Cross Power Density Spectrum and Cross-Correlation Function

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} R_{XY}(t, t+\tau) dt \right\} e^{-j\omega\tau} d\tau$$

Definition

$$X_T(\omega) = \int_{-T}^{+T} X(t) e^{-j\omega t} dt$$

$$Y_T(\omega) = \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1$$

$$X_T^*(\omega) Y_T(\omega) = \int_{-T}^{+T} X(t) e^{+j\omega t} dt \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1$$

Definition

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E[X_T^*(\omega) Y_T(\omega)]$$

$$= \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{+T} X(t) e^{+j\omega t} dt \int_{-T}^{+T} Y(t_1) e^{-j\omega t_1} dt_1\right]$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XY}(t, t_1) e^{-j\omega(t_1 - t)} dt dt_1$$

Definition

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{+j\omega\tau}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XY}(t,t_1)e^{-j\omega(t_1-t)}dt\,dt_1\,e^{+j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XY}(t,t_1)\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1+t)}d\omega\,dt_1\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XY}(t,t_1)\delta(t_1-\tau-t)dt_1\,dt$$

Definition

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{+j\omega\tau}d\omega$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} R_{XY}(t,t_1)\delta(t_1-\tau-t)dt_1 dt$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega)e^{+j\omega\tau}d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{XY}(t,t+\tau)dt$$