Random Process Background (1C)

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Outline

Open set examples

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
  the *circle* set is its **boundary set**
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
  The *disk* set is an **open set**
- the union of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a set along with a distance defined between any two points
- in a **metric space**,
  an **open set** is a set that, along with every point P, contains all points that are sufficiently near to P
  - all points whose distance to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2-1)

- more generally, an **open set** is
  a member of a given collection of subsets of a given set

> - a given set
> - subsets of a given set
> - a given collection of subsets of a given set

https://en.wikipedia.org/wiki/Open_set

Open set (2-2)

- a **collection** has the following property of containing

> - a **collection** contains
>   - every **union** of its members
>   - every finite **intersection** of its members
>   - the **empty set**
>   - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (3)

- These conditions are very loose,
  and allow enormous flexibility in the choice of open sets.
- For example,
  - every subset can be open (the **discrete topology**)
  - no subset can be open (the **indiscrete topology**)
    except
    - the space itself and
    - the empty set

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A set in which such a collection is given is called a **topological space**, and the collection is called a **topology**.
  - A **set** is a collection of distinct objects.
  - Given a **set** A, we say that a is an **element** of A

    if a is one of the distinct objects in A,
    and we write $a \in A$ to denote this
  - Given two **sets** A and B, we say that A is a **subset** of B

    if every element of A is also an element of B
    write $A \subseteq B$ to denote this.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (5) Open Balls

- An **open ball** $B_r(\boldsymbol{a})$ in $\mathbb{R}^n$ centered at $\boldsymbol{a} = (a_1, \ldots a_n) \in \mathbb{R}^n$ with radius r

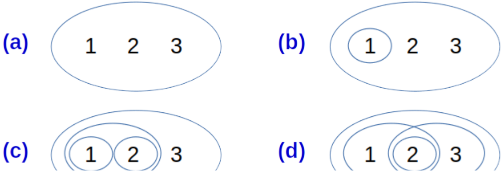
  is the set of all points $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n$ such that the distance between x and $\boldsymbol{a}$ is less than r
- In $\mathbb{R}^2$ an **open ball** is often called an **open disk**

We give these definitions in general, for when one is working in $\mathbb{R}^n$ since they are really not all that different to define in $\mathbb{R}^n$ than in $\mathbb{R}^2$

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

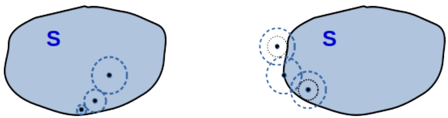
Open set (6) Interior points

- Suppose that $S \subseteq \mathbb{R}^n$
- A **point** $\boldsymbol{p} \in S$ is an **interior point** of S if there exists an open ball $B_r(\boldsymbol{p}) \subseteq S$
- Intuitively, $\boldsymbol{p}$ is an **interior point** of S if we can *squeeze* an entire open ball centered at $\boldsymbol{p}$ within S



Open set (7) Boundary points

- A point $\boldsymbol{p} \in \mathbb{R}^n$ is a **boundary point** of S if all open balls centered at $\boldsymbol{p}$ contain both points in S and points not in S
- The **boundary** of S is the set ∂S that consists of all of the **boundary points** of S.



an interior point　　a boundary point

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Open set (8) Open and Closed Sets

- A set $O \subseteq \mathbb{R}^n$ is **open**
  if every point in O is an interior point.
- A set $C \subseteq \mathbb{R}^n$is **closed**
  if it contains all of its boundary points.

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd(

Open set (9) Bounded and Unbounded

- A set S is **bounded** if there is an open ball $B_M(0)$ such that

$$S \subseteq B.$$

intuitively, this means that we can enclose all of the set S within a large enough ball centered at the origin, $B_M(0)$

- A set that is not **bounded** is called **unbounded**

https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd

Family of sets (1)

- a **collection** F of subsets of a given set S is called
  a **family** of **subsets** of S, or
  a **family** of **sets** over S.
- More generally,
  a **collection** of any sets whatsoever is called
  a **family** of **sets**,
  set family, or
  a **set system**

https://en.wikipedia.org/wiki/Family_of_sets

Family of sets (2)

- The term "**collection**" is used here because,
  - in some contexts,
    a **family** of **sets** may be allowed
    to contain repeated copies of any given member, and
  - in other contexts
    it may form a proper class rather than a set.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (1)

- The set of all subsets of a given set S is called the **power set** of S and is denoted by $\wp(S)$.

  The **power set** $\wp(S)$ of a given set S is a **family** of **sets** over S.

- A subset of S having k elements is called a k-subset of S.

  The k-subset $S^{(k)}$ of a set S form a **family** of **sets**.

https://en.wikipedia.org/wiki/Family_of_sets

Examples of family of sets (2)

- Let $S = \{a, b, c, 1, 2\}$.
  An example of a **family** of **sets** over S
  (in the multiset sense) is given by $F = \{A_1, A_2, A_3, A_4\}$, where
  $A_1 = \{a, b, c\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$, and $A_4 = \{a, b, 1\}$.

https://en.wikipedia.org/wiki/Family_of_sets

Neighbourhood basis (1)

- A **neighbourhood basis** or **local basis** (or **neighbourhood base** or **local base**) for a point x is a filter base of the neighbourhood filter;
- this means that it is a subset $\mathscr{B} \subseteq \mathscr{N}(x)$ such that for all $V \in \mathscr{N}(x)$, there exists some $B \in \mathscr{B}$ such that $B \subseteq V$. That is, for any **neighbourhood** V we can find a **neighbourhood** B in the **neighbourhood basis** that is contained in V.

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

Neighbourhood basis (2)

- Equivalently, $\mathscr{B}$ is a local basis at x if and only if the neighbourhood filter $\mathscr{N}$ can be recovered from $\mathscr{B}$ in the sense that the following equality holds:

$$\mathscr{N}(x) = \{V \subseteq X \; : \; B \subseteq V \text{ for some } B \in \mathscr{B}\}$$

- A family $\mathscr{B} \subseteq \mathscr{N}(x)$ is a neighbourhood basis for x if and only if $\mathscr{B}$ is a cofinal subset of $(\mathscr{N}(x), \supseteq)$ with respect to the partial order $\supseteq$ (importantly, this partial order is the superset relation and not the subset relation).

https://en.wikipedia.org/wiki/Neighbourhood_system#Neighbourhood_basis

A collection of sets around x

- In general, one refers to the family of sets containing 0, used to **approximate** 0, as a neighborhood basis;
- a member of this neighborhood basis is referred to as an **open set**.
- In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers.
- In this case, given a point (x) of that set (X), one may define a **collection** of sets "around" (that is, containing) x, used to **approximate** x.

https://en.wikipedia.org/wiki/Open_set

Smaller sets containing x

- Of course, this collection would have to *satisfy* certain properties (known as axioms) for otherwise we may not have a *well-defined method* to measure distance.
- For example, every point in X should **approximate** x to some degree of accuracy.
- Thus X should be in this family.
- Once we begin to define "smaller" sets containing x, we tend to **approximate** x to a greater degree of accuracy.
- Bearing this in mind, one may define the remaining axioms that the family of sets about x is required to satisfy.

https://en.wikipedia.org/wiki/Open_set

Outline

Open ball (1)

- a **ball** is the solid figure bounded by a **sphere**;
  it is also called a **solid sphere**.
  - a **closed ball**
    includes the *boundary points* that constitute the sphere
  - an **open ball**
    excludes them

https://en.wikipedia.org/wiki/Ball_(mathematics)

Open ball (2)

- A **ball** in n dimensions is called a **hyperball** or **n-ball** and is bounded by a **hypersphere** or $(\boldsymbol{n}-1)$**-sphere**
- One may talk about **balls** in any topological space X, not necessarily induced by a metric.
- An n-dimensional **topological ball** of X is any subset of X which is homeomorphic to an **Euclidean n-ball**.

https://en.wikipedia.org/wiki/Ball_(mathematics)

Neighborhood (1)

- a neighbourhood is one of the basic *concepts* in a topological space.
- It is closely related to the *concepts* of open set and interior.
- Intuitively speaking, a neighbourhood of a point is
  a set of points containing that point
  where one can move some amount in any direction
  away from that point without leaving the set.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Interior

- the interior of a subset S of a topological space X is the union of *all* subsets of S that are open in X.
- A point that is in the interior of S is an interior point of S.
- The interior of S is the complement of the closure of the complement of S.
  the closure of (boundary + exterior)
- In this sense, interior and closure are dual notions.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Exterior

- The exterior of a set S is the complement of the closure of $\underline{S}$;
  the closure of S = boundary + interior
- it consists of the points that are
  in neither the set nor its boundary.
- The interior, boundary, and exterior of a subset together partition the whole space into three blocks
- fewer when one or more of these is empty

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (1)

- If S is a subset of a Euclidean space,
  then x is an interior point of S

  if there exists an open ball centered at x
  which is completely contained in S.

- This definition generalizes to
  any subset S of a metric space X with metric d:

  x is an interior point of S if there exists a real number $r > 0$,
  such that y is in S whenever the distance $d(x, y) < r$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (2)

- This definition generalizes to topological spaces by replacing "open ball" with "open set".
  - if there exists an *open ball* centered at x which is completely contained in S.
  - if x is contained in an *open subset* of X that is completely contained in S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior Point (3)

- If S is a subset of a topological space X
  then x is an interior point of S in X

  if x is contained in an open subset of X
  that is completely contained in S.

- Equivalently, x is an interior point of S
  if S is a neighbourhood of x.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior of a Set (1)

- The interior of a subset S of a topological space X, can be defined in any of the following equivalent ways:
  - the largest open subset of X contained in S.
  - the union of all open sets of X contained in S.
  - the set of all interior points of S.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Interior of a Set (2)

- The interior of a subset S of a topological space X, denoted by $int_X S$ or $int S$ or S°
- If the space X is understood from context then the shorter notation $int S$ is usually preferred to $int_X S$.

https://en.wikipedia.org/wiki/Interior_(topology)#Interior_point

Neighborhood of a point (1-1)

- If X is a topological space and p is a point in X, then a neighbourhood of p is a subset V of X that includes an open set U containing p,

$$p \in U \subseteq V \subseteq X.$$

  - X : a topological space
  - V : a subset of X
  - U : an open set containing p
  - p : a point in X
  - V : a neighbourhood of p

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a point (1-2)

- This is also equivalent to the point $p \in X$ belonging to the topological interior of V in X.
- The neighbourhood V need not be an open subset of X, but when V is open in X then it is called an open neighbourhood.
- Some authors have been known to require neighbourhoods to be open, so it is important to note conventions.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a point (2)

- A set that is a neighbourhood of each of its points is open since it can be expressed as the union of open sets containing each of its points.
- A closed rectangle, as illustrated in the figure, is not a neighbourhood of all its points;
  - points on the edges or corners of the rectangle are not contained in any open set that is contained within the rectangle.
- The collection of all neighbourhoods of a point is called the neighbourhood system at the point.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a set (1-1)

- If S is a subset of a topological space X, then a neighbourhood of S is a set V that includes an open set U containing S,

$$S \subseteq U \subseteq V \subseteq X.$$

- It follows that a set V is a neighbourhood of S if and only if it is a neighbourhood of all the points in S.
- Furthermore, V is a neighbourhood of S if and only if S is a subset of the interior of V.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood of a set (1-2)

- A neighbourhood of S that is also an open subset of X is called an open neighbourhood of S.
- The neighbourhood of a point is just a special case of this definition.

https://en.wikipedia.org/wiki/Neighbourhood_(mathematics)

Neighborhood definition (1)

- the open set axioms are often taken as
  the definition of a topology,
  when they are quite *unintuitive*,
  though extremely useful in the long run.
- the neighbourhood definition,
  while somewhat *cumbersome*,
  has the advantage of being closely related to ideas from
  analysis, and has a *historical basis*

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (2-1)

- A neighbourhood topology on a set X
  assigns to each element $x \in X$
  a non empty set $N(x)$ of subsets of X,
  called neighbourhoods of x
- with the following properties:

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (2-2)

- the properties of a neighbourhood topology:
  - If N is a neighbourhood of x then $x \in X$
  - If M is a neighbourhood of x and $M \subseteq N \subseteq X$, then N is a neighbourhood of x
  - The intersection of two neighbourhoods of x is a neigbourhood of x
  - If N is a neighbourhood of x, then N contains a neighbourhood M of x such that N is a neighbourhood of each point of M.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (3-1)

- Then one says a function $f : X \to Y$ is continuous wrt neighbourhoods on X and Y if for each $x \in X$ and neighbourhood N of $f(x)$ there is a neighbourhood M of x such that $f(M) \subseteq N$.
- The open set definition of continuity is then justified as being equivalent to this definition in terms of neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Neighborhood definition (3-2)

- One also says a set U in X is open
  if U is a neighbourhood of all of its points.
  THEN one can develop the open set axioms
  and show that one can recover the neighbourhoods.

https://math.stackexchange.com/questions/157735/definition-of-neighborhood-and-open-set-in-topology

Outline

Class (1)

- a **class** is a collection of sets
  (or sometimes other mathematical objects)
  that can be unambiguously defined
  by a property that all its members share.
- **Classes** act as a way to have set-like collections
  while differing from sets so as to avoid **Russell's paradox**

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (2)

- A **class** *that is not a set* is called a **proper class**, and
- a **class** *that is a set* is sometimes called a **small class**.
- the followings are **proper classes** in many formal systems
  - the **class** of all sets
  - the **class** of all ordinal numbers
  - the **class** of all cardinal numbers

https://en.wikipedia.org/wiki/Class_(set_theory)

Class (3)

- consider "the set of all sets with property X."
- especially when dealing with categories, since the objects of a concrete category are all sets with certain additional structure.
- However, **if** we're <u>not</u> *careful* about this we can get into serious *trouble* –

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (4)

- let X be the set of all sets which do not contain *themselves*
- Since X is a set, we can ask whether X is an element of *itself*.
- But then we run into a paradox – **if** X contains *itself* as an element, **then** it does not, and vice versa.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (5)

- In order to avoid this paradox,
  we cannot consider the collection of all sets
  to be itself a set.
- This means we have to *throw out*
  the whole "the set of all sets with property X" construction.
  But we wanted that.
- So the way we get around it is to say that
  there's something called a **class**,
  which is like a set but not a set.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class (6)

- Then we can talk about
  "the **class** X of all sets with property Y."
- Since X is not a set,
  it can't be an element of itself, and we're fine.
- Of course, if we need to talk about the collection of all classes,
  we need to create another name that goes another step back,
  and so forth.

https://www.quora.com/In-set-theory-what-is-the-difference-between-a-set-of-objects-and-a-class-of-objects

Class Examples (1)

- The **collection** of all algebraic structures of a given type will usually be a **proper class**.
  (a **class** *that is not a set* is called a **proper class**)
  - the **class** of all groups
  - the **class** of all vector spaces
  - and many others.
- Within set theory, many **collections** of sets turn out to be proper classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Examples (2)

- One way to *prove* that a **class** is **proper** is to place it in bijection with the class of all ordinal numbers.
  - **Cardinal numbers** indicate an amount how many of something we have: one, two, three, four, five.
  - **Ordinal numbers** indicate position in a series: first, second, third, fourth, fifth.

https://en.wikipedia.org/wiki/Class_(set_theory)
https://editarians.com/cardinals-ordinals/

Class Paradoxes (1)

- The **paradoxes** of naive set theory can be explained in terms of the *inconsistent tacit assumption* that "all classes are sets".
- These paradoxes do not arise with classes because there is no notion of classes containing classes.
- Otherwise, one could, for example, define a class of all classes that do not contain themselves, which would lead to a Russell paradox for classes.

https://en.wikipedia.org/wiki/Class_(set_theory)

Class Paradoxes (2)

- With a rigorous foundation, these **paradoxes** instead *suggest proofs* that certain classes are proper (i.e., that they are not sets).
  - **Russell's paradox** *suggests* a *proof* that the class of all sets which do not contain themselves is proper
  - the **Burali-Forti paradox** *suggests* that the class of all ordinal numbers is proper.

https://en.wikipedia.org/wiki/Class_(set_theory)

Russell's Paradox (1)

- According to the unrestricted comprehension principle, for any sufficiently well-defined property,
  there is the set of all
  and only the objects that have that property.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (2)

- Let R be the set of all sets ($R = \{x \mid x \notin x\}$) that are not members of themselves ($R \notin R$).
  - ***if*** R is not a member of itself ($R \notin R$), ***then*** its definition (the set of all sets) entails that it is a member of itself ($R \in R$);
  - yet, ***if*** it is a member of itself ($R \in R$), ***then*** it is not a member of itself ($R \notin R$), since it is the set of all sets that are not members of themselves ($R \notin R$)
- the resulting contradiction is **Russell's paradox**.
- Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (3)

- *Most sets* commonly encountered are not members of themselves.
- For example, consider the set of all squares in a plane.
- This set is not itself a square in the plane, thus it is not a member of itself.
- Let us call a set "**normal**" if it is not a member of itself, and "**abnormal**" if it is a member of itself.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (4)

- Clearly *every* set must be either normal or abnormal.
- The set of squares in the plane is normal.
- In contrast, the complementary set
  that contains everything which is not a square in the plane
  is itself not a square in the plane,
  and so it is one of its own members
  and is therefore abnormal.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Russell's Paradox (5)

- Now we consider the set of all normal sets, R, and try to determine whether R is normal or abnormal.
  - ***If*** R were normal,
    it would be contained
    in the set of all normal sets (itself),
    and therefore be abnormal;
  - on the other hand ***if*** R were abnormal,
    it would not be contained
    in the set of all normal sets (itself),
    and therefore be normal.
- This leads to the conclusion that R is neither normal nor abnormal: **Russell's paradox**.

https://en.wikipedia.org/wiki/Russell%27s_paradox

Outline

Binary Relation (1)

- a **binary relation** associates elements of one set, called the domain, with elements of another set, called the codomain.
- A **binary relation** over sets X and Y is a new set of ordered pairs (x, y) consisting of elements x from X and y from Y.

https://en.wikipedia.org/wiki/Binary_relationelation

Binary Relation (2)

- It is a generalization of a unary function.
- It encodes the common concept of relation:
- an element x is related to an element y,
  if and only if the pair (x, y) belongs
  to the set of ordered pairs that defines the binary relation.
- A binary relation is the most studied special case $n = 2$
  of an n-ary relation over sets $X_1, ..., X_n$,
  which is a subset of the Cartesian product $X_1 \times \cdots \times X_n$.

https://en.wikipedia.org/wiki/Binary_relationelation

Homogeneous Relation

- a **homogeneous relation** (also called endorelation) on a set X is a binary relation between X and itself, i.e.
  it is a subset of the Cartesian product $X \times X$.
- This is commonly phrased as "a **relation** on X"
  or "a (**binary**) **relation** over X".
- An example of a **homogeneous relation** is the relation of kinship, where the relation is between people.

https://en.wikipedia.org/wiki/Homogeneous_relation

Partially Ordered Set (1-1)

- a **partial order** on a set is an arrangement such that, for certain pairs of elements, one precedes the other.
- The word **partial** is used to indicate that not every pair of elements needs to be comparable; that is, there may be pairs for which neither element precedes the other.
- **Partial orders** thus generalize **total orders**, in which every pair is comparable.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (1-2)

- Formally, a **partial order** is a homogeneous binary relation that is reflexive, transitive and antisymmetric.
- A **partially ordered set** (**poset** for short) is a set on which a **partial order** is defined.
- A reflexive, weak, or non-strict **partial order**, commonly referred to as simply as a **partial order**, is a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive.

https://en.wikipedia.org/wiki/Partially_ordered_set

Partially Ordered Set (2)

- a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive.
- That is, for all $a, b, c \in P$, it must satisfy:
  - Reflexivity:
    $a \leq a$, i.e. every element is related to itself.
  - Antisymmetry:
    if $a \leq b$ and $b \leq a$ then $a = b$,
    i.e. no two distinct elements precede each other.
  - Transitivity:
    if $a \leq b$ and $b \leq c$ then $a \leq c$.
- A non-strict **partial order** is also known as an antisymmetric preorder.

https://en.wikipedia.org/wiki/Partially_ordered_set

Filter in Set Theory (1-1)

- A **filter** on a set may be thought of
  as representing a "collection of *large* subsets",
  one intuitive example being the **neighborhood filter**.

- keep *large* grains excluding *small* impurities

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-2)

- When you put a filter in your sink,
  the idea is that you filter out the *big* chunks of food,
  and let the water and the *smaller* chunks go through
  (which can, in principle, be washed through the pipes) .
- You filter out the *larger parts*.
- A filter filters out the *larger* sets.
- It is a way to say "these sets are *'large'*"

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-3)

- a **filter** on a set X is a family $\mathscr{B}$ of subsets such that:

1. $X \in \mathscr{B}$ and $\emptyset \notin \mathscr{B}$
2. if $A \in \mathscr{B}$ and $B \in \mathscr{B}$, then $A \cap B \in \mathscr{B}$
3. If $A, B \subset X, A \in \mathscr{B}$, and $A \subset B$, then $B \in \mathscr{B}$

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-4)

- The set of "everything" is definitely *large*

$\boxed{X \in \mathscr{B}}$

- and "nothing" is definitely not;

$\boxed{\emptyset \notin \mathscr{B}}$

- if something is *larger* than a *large* set,
  then it is also *large*;

$\boxed{\text{If } A, B \subset X, A \in \mathscr{B}, \text{ and } A \subset B, \text{ then } B \in \mathscr{B}}$

- and two *large* sets *intersect* on a *large* set.

$\boxed{\text{If } A \in \mathscr{B} \text{ and } B \in \mathscr{B}, \text{ then } A \cap B \in \mathscr{B}}$

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Filter in Set Theory (1-5)

- you can think about this as
  - being **co-finite**,
  - or being of **measure 1** on the unit **interval**,
  - or having a **dense open subset** (again on the unit interval).
- These are examples of ways
  where a set can be thought of as "almost everything".
  and that is the idea behind a filter.

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Co-finite

- a **cofinite** subset of a set X is
  a subset A whose complement in X is a finite set.
- a subset A contains all but *finitely many* elements of X
- If the complement is not finite, but is countable,
  then one says the set is **cocountable**.
- These arise naturally when generalizing structures
  on finite sets to infinite sets,
  particularly on infinite products,
  as in the product topology or direct sum.
- This use of the prefix "**co**" to describe a property
  possessed by a set's complement
  is *consistent* with its use in other terms such as "comeagre set".

https://en.wikipedia.org/wiki/Cofiniteness

Unit interval

- the **unit interval** is the closed interval [0,1],
  that is, the set of all real numbers
  that are greater than or equal to 0
  and less than or equal to 1.
- It is often denoted **I** (capital letter I).
- In addition to its role in real analysis,
  the **unit interval** is used to study homotopy theory
  in the field of topology.
- the term "**unit interval**" is sometimes applied
  to the other shapes that an interval from 0 to 1 could take:
  (0,1], [0,1), and (0,1).
- However, the notation **I** is most commonly reserved
  for the closed interval [0,1].

https://en.wikipedia.org/wiki/Unit_interval

Dense set

- In topology, a subset A of a topological space X is said to be **dense** in X if *every* point of X either belongs to A or else is arbitrarily "close" to a member of A
  - for instance, the rational numbers are a **dense** subset of the real numbers because every real number either is a rational number or has a rational number arbitrarily close to it (see **Diophantine approximation**).
- Formally, A is **dense** in X if the *smallest* closed subset of X containing A is X itself.
- The **density** of a topological space X is the least cardinality of a **dense subset** of X.

https://en.wikipedia.org/wiki/Dense_set

Proper Subset

- a set A is a subset of a set B
  if all elements of A are also elements of B;
- B is then a superset of A.
- It is possible for A and B to be equal;
- if they are unequal, then A is a proper subset of B.
- The relationship of one set being a subset of another
  is called inclusion (or sometimes containment).
- A is a subset of B may also be expressed
  as B includes (or contains) A or A is included (or contained) in B.
- A k-subset is a subset with k elements.

https://en.wikipedia.org/wiki/Subset

Outline

Proper Filter (1-1)

- Fix a partially ordered set (poset) P.
- Intuitively, a filter F is a subset of P

  whose members are elements large enough
  to satisfy some *criterion*.
- For instance, if $x \in P$, then the set of elements above x is a filter, called the principal filter at x.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (1-2)

- If $x$and y are incomparable elements of P,
  then neither the principal filter at x nor y
  is contained in the other
  - two elements x and y of a set P are said to be **comparable**
    with respect to a binary relation $\leq$
    if at least one of $x \leq y$ or $y \leq x$ is true.
    They are called **incomparable** if they are not **comparable**.
  - Hasse diagram of the natural numbers,
    partially ordered by "$x \leq y$ if x divides y".
    The numbers 4 and 6 are **incomparable**,
    since neither divides the other.

https://en.wikipedia.org/wiki/Filter_(mathematics)
https://en.wikipedia.org/wiki/Comparability

Proper Filter (1-3)

- Similarly, a filter on a set S contains those subsets that are sufficiently large to contain some given *thing*.
- For example, if S is the *real line* and $x \in S$, then the family of sets including x *in their interior* is a filter, called the neighborhood filter at x.
- The *thing* in this case is *slightly larger* than x, but it still does not contain any other specific point of the line.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (2)

- The above considerations motivate
  the upward closure requirement in the definition below:
  "*large enough*" objects can always be made *larger*.
- To understand the other two conditions,
  reverse the roles and instead consider F
  as a "*locating scheme*" to find x.
- In this interpretation, one searches in some space X,
  and expects F to describe those subsets of X that contain the goal.
- The goal must be located somewhere;
  thus the empty set $\varnothing$ can never be in F.
- And if two subsets both contain the goal,
  then should "zoom in" to their common region.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Proper Filter (3)

- An ultrafilter describes a "perfect locating scheme" where each scheme component gives new information (either "look here" or "look elsewhere").
- Compactness is the property that "every search is fruitful," or, to put it another way, "every locating scheme ends in a search result."
- A common use for a filter is to define properties that are satisfied by "generic" elements of some topological space.
- This application generalizes the "locating scheme" to find points that might be hard to write down explicitly.

https://en.wikipedia.org/wiki/Filter_(mathematics)

Neighborhood Filter (1-1)

- Let X be a set;
- the elements of X are usually called points
- We allow X to be empty.

- Let $\mathcal{N}$ be a function
  assigning to each x (point) in X
  a non-empty collection $\mathcal{N}(x)$ of subsets of X.

- The elements of $\mathcal{N}(x)$ will be called
  neighbourhoods of x with respect to $\mathcal{N}$
  (or, simply, neighbourhoods of x).

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-2)

- Let X be a set;
- $\mathscr{N}$: a function assigning to each point x in X
- $\mathscr{N}(x)$: a non-empty collection of subsets of X.

- The elements of $\mathscr{N}(x)$
  - subsets of X
  - neighbourhoods of x with respect to $\mathscr{N}$

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-3)

- The function $\mathscr{N}$ is called
  a neighbourhood topology
  if *some axioms* are satisfied;
- then X with $\mathscr{N}$ is called
  a topological space – $(X, \mathscr{N})$

https://en.wikipedia.org/wiki/Topological_space

Neighborhood Filter (1-4)

- If $(X, \mathscr{T})$ is a topological space and $p \in X$,
  a neighbourhood of p is a subset V of X,
  in which $p \in U \subseteq V$, and U is open.
- We say that V is a $\mathscr{T}-$ neighbourhood of $x \in X$
  or that V is a neighborhood of x
- The set of all neighbourhoods of $x \in X$, denoted $\mathscr{N}_{\mathscr{X}}$
  is called the neighbourhood filter of x

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (1-4)

- An example of Neighborhood Filters on a Topological space.
- Let $X = \{a, b, c\}$ and let $\mathscr{T} = \{\varnothing, \{a\}, \{b\}, \{b, c\}, \{a, b\}, X\}$
- Let
  $\mathscr{N}_a = \{\{a\}, \{a, b\}, \{a, c\}, X\}$
  $\mathscr{N}_b = \{\{b\}, \{a, b\}, \{b, c\}, X\}$
  $\mathscr{N}_c = \{\{b, c\}, X\}$.
- In this example a, c is a neighborhood of a but not of c.
- Thus a set does not have to be a neighborhood of all of its points.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (2)

- One can specify a topology in more than four different ways.
  1. The standard definition specifies the open sets, what we usually call a "topology."
  2. to specify the close sets - this is of course only a *trivial difference*.
  3. to specify a closure operation on subsets of your space
  4. to specify a neighborhood filter for every point satisfying the natural axiom that every neighborhood of x is a neighborhood of every point of one of its subsets

  - So in this sense neighborhood filters tell you everything they possibly could about a topological space.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (3-1)

- Probably the best way to think about the neighborhood filter of x: is that it contains all information regarding convergence to x.
- In the first topological spaces one encounters, convergence is usually of sequences.
- But this isn't enough to describe the topology in arbitrary spaces, for instance the infinite-dimensional spaces of functional analysis.
- It becomes important to speak of convergence of nets, or of filters.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (3-2)

- A filter on X is just a nontrivial subset of the powerset of X closed under finite intersection and superset, and a filter converges to a point x if and only if it contains the neighborhood filter of x.
- In contrast to the case with sequences, this is enough to specify a topology: in fact it's enough to describe how ultrafilters, that is, maximal filters, converge.
- So in this sense the neighborhood filter encapsulates the viewpoint that topology generalizes the study of convergent sequences.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (4-1)

- in a sense the neighborhood filter describes
  the smallest neighborhood of a point
  - except that there is no smallest neighborhood!
- That's true, at least, in many of the most interesting spaces,
  and is the main reason to worry about a whole filter of neighborhoods
  - if there were a smallest neighborhood then in any hypothesis
  requiring something to hold on a sufficiently small neighborhood of x
  we could just pick the smallest neighborhood.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Neighborhood Filter (4-2)

- But the smallest neighborhood of a point must be contained in the intersection of all its neighborhoods, and in, say, a Hausdorff space the intersection of all neighborhoods of x is x, which is not a neighborhood of x when x is not isolated.
- So the filter functions as a virtual smallest neighborhood of x: it doesn't converge to a neighborhood of x, so we can't think about its limit, but functionally we do just that.

https://math.stackexchange.com/questions/799732/neighborhood-vs-neighborhood-filter

Ultrafilter (1)

- an **ultrafilter** on a given partially ordered set (or "poset") P is a certain subset of P, namely a maximal filter on P; that is, a proper filter on P that cannot be enlarged to a bigger proper filter on P.
- If X is an arbitrary set, its power set $\mathscr{P}(X)$, ordered by set inclusion, is always a Boolean algebra and hence a poset, and ultrafilters on $\mathscr{P}(X)$ are usually called ultrafilter on the set X.

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (2)

- In order theory, an **ultrafilter** is
  a subset of a partially ordered set
  that is maximal among all proper filters.
- This implies that any filter that properly contains an ultrafilter has to be equal to the whole poset.

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (3)

- An ultrafilter on a set X may be considered as a finitely additive measure on X.
- In this view, every subset of X is either considered "*almost everything*" (has measure 1) or "*almost nothing*" (has measure 0), depending on whether it belongs to the given ultrafilter or not

https://en.wikipedia.org/wiki/Ultrafilter

Ultrafilter (4)

- Formally, if P is a set, partially ordered by $\leq$ then
- a subset $F \subseteq P$ is called a filter on P
  if F is nonempty, for every $x, y \in F$,
  there exists some element $z \in F$
  such that $z \leq x$ and $z \leq y$,
  and for every $x \in F$ and $y \in P$,
  $x \leq y$ implies that y is in F too;
- a proper subset U of P is called
  an ultrafilter on P if U is a filter on P,
  and there is no proper filter F on P
  that properly extends U
  (that is, such that U is a proper subset of F).

https://en.wikipedia.org/wiki/Ultrafilter

Outline

Filter Examples (1)

- Let $X = 1, 2, 3$
  Choose some element from X say $F = 1, 1, 2, 1, 3, 1, 2, 3$
- Then every intersection of an element of F
  with another element in F is in F again.
  Examples: $1 \bigcap 1, 2, 3 = 1 \quad 1, 2 \bigcap 1, 2, 3 = 1, 2$
  $1, 3 \bigcap 1, 2, 3 = 1, 3 \quad 1, 2, 3 \bigcap 1, 2, 3 = 1, 2, 3$
- Also the original $X = 1, 2, 3$ is also in F.
  Here $F = 1, 1, 2, 1, 3, 1, 2, 3$ is called the filter on $X = 1, 2, 3$

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (2)

- .Suppose we have the collection $G = 1, 1, 2, 1, 3, 2, 3, 1, 2, 3$
- Then we have $1, 3 \bigcap 2, 3 = 3$ but 3 isn't in G.
  So this G is not called a filter.
- Now with $F = 1, 1, 2, 1, 3, 1, 2, 3$
  can we put as any other element in it
  so that after placing the extra element it is still a filter?
  Probably not in this case.
  So on $X = 1, 2, 3$, $\quad F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (3)

- If we have started say with $H = 1, 1, 2, 1, 2, 3$
  this is still a filter on $X = 1, 2, 3$
  but we can still add $1, 3$
  and it will still be classified as filter.
- So on $X = 1, 2, 3$
  $F = 1, 1, 2, 1, 3, 1, 2, 3$ is an Ultrafilter
  but $H = 1, 1, 2, 1, 2, 3$ is a filter but not an Ultrafilter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (4)

- Now suppose we have $X = 1,2,3,4$
  Let $F = 1,4,1,2,4,1,3,4,1,2,3,4$
- Every in intersection of element of F is in F again.
  We have as examples $1,4 \bigcap 1,4 = 1,4 \quad 1,4 \bigcap 1,2,4 = 1,4$
  $1,4 \bigcap 1,3,4 = 1,4 \quad 1,2,4 \bigcap 1,2,4 = 1,2,4 \quad 1,2,4 \bigcap 1,3,4 = 1,4$
  $1,3,4 \bigcap 1,3,4 = 1,3,4 \quad 1,2,3,4 \bigcap 1,2,3,4 = 1,2,3,4$
- Also $X = 1,2,3,4$ is also in $F = 1,4,1,2,4,1,3,4,1,2,3,4$
  and the null element $\varnothing =$ is not in F.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (5)

- We call F a filter but not an Ultrafilter on $X = 1, 2, 3, 4$
- We can still add element in it and it will still be a filter
  for instance by adding the element 1 from $X = 1, 2, 3, 4$
  we can have the filter $F = 1, 1, 4, 1, 2, 4, 1, 3, 4, 1, 2, 3, 4$
- This is an Ultrafilter on $X = 1, 2, 3, 4$
  as we cannot add any further element from $X = 1, 2, 3, 4$
  that satisfies closures on intersection.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (6)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we cannot call this as Ultrafilter.
- This is not a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

https://math.stackexchange.com/questions/2816362/meaning-behind-filter-in-set-theory

Filter Examples (7)

- There is another collection of sets taken from $X = 1, 2, 3, 4$
- which is the powerset
  $P =, 1, 2, 3, 4, 1, 2, 1, 3, 1, 4, 2, 3, 2, 4, 3, 4, 1, 2, 3, 1, 2, 4, 1, 3, 4, 2, 3, 4, 1, 2, 3, 4$
- This contain the null element $\varnothing =$ so we cannot call this as Ultrafilter.
- This is not a proper filter according to the article in Wikipedia.
- In the powerset every intersection of element is again in the powerset again but it contains t
- he null element $\varnothing =$ and isn't classified as proper filter.

https://en.wikipedia.org/wiki/Filter_(set_theory)#filter_base

Outline

Topology

- **topology**
  from the Greek words
  τόπος, 'place, location',
  and λόγος, 'study'

https://en.wikipedia.org/wiki/Topology

Topology (2)

- **topology** is concerned with
  the *properties* of a geometric object that are *preserved*
  - under continuous deformations such as
    - stretching
    - twisting
    - crumpling
    - bending
  - that is, *without*
    - closing holes
    - opening holes
    - tearing
    - gluing
    - passing through itself

https://en.wikipedia.org/wiki/Topology

Topological space (1)

- a **topological space** is, roughly speaking,

  a geometrical space
  in which closeness is defined

  but cannot necessarily be measured
  by a numeric distance.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
  - a set whose elements are called points,
  - along with an additional structure called a topology,
- which can be defined as
  - a set of neighbourhoods for each point
  - that satisfy some axioms
    formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several *equivalent* definitions of a **topology**,

  the most commonly used of which is
  the definition through open sets,

  which is easier than the others to manipulate.

  https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is
  the most general type of a mathematical space
  that allows for the definition of
  - limits
  - continuity
  - connectedness
- Although very general,
  the concept of **topological spaces** is fundamental,
  and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right
  is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Topological_space

Topological space (5)

- Common types of **topological spaces** include
  - Euclidean spaces : a set of points satisfying certain relationships, expressible in terms of distance and angles.
  - metric spaces : a set together with a notion of distance between points. The distance is measured by a function called a metric or distance function.
  - manifolds : a topological space that *locally* resembles Euclidean space near each point. More precisely, an n-manifold is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of n-dimensional Euclidean space.

https://en.wikipedia.org/wiki/Topological_space

Discrete Topology

- a **discrete space** is a topological space,

  in which the points form a discontinuous sequence,
  meaning they are isolated from each other in a certain sense.
- The **discrete topology** is
  the finest topology that can be given on a set.
  - every subset is open
  - every singleton subset is an open set

https://en.wikipedia.org/wiki/Discrete_space

Singletone

- a **singleton**, also known as a **unit set** or **one-point set**, is a set with exactly one element.
- for example, the set $\{0\}$ is a **singleton** whose single element is 0

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (1)

- a **topological space** with the **trivial topology** is one where the only open sets are the empty set and the entire space.
- Such spaces are commonly called **indiscrete**, **anti-discrete**, **concrete** or **codiscrete**.

  - every subset can be open (the **discrete topology**), or
  - no subset can be open (the **indiscrete topology**) except the space itself and the empty set .

https://en.wikipedia.org/wiki/Discrete_space

Indiscrete Space (2)

- Intuitively, this has the consequence that all points of the space are "lumped together" and cannot be distinguished by topological means (not topologically distinguishable points)
- Every **indiscrete space** is a **pseudometric space** in which the distance between any two points is zero.

https://en.wikipedia.org/wiki/Discrete_space

T_0 Space

- a topological space X is a T_0 **space** or
  if for every pair of distinct points of X,
  at least one of them has a neighborhood
  not containing the other.
- In a T_0 **space**, all points are topologically distinguishable.
- This condition, called the T_0 **condition**,
  is the weakest of the separation axioms.
- Nearly all topological spaces *normally*
  studied in mathematics are T_0 **space**.

https://en.wikipedia.org/wiki/Kolmogorov_space

Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.
- two points in a topological space,
  there exists an open set
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Topologically distinguishable points

- Intuitively, an open set provides a *method* to *distinguish* two points.
- two points in a topological space,
  there exists an open set
  - containing one point but
  - not containing the other (distinct) point
  - the two points are **topologically distinguishable**.

https://en.wikipedia.org/wiki/Open_set

Metric spaces

- In this manner, one may speak of whether two points, or more generally two subsets, of a topological space are "**near**" without concretely defining a distance.
- Therefore, **topological spaces** may be seen as a generalization of **spaces** equipped with a notion of distance, which are called **metric spaces**.

https://en.wikipedia.org/wiki/Open_set

Outline

Why called a discrete topology? (1)

- the **discrete topology** is the finest topology
  - it cannot be subdivided further.
- if you think of the elements of the set
  as indivisible "discrete" atoms,
  each one appears as a singleton set.
- can effectively "see" the individual points
  in the topology itself.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (2)

- the **indiscrete topology** consists only of X itself and $\varnothing$.
- This topology obscures everything about *how many points* were in the original set.
- It fully agglomerates the points of the set together.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (3)

- helpful to think of topologies
  as obscuring or blurring together
  the *underlying points* of the set.
- topologies are all about nearness relations:
  points in an open set are in the vicinity of one another.
- topologically indistinguishable points
  points that never appear alone in an open set,
  - they are so close as to be identical,
    from the perspective of the topology,

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (4)

- the discrete topology
  - has no indistinguishable points.
  - obscures nothing about the underlying set.
  - each point in the set is
    - clearly highlighted
    - distinguishable
    - recoverable as an open singleton set in the topology.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (5)

- If you think of topologies that can arise from metrics, the discrete topology arises from metrics such as

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

  - This metric "*shatters*" the points X, *isolating* each one within its own unit ball.
    - In such a space, the only convergent sequences are the ones that are eventually constant;
    - you can't find points arbitrarily close to any other points.
    - because points are isolated in this way,
    - it makes sense to call the space "discrete".

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (6-1)

- Every function from a discrete space is automatically continuous.
- for this reason, the discrete topology is the one that best "represents" X in topological space.
- the nature of a set is characterized by its functions,
- the nature of a topological space is characterized by its continuous functions.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (6-2)

- So, note that if T is any topological space, there's a natural bijective correspondence between functions $f : X \to set(T)$ and continuous morphisms $g : discrete(X) \to T$.
- For every function on X, you can find a continuous function on $discrete(X)$, and given any continuous function on $discrete(X)$, you can uniquely recover a function on X
- The discrete topology best represents the structure of the set X which, as you say, is discretized into individual points.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (7-1)

- Throughout abstract algebra, isomorphisms describe which structures are "the same".
- A topological isomorphism (a homeomorphism) between two topologies says that they are essentially the same topology.
- An isomorphism of sets is just a bijection;
- it says that the sets contain the same number of elements.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (7-3)

- Continuing the discussion of functions above,
  two discrete topologies are topologically isomorphic (homeomorphic) if and only if their underlying sets are isomorphic as sets (bijective).
- Put casually, this means that the discrete-topology-creating process maintains the similarity and differences between the underlying sets: discrete topologies are the same if and only if their underlying sets are the same.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Why called a discrete topology? (8)

- This is all the more important
  when we realize that sets are the same
  when they have the same number of points.
- Hence discrete topologies are the same
  when (and only when) their underlying sets
  have "discrete points" in the same quantity.
- You can count the points in a discrete topology through isomorphisms,
  and the discrete topology is the only topology for which this is possible.

https://math.stackexchange.com/questions/2614268/why-is-a-discrete-topology-...

Topological Space Definition by Neighbourhood (1)

- This axiomatization is due to Felix Hausdorff. Let X be a (possibly empty) set.
- The elements of X are usually called points,
  though they can be any mathematical object.
  Let $\mathscr{N}$ be a function assigning to each x (point) in X
  a non-empty collection $\mathscr{N}(x)$ of subsets of X.
- The elements of $\mathscr{N}(x)$ will be called neighbourhoods of x with respect to $\mathscr{N}$
  (or, simply, neighbourhoods of x).
- The function $\mathscr{N}$ is called a neighbourhood topology
  if the axioms below are satisfied;
  and then X with $\mathscr{N}$ is called a topological space.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (2)

- If N is a neighbourhood of x (i.e., $N \in \mathcal{N}(x)$), then $x \in N$.
- In other words, each point of the set X belongs to every one of its neighbourhoods with respect to $\mathcal{N}$.
- If N is a subset of X and includes a neighbourhood of x, then N is a neighbourhood of x.
- I.e., every superset of a neighbourhood of a point $x \in X$ is again a neighbourhood of x.
- The intersection of two neighbourhoods of x is a neighbourhood of x.
- Any neighbourhood N of x includes a neighbourhood M of x such that N is a neighbourhood of each point of M.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (3)

- The first three axioms for neighbourhoods have a clear meaning.
- The fourth axiom has a very important use in the structure of the theory,
  that of linking together the neighbourhoods of different points of X.
- A standard example of such a system of neighbourhoods is
  for the real line $\mathbb{R}$, where a subset N of $\mathbb{R}$ is defined
  to be a neighbourhood of a real number x
  if it includes an open interval containing x.

https://en.wikipedia.org/wiki/Topological_space

Topological Space Definition by Neighbourhood (4)

- Given such a structure, a subset U of X is defined to be open if U is a neighbourhood of all points in U.
- The open sets then satisfy the axioms given below in the next definition of a topological space.
- Conversely, when given the open sets of a topological space, the neighbourhoods satisfying the above axioms can be recovered by defining N to be a neighbourhood of x if N includes an open set U such that $x \in U$.

https://en.wikipedia.org/wiki/Topological_space

Continuous Functions (1)

- In category theory, one of the fundamental categories is Top, which denotes the category of topological spaces whose objects are topological spaces and whose morphisms are continuous functions. The attempt to classify the objects of this category (up to homeomorphism) by invariants has motivated areas of research, such as homotopy theory, homology theory, and K-theory.

https://en.wikipedia.org/wiki/Topological_space

Continuous Functions (2)

- A function f : X → Y {\displaystyle f:X\to Y} between topological spaces is called continuous if for every x ∈ X {\displaystyle x\in X} and every neighbourhood N {\displaystyle N} of f (x) {\displaystyle f(x)} there is a neighbourhood M {\displaystyle M} of x {\displaystyle x} such that f (M) ⊆ N . {\displaystyle f(M)\subseteq N.} This relates easily to the usual definition in analysis. Equivalently, f {\displaystyle f} is continuous if the inverse image of every open set is open.[11] This is an attempt to capture the intuition that there are no "jumps" or "separations" in the function. A homeomorphism is a bijection that is continuous and whose inverse is also continuous. Two spaces are called homeomorphic if there exists a homeomorphism between them. From the standpoint of topology, homeomorphic spaces are essentially identical

https://en.wikipedia.org/wiki/Topological_space

Characterization (1-1)

- a characterization of an object is a set of conditions that, while different from the definition of the object, is logically equivalent to it.
- "Property P characterizes object X"
  - not only does X have property P
  - but that object X is the only thing that has property P
  - i.e., P is a defining property of object X

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (1-2)

- Similarly, a set of properties P is said to characterize object X, when these properties distinguish object X from all other objects.
- Even though a characterization identifies an object in a unique way, several characterizations can exist for a single object.
- Common mathematical expressions

  for a characterization of object X in terms of a set of properties P include "a set of properties P is necessary and sufficient for object X",

  and "object X holds if and only if a set of properties P".

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (2-1)

- It is also common to find statements such as "Property Q characterizes object Y up to isomorphism".
- The first type of statement says in different words that the extension of P is a singleton set, while the second says that the extension of Q is a single equivalence class (for isomorphism, in the given example — depending on how up to is being used, some other equivalence relation might be involved).

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (2-2)

- A reference on mathematical terminology notes that characteristic originates from the Greek term kharax, "a pointed stake":
- From Greek kharax came kharakhter, an instrument used to mark or engrave an object.
- Once an object was marked, it became distinctive, so the character of something came to mean its distinctive nature.
- The Late Greek suffix -istikos converted the noun character into the adjective characteristic, which, in addition to maintaining its adjectival meaning, later became a noun as well.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (3-1)

- Just as in chemistry, the characteristic property of a material will serve to identify a sample, or in the study of materials, structures and properties will determine characterization, in mathematics there is a continual effort to express properties that will distinguish a desired feature in a theory or system.
- Characterization is not unique to mathematics, but since the science is abstract, much of the activity can be described as "characterization".

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (3-2)

- For instance, in Mathematical Reviews, as of 2018, more than 24,000 articles contain the word in the article title, and 93,600 somewhere in the review.
- In an arbitrary context of objects and features, characterizations have been expressed via the heterogeneous relation aRb, meaning that object a has feature b.
- For example, b may mean abstract or concrete.
- The objects can be considered the extensions of the world, while the features are expression of the intensions.
- A continuing program of characterization of various objects leads to their categorization.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-1)

- A rational number, generally defined as a ratio of two integers, can be characterized as a number with finite or repeating decimal expansion.
- A parallelogram is a quadrilateral whose opposing sides are parallel. One of its characterizations is that its diagonals bisect each other. This means that the diagonals in all parallelograms bisect each other, and conversely, that any quadrilateral whose diagonals bisect each other must be a parallelogram.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-2)

- "Among probability distributions on the interval from 0 to ∞ on the real line, memorylessness characterizes the exponential distributions."
  This statement means that the exponential distributions are the only probability distributions that are memoryless, provided that the distribution is continuous as defined above (see Characterization of probability distributions for more).

https://en.wikipedia.org/wiki/Characterization_(mathematics)

Characterization (4-3)

- "According to Bohr–Mollerup theorem,
  among all functions f such that $f(1) = 1$ and $xf(x) = f(x+1)$ for $x > 0$,
  log-convexity characterizes the gamma function."
  This means that among all such functions,
  the gamma function is the only one that is log-convex.
- The circle is characterized as a manifold by being one-dimensional, compact and connected; here the characterization,
  as a smooth manifold, is up to diffeomorphism.

https://en.wikipedia.org/wiki/Characterization_(mathematics)

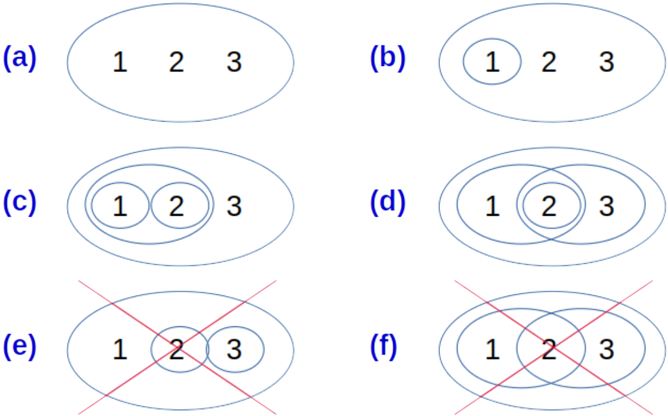
Outline

Examples of topoloy (1)

- Let τ be denoted with the circles,
  here are four examples **(a)**, **(b)**, **(c)**, **(d)**, and
  two non-examples **(e)**, **(f)** of topologies
  on the three-point set $\{1, 2, 3\}$.
- **(e)** is not a topology
  because the union of $\{2\}$ and $\{3\}$
  [i.e. $\{2, 3\}$] is missing;
- **(f)** is not a topology
  because the intersection of $\{1, 2\}$ and $\{2, 3\}$
  [i.e. $\{2\}$], is missing.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (2)



Every union of (c)

(c) is a topology $\{\{\}, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$

every union of (c)

$\cup$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1\}$	$\{1\}$	$\{1\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (c)

(c) is a topology $\{\{\},\{1\},\{2\},\{1,2\},\{1,2,3\}\}$

every intersection of (c)

$\cap$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1\}$	$\{\}$	$\{1\}$	$\{\}$	$\{1\}$	$\{1\}$
$\{2\}$	$\{\}$	$\{\}$	$\{2\}$	$\{2\}$	$\{2\}$
$\{1,2\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2\}$
$\{1,2,3\}$	$\{\}$	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every union of (f)

(f) is not a topology $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$

every union of (f)

$\cup$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$
$\{2,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Every intersection of (f)

(f) is not a topology $\{\{\},\{1,2\},\{2,3\},\{1,2,3\}\}$

every intersection of (f)

$\cap$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$
$\{\}$	$\{\}$	$\{\}$	$\{\}$	$\{\}$
$\{1,2\}$	$\{\}$	$\{1,2\}$	$\{2\}$	$\{1,2\}$
$\{2,3\}$	$\{\}$	$\{2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,2,3\}$	$\{\}$	$\{1,2\}$	$\{2,3\}$	$\{1,2,3\}$

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (3)

- Given $X = \{1, 2, 3, 4\}$,
  the *trivial* or *indiscrete* topology on X is
  the family $\tau = \{\{\}, \{1, 2, 3, 4\}\} = \{\varnothing, X\}$
  consisting of only the two subsets of X
  required by the axioms
  forms a topology of X.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (4)

- Given $X = \{1, 2, 3, 4\}$,
  the family $\tau = \{\{\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$
  $= \{\varnothing, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, X\}$
  of six subsets of X forms another topology of X.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (5)

- Given $X = \{1, 2, 3, 4\}$,
  the *discrete* topology on X is
  the power set of X, which is the family $\tau = \wp(X)$
  consisting of *all possible* subsets of X.
  the family

$$\begin{aligned} \tau = \quad & \{\{\}, \{1\}, \{2\}, \{3\}, \{4\} \\ & \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ & \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\} \end{aligned}$$

  - In this case the topological space (X, τ)
    is called a *discrete* space.

https://en.wikipedia.org/wiki/Topological_space

Examples of topoloy (6)

- Given $X = \mathbb{Z}$, the set of integers,
  the family τ of
  all finite subsets of the integers
  plus $\mathbb{Z}$ itself
  is not a topology,
  because (for example)
  the union of *all finite* sets
  not containing zero is not finite
  but is also not all of $\mathbb{Z}$,
  and so it cannot be in τ .

https://en.wikipedia.org/wiki/Topological_space