

# Cross Power Density Spectrum and Cross-Correlation Function

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi



# Cross Power Spectrum and Cross Correlation

$N$  Gaussian random variables

## Definition

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XY}(t, t + \tau) dt \right\} e^{-j\omega\tau} d\tau$$

# Two truncated processes

$N$  Gaussian random variables

## Definition

$$X_T(\omega) = \int_{-T}^{+T} X(t) e^{-j\omega t} dt$$

$$Y_T(\omega) = \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1$$

$$X_T^*(\omega) Y_T(\omega) = \int_{-T}^{+T} X(t) e^{+j\omega t} dt \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1$$

# Cross Power Density Spectrum

$N$  Gaussian random variables

## Definition

$$\begin{aligned} S_{XY}(\omega) &= \lim_{T \rightarrow \infty} \frac{1}{2T} E[X_T^*(\omega) Y_T(\omega)] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[ \int_{-T}^{+T} X(t) e^{+j\omega t} dt \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1 \right] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) e^{-j\omega(t_1-t)} dt dt_1 \end{aligned}$$

# Inverse Fourier Transform

$N$  Gaussian random variables

## Definition

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) e^{+j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) e^{-j\omega(t_1-t)} dt dt_1 e^{+j\omega\tau} d\omega \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1+t)} d\omega dt_1 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) \delta(t_1 - \tau - t) dt_1 dt \end{aligned}$$

# Impulse Function Definition

$N$  Gaussian random variables

## Definition

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) e^{+j\omega\tau} d\omega$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) \delta(t_1 - \tau - t) dt_1 dt$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XY}(\omega) e^{+j\omega\tau} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XY}(t, t + \tau) dt$$





