 0110	0 1 0 $\bar{1}$ 0	$8-2=6$
7	0 0111	0 1 0 0 $\bar{1}$	$8-1=7$
8	0 1000	1 $\bar{1}$ 0 0 0	$16-8=8$
9	0 1001	1 $\bar{1}$ 0 1 $\bar{1}$	$16-8+2-1=9$
10	0 1010	1 $\bar{1}$ 1 $\bar{1}$ 0	$16-8+4-2=10$
11	0 1011	1 $\bar{1}$ 1 0 $\bar{1}$	$16-8+4-1=11$
12	0 1100	1 0 $\bar{1}$ 0 0	$16-4=12$
13	0 1101	1 0 $\bar{1}$ 1 $\bar{1}$	$16-4+2-1=13$
14	0 1110	1 0 0 $\bar{1}$ 0	$16-2=14$
15	0 1111	1 0 0 0 $\bar{1}$	$16-1=15$

Booth 2 code :

Signed Digit Number

→ Need special treatment

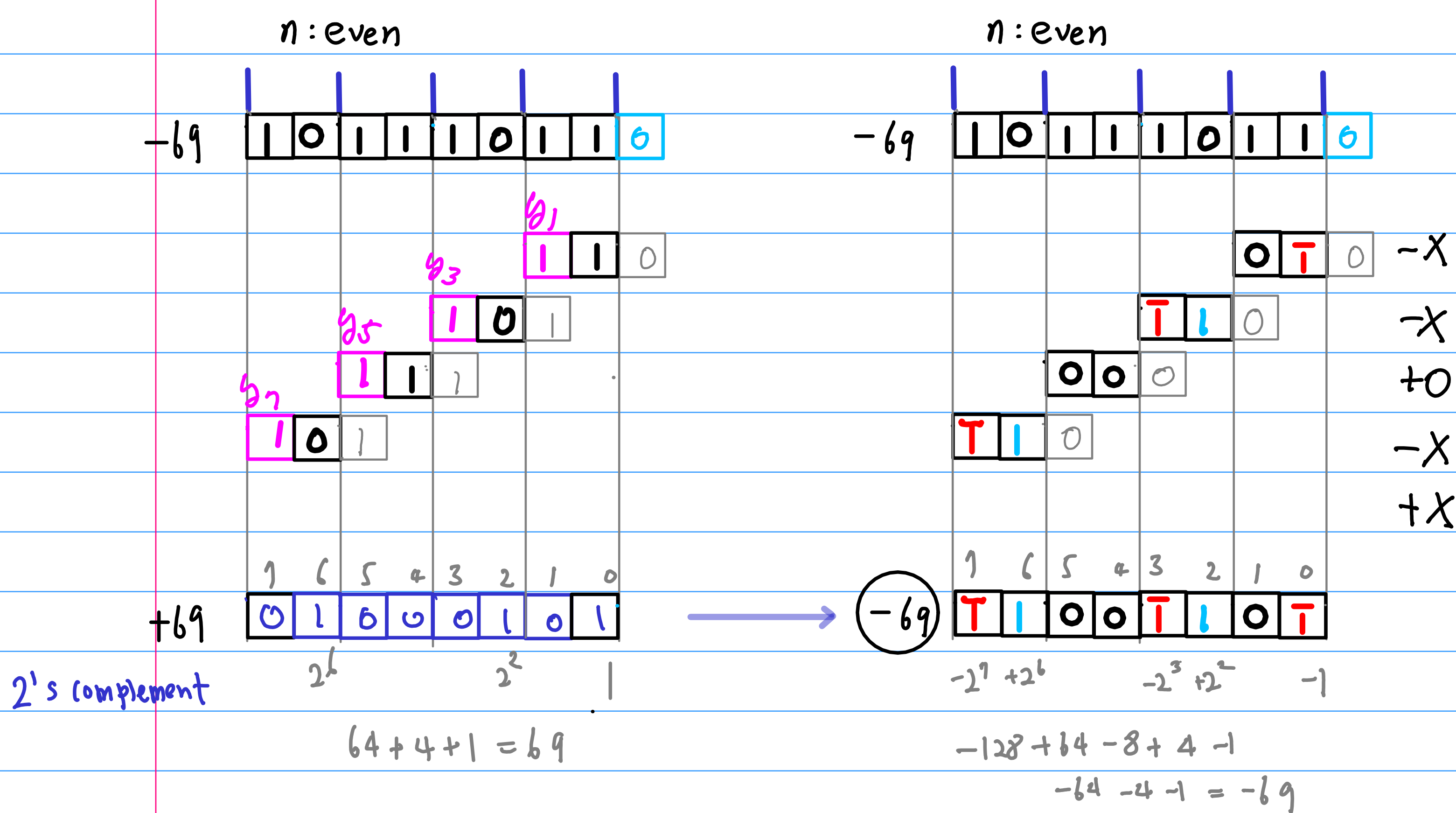
to make all positive number

4-Bit 2's complement number & Booth 2 Code (Signed)

	2's complement	Booth 2 code	
0	0 0 0 0	0 0 0 0	0
1	0 0 0 1	0 0 1 $\bar{1}$	2 − 1 = 1
2	0 0 1 0	0 1 $\bar{1}$ 0	4 − 2 = 2
3	0 0 1 1	0 1 0 $\bar{1}$	4 − 1 = 3
4	0 1 0 0	1 $\bar{1}$ 0 0	8 − 4 = 4
5	0 1 0 1	1 $\bar{1}$ 1 $\bar{1}$	8 − 4 + 2 − 1 = 5
6	0 1 1 0	1 0 $\bar{1}$ 0	8 − 2 = 6
7	0 1 1 1	1 0 0 $\bar{1}$	8 − 1 = 7
−8	1 0 0 0	$\bar{1}$ 0 0 0	−8
−7	1 0 0 1	$\bar{1}$ 0 1 $\bar{1}$	−8 + 2 − 1 = −7
−6	1 0 1 0	$\bar{1}$ 1 $\bar{1}$ 0	−8 + 4 − 2 = −6
−5	1 0 1 1	$\bar{1}$ 1 0 $\bar{1}$	−8 + 4 − 1 = −5
−4	1 1 0 0	0 $\bar{1}$ 0 0	−4
−3	1 1 0 1	0 $\bar{1}$ 1 $\bar{1}$	−4 + 2 − 1 = −3
−2	1 1 1 0	0 0 $\bar{1}$ 0	−2
−1	1 1 1 1	0 0 0 $\bar{1}$	−1

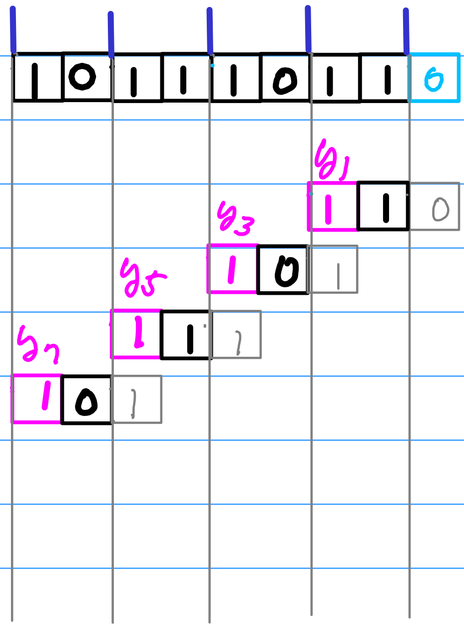
Booth 2 code : Signed Digit Number

→ no special treatment for the sign bit y_n

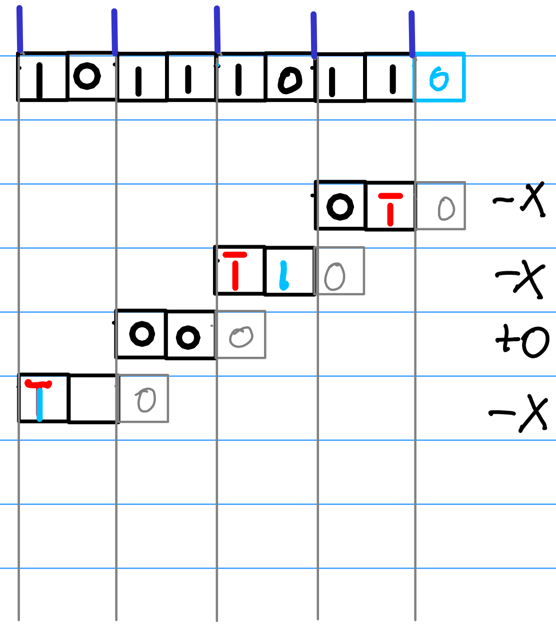


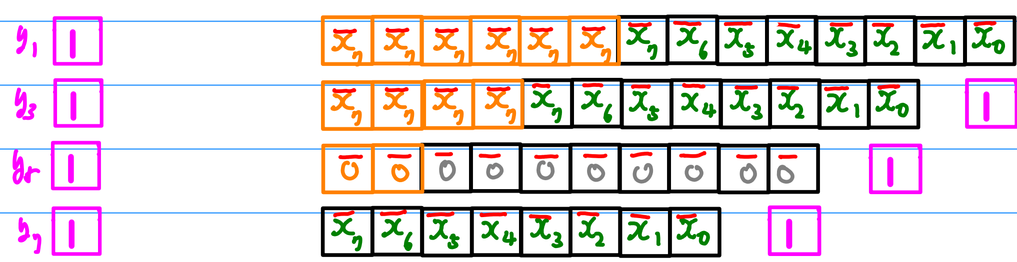
no problem in Booth encoding signed numbers!

n : even



n : even

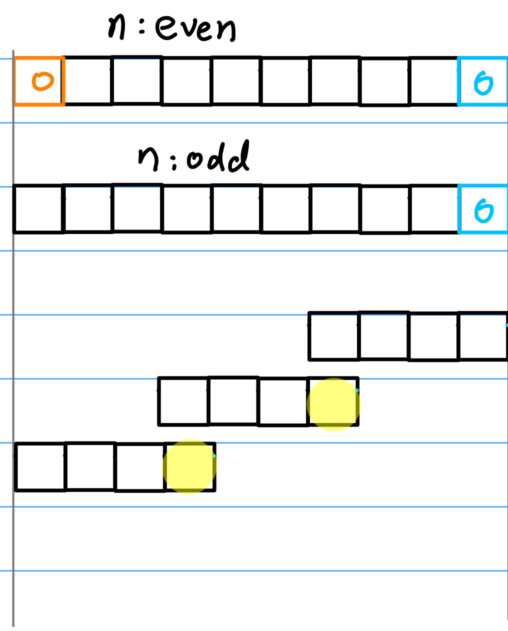




Modified Booth 3 (unsigned)

the worst case $\frac{N}{3}+1$ partial products

$\frac{8}{3}+1=3$



3 partial products

overlapped 4-bit partitions

Scale factor

0 0 0 0	→	0 0 0 0	+0
0 0 0 1	→	0 0 1 0	+X
0 0 1 0	→	0 1 $\bar{1}$ 0	+X
0 0 1 1	→	0 1 0 0	+2X
0 1 0 0	→	1 $\bar{1}$ 0 0	+2X
0 1 0 1	→	1 $\bar{1}$ 1 0	+3X
0 1 1 0	→	1 0 $\bar{1}$ 0	+3X
0 1 1 1	→	1 0 0 0	+4X
1 0 0 0	→	$\bar{1}$ 0 0 0	−4X
1 0 0 1	→	$\bar{1}$ 0 1 0	−3X
1 0 1 0	→	$\bar{1}$ 1 $\bar{1}$ 0	−3X
1 0 1 1	→	$\bar{1}$ 1 0 0	−2X
1 1 0 0	→	0 $\bar{1}$ 0 0	−2X
1 1 0 1	→	0 $\bar{1}$ 1 0	− X
1 1 1 0	→	0 0 $\bar{1}$ 0	− X
1 1 1 1	→	0 0 0 0	− 0