Random Waveform

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Definition

$$X(t) = \sum_{k=-\infty}^{\infty} A_k rect\left[\frac{t - kT_b}{T_b}\right]$$

$$E\left[A_k\right] = 0 \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$E\left[A_k\right] = \begin{cases} A^2 & k = m \\ 0 & k \neq m \end{cases}$$

Definition

$$X(t) = \sum_{k=-\infty}^{\infty} A_k rect\left[\frac{t-kT_b}{T_b}\right]$$

$$2T = (2T+1)T_b$$

$$X_T(t) = \sum_{k=-K}^{K} A_k rect\left[\frac{t-kT_b}{T_b}\right]$$

Definition

$$X_T(t) = T_b \sum_{k=-K}^{K} A_k Sa(\omega T_b/2) e^{-jk\omega T_b}$$

$$= T_b Sa(\omega T_b/2) \sum_{k=-K}^{K} A_k e^{-jk\omega T_b}$$

$$\frac{E\left[|X_T(\omega)|^2\right]}{2T} = A^2 T_b Sa^2(\omega T_b/2)$$

$$= \frac{T_b Sa^2(\omega T_b/2)}{(2K+1)} \sum_{k=-K}^{K} \sum_{m=-K}^{K} E\left[A_k A_m\right] e^{-j(k-m)\omega T_b}$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} = A^2 T_b Sa^2(\omega T_b/2)$$

Definition

$$s(t) = A_0 \cos(\omega_o t + \theta_0)$$
$$= A_0 \cos(\theta_0) \cos(\omega_0 t) - A_0 \sin(\theta_0) \sin(\omega_0 t)$$

$$s(t) + N(t)$$
$$= [A_0 \cos(\theta_0) + X(t)] \cos(\omega_0 t) - [A_0 \sin(\theta_0) + Y(t)] \sin(\omega_0 t)$$
$$= R(t) \cos[\omega_0 t + \Theta(t)]$$

Definition

$$R = T_1(X, Y) = \left\{ [A_0 \cos(\theta_0) + X]^2 + [A_0 \sin(\theta_0) + Y]^2 \right\}^{1/2}$$

$$\Theta = T_2(X, Y) = \tan^{-1} \left[\frac{A_0 \sin(\theta_0) + Y}{A_0 \cos(\theta_0) + X} \right]$$

$$X = T_1^{-1}(R, \Theta) = R \cos(\Theta) - A_0 \cos(\theta_0)$$

$$Y = T_2^{-1}(R, \Theta) = R \sin(\Theta) - A_0 \sin(\theta_0)$$

Definition

$$f_{X,Y}(x,y) = \frac{e^{-(x^2+y^2)/2\sigma^2}}{2\pi\sigma^2}$$

$$f_{R,\Theta}(r,\theta) = \frac{u(r)r}{2\pi\sigma^2} exp\left\{-\frac{1}{2\sigma^2}\left[r^2 - 2rA_0\cos(\theta-\theta_0) + A_0^2\right]\right\}$$

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)d\theta$$

$$= \frac{u(r)}{\sigma^2} e^{-(r+A_0^2)/2\sigma^2} \frac{1}{2\pi}\int_0^{2\pi} e^{rA_0\cos(\theta-\theta_0)/\sigma^2} d\theta$$

Definition

$$f_{R,\Theta}(r,\theta) = \frac{u(r)}{\sigma^2} e^{-(r+A_0^2)/2\sigma^2} \frac{1}{2\pi} \int_0^{2\pi} e^{rA_0 \cos(\theta-\theta_0)/\sigma^2} d\theta$$

$$I_0(\beta) = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta cos(\theta)} d\theta$$

$$I_0\left(\frac{rA_0}{\sigma^2}\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{rA_0}{\sigma^2} cos(\theta)} d\theta$$

$$f_{R,\Theta}(r,\theta) = \frac{u(r)}{\sigma^2} r I_0\left(\frac{rA_0}{\sigma^2}\right) e^{-(r+A_0^2)/2\sigma^2}$$

Definition

$$I_0(\beta) = \frac{e^{\beta}}{\sqrt{2\pi\beta}} \qquad \beta \gg 1$$

$$f_{R,\Theta}(r,\theta) \cong u(r)\sqrt{\frac{r}{2\pi A_0 \sigma^2} exp\left[\frac{-(r-A_0)^2}{2\sigma^2}\right]}$$

$$f_{R,\Theta}(r,\theta) = \frac{u(r)}{\sigma^2} r I_0\left(\frac{rA_0}{\sigma^2}\right) e^{-(r+A_0^2)/2\sigma^2}$$

$$f_R(r) = \frac{e^{-(r+A_0^2)/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

Definition

$$f_\Theta(\theta) = \frac{1}{2\pi} exp\left[-\frac{A_0^2}{2\sigma^2}\right]$$

$$+\frac{A_0 \cos(\theta-\theta_0)}{\sqrt{2\pi\sigma}} exp\left[-\frac{A_0^2 \sin^2(\theta-\theta_0)}{2\sigma^2}\right] \cdot F\left[\frac{A_0 \cos(\theta-\theta_0)}{\sigma}\right]$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2/2} d\xi$$

$$\lim_{A_0\sigma\to\infty} f_\Theta(\theta) = \delta(\theta-\theta_0)$$