First Order ODE's (1A)

Young Won Lim

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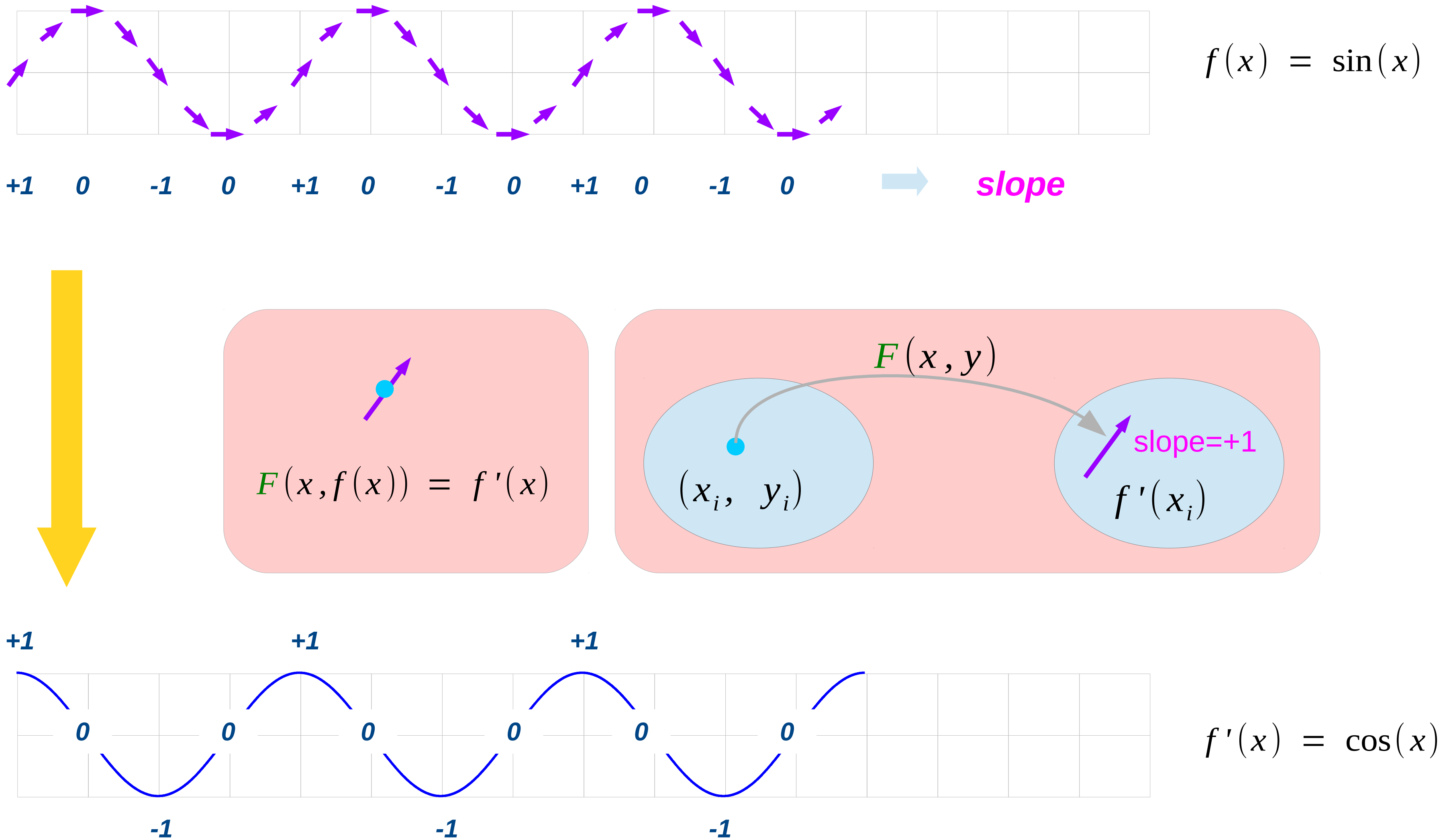
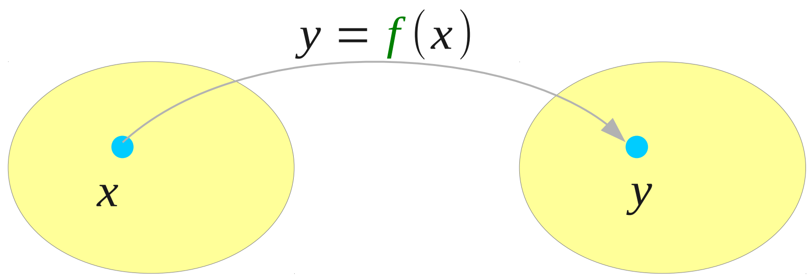
Direction Fields

Plot of F(x,y)=cos(x)

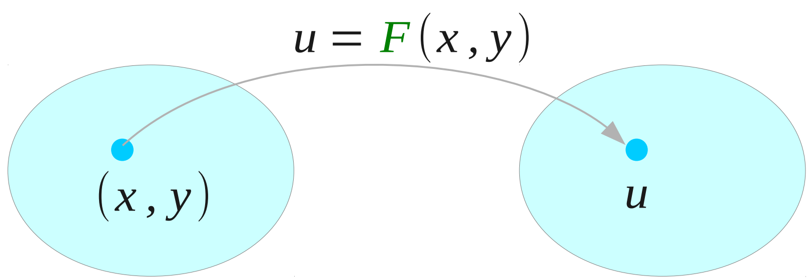


A function of two variables



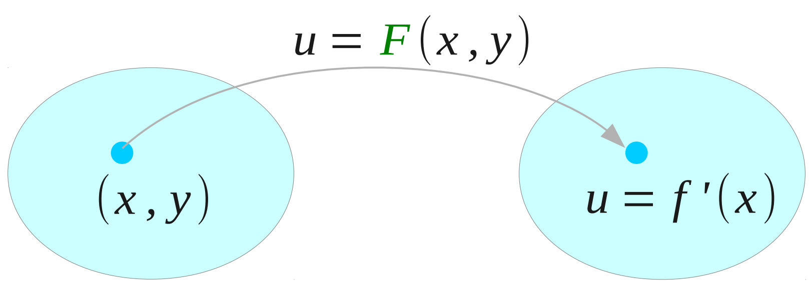
$y = f(x)$

f maps x to y



$u = F(x, y)$

F maps (x,y) to u



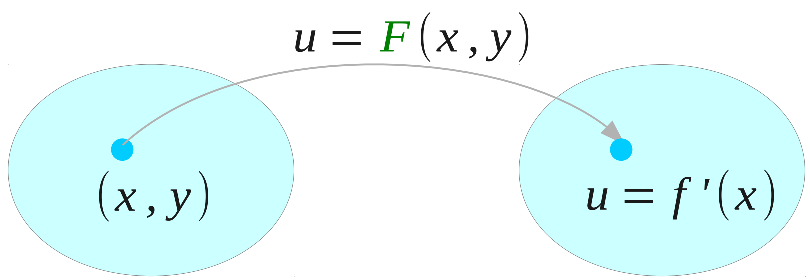
$u = F(x, y)$

F maps (x,y) to f'(x)

the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

Direction Field



$u = F(x, y)$ **F maps (x,y) to f'(x)**

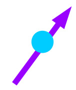
the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

$f'(x) = F(x, y)$ **F maps (x,y) to f'(x)**

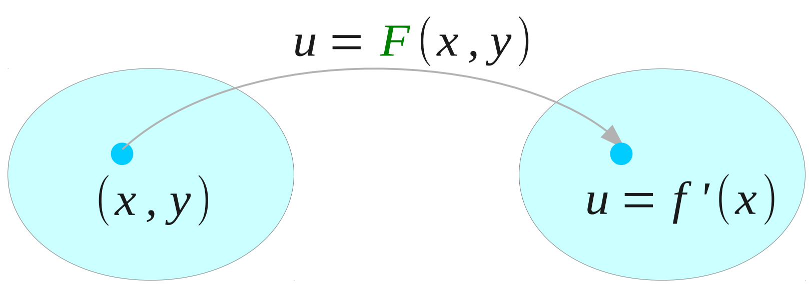
Direction Field
Slope Field

A 2-d plot representing f'(x) by a lineal element at (x, y)



The slope of the tangent line at (x, f(x))

Direction Field, First Order ODE



$y = f(x)$

$u = F(x, y)$ **F maps (x,y) to u**

$f'(x) = F(x, y)$ **F maps (x,y) to f'(x)**

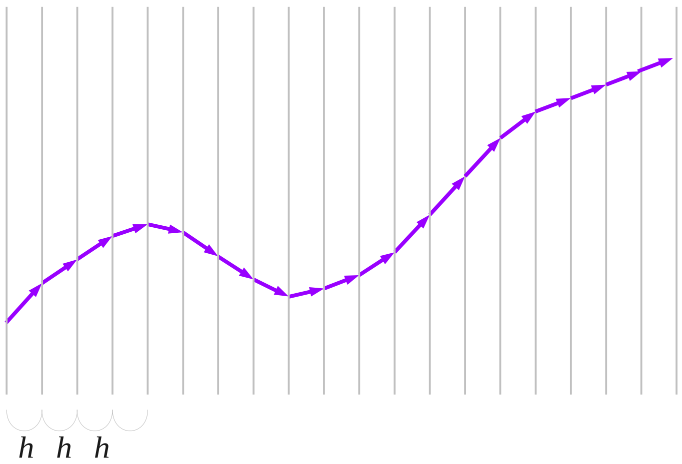
the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

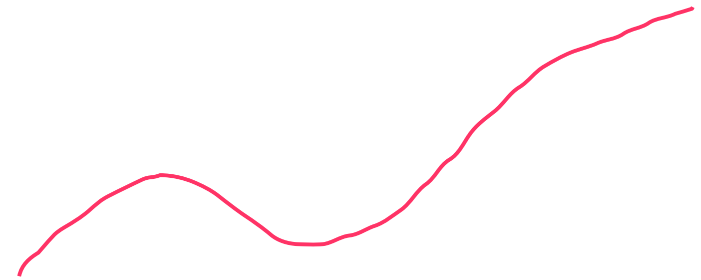
First Order ODE	**Find solution y=f(x)**	$\frac{dy}{dx} = g(x, y)$	**where the first derivative y' is given by some formula g(x,y) containing variable x, y**
Direction Field Slope Field	**A 2-d plot of y'=f'(x) at (x, y)**	$g(x, y) = \frac{dy}{dx}$	**Now, it also can be viewed as a function** **g maps (x,y) to f'(x)**

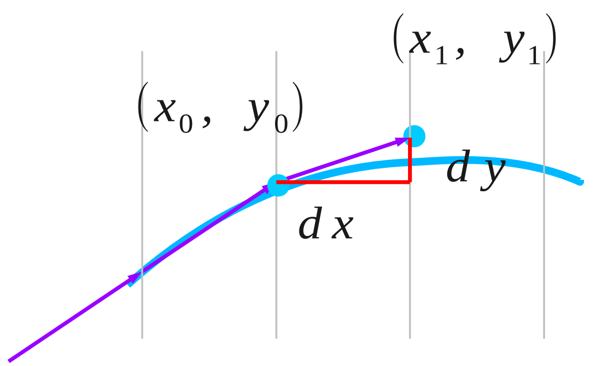
Euler's Method

$F(x, y) = f'(x)$



$y = f(x)$





$dy = f'(x)dx$

$x_1 - x_0 = dx = h$

$y_1 - y_0 = dy = f'(x)dx = F(x_0, y_0)dx$

$y_1 = y_0 + F(x_0, y_0)h$

$$y_{i+1} = y_i + F(x_i, y_i)h$$

Types of First Order ODEs

First & Second Order IVPs (y=f(x))

First Order Initial Value Problem

$$\frac{dy}{dx} = g(x, y)$$
$$y(x_0) = y_0$$

$$y' = g(x, y)$$
$$y(x_0) = y_0$$

Second Order Initial Value Problem

$$\frac{d^2y}{dx^2} = g(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y'' = g(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

$$a_1(x) y' + a_0(x) y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y) dx + N(x, y) dy = 0$$

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0$$

$$z = f(x, y)$$

Separable First Order ODEs

Separable ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

$$\frac{1}{g_2(y)} \frac{dy}{dx} = g_1(x)$$

$$\frac{1}{g_2(y)} y' = g_1(x)$$

$$p(y) \frac{dy}{dx} = q(x)$$

$$p(y) y' = q(x)$$

Solving Separable ODEs (1)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx}dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$p(y)y' = q(x)$$

$$p(y)y' = q(x)$$

$$p(y)y'dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$y = f(x)$$

not a ratio

$$dy = \frac{df}{dx}dx$$

$$\int p(y)dy$$

includes a constant

$$\int q(x)dx$$

includes another constant

implicit function y

Solving Separable ODEs (2)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\,y' = q(x)$$

given a composite function
p(y(x)) find y=f(x)

$$P(y) = \int p(y)\,dy + c_1$$

$$\frac{d}{dy}[P(y)] = p(y)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right] \cdot \frac{dy}{dx} = q(x)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right] \cdot y' = q(x)$$

$$\frac{d}{dy}[P(y)] \cdot \frac{dy}{dx} = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}[P(y)] \cdot = q(x)$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$p(y) = \frac{d}{dy}\left[\int p(y)\,dy + c_1\right]$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

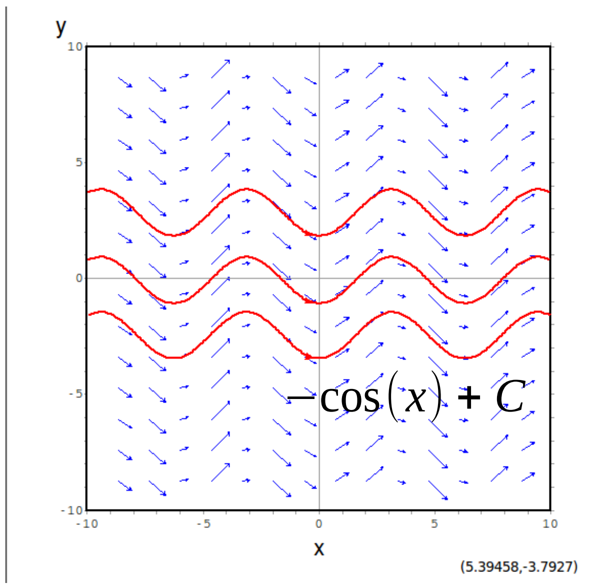
$$P(y) = Q(x) + C$$

$$P(y) = Q(x) + C$$

Special Cases of Separable Equations

$$\frac{dy}{dx} = g_1(x)$$

$$\frac{dy}{dx} = \sin(x)$$



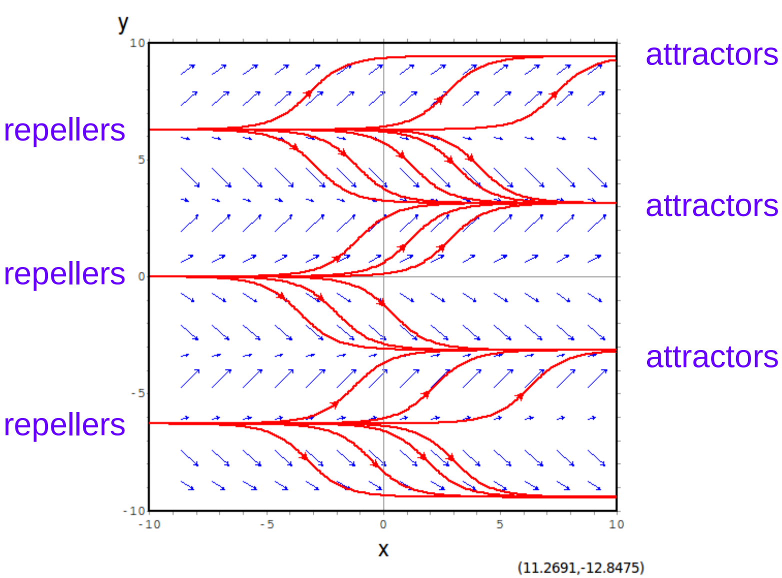
solution curves y(x):
translated in the **y** direction

$$\frac{dy}{dx} = g_2(y)$$

$$\frac{dy}{dx} = \sin(y)$$

the only literal y

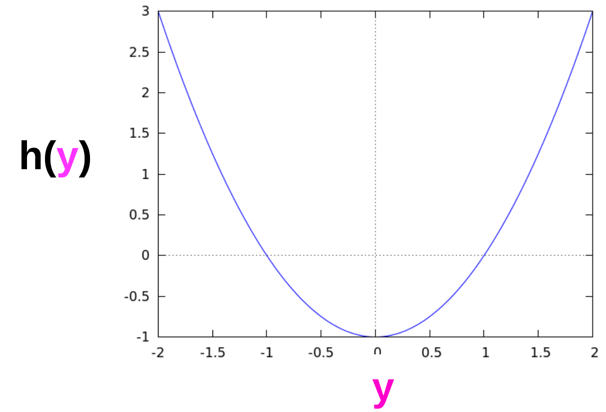
Autonomous ODEs



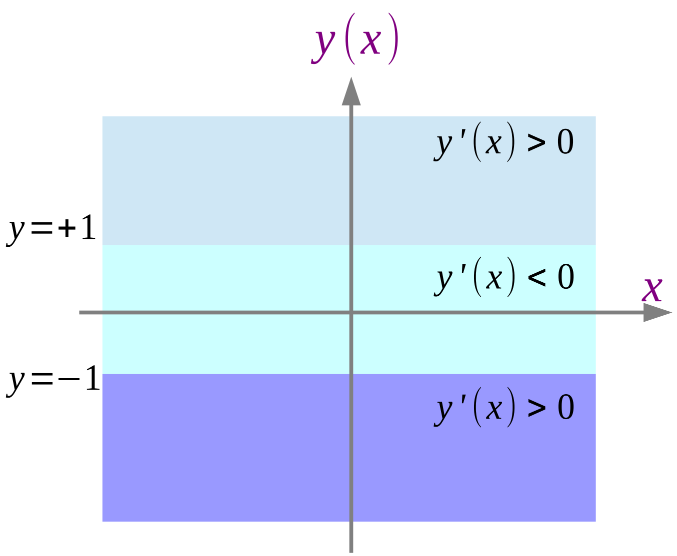
solution curves y(x):
translated in the **x** direction

Autonomous First Order ODEs

$$\frac{dy}{dx} = (y-1)(y+1) \qquad \Rightarrow \; h(y)$$



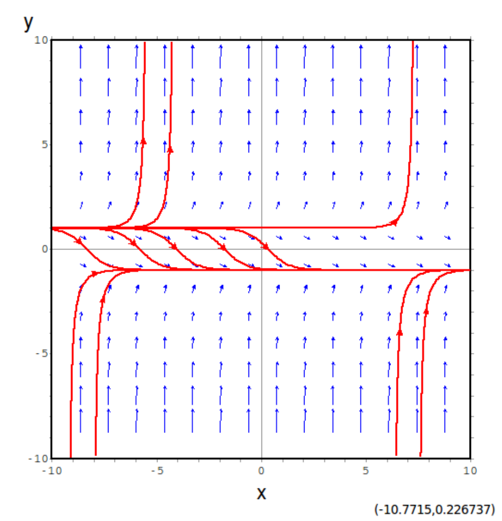
$y'(x) = h(y) > 0$	$y > +1$
$y'(x) = h(y) < 0$	$-1 < y < +1$
$y'(x) = h(y) > 0$	$y < -1$



increasing $y(x)$

decreasing $y(x)$

increasing $y(x)$



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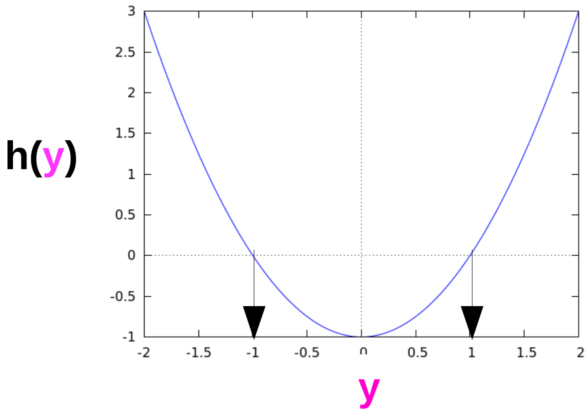
Critical Points of Autonomous ODEs

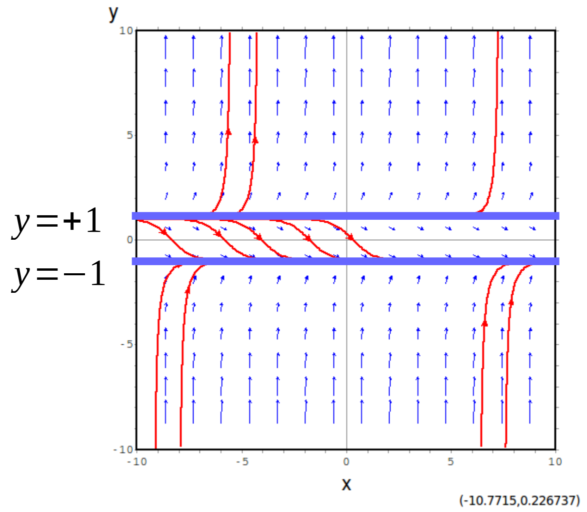
$$\frac{dy}{dx} = (y-1)(y+1) \quad \Rightarrow \quad h(y)$$

$$h(y) = 0$$
$$(y-1)(y+1) = 0 \quad \Rightarrow \quad \begin{cases} y = +1 \\ y = -1 \end{cases}$$



critical points (equilibrium, stationary points) $+1, \; -1$

constant solutions $\begin{cases} y = +1 \\ y = -1 \end{cases}$



Examples of Autonomous ODEs (1)

$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\frac{1}{(y^2-1)}\frac{dy}{dx}dx = 1\,dx$$

assumed
$(y \neq -1) \wedge (y \neq +1)$

$$\frac{1}{(y^2-1)}dy = 1\,dx$$

$$\frac{1}{2}\left(\frac{1}{(y-1)} - \frac{1}{(y+1)}\right)dy = 1\,dx$$

$$\frac{1}{2}\left(\int \frac{1}{(y-1)}dy - \int \frac{1}{(y+1)}dy\right) = \int 1\,dx$$

$$\frac{1}{2}(\ln|y-1| - \ln|y+1|) = x + c_1$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2x + c_2$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2x + c_2 \quad \Rightarrow \quad \left|\frac{y-1}{y+1}\right| = e^{2x+c_2} = c_3 e^{2x}$$

$$\frac{y-1}{y+1} = +c_3 e^{2x} \qquad \frac{y-1}{y+1} = -c_3 e^{2x}$$

$$y = \frac{1+c_3 e^{2x}}{1-c_3 e^{2x}} \qquad y = \frac{1-c_3 e^{2x}}{1+c_3 e^{2x}}$$

$+c_3 \Rightarrow c$ $\qquad$ $-c_3 \Rightarrow c$

$$y = \frac{1+c\,e^{2x}}{1-c\,e^{2x}}$$

constant solutions
$y = +1$ $\quad c \Rightarrow 0$
$y = -1$

: excluded in this method

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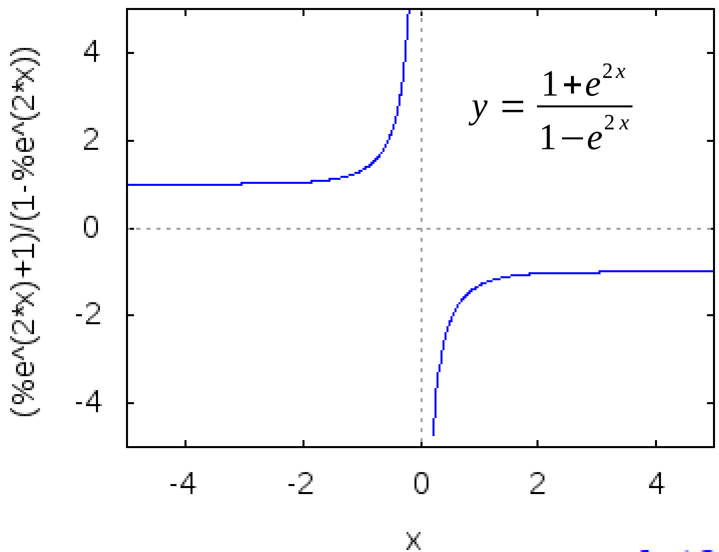
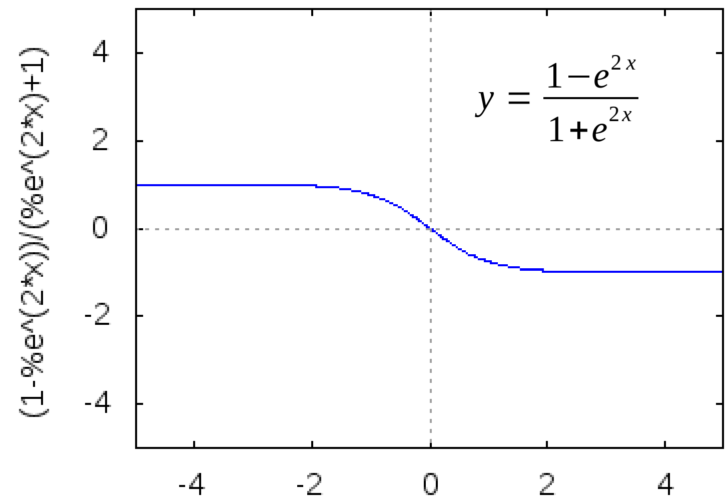
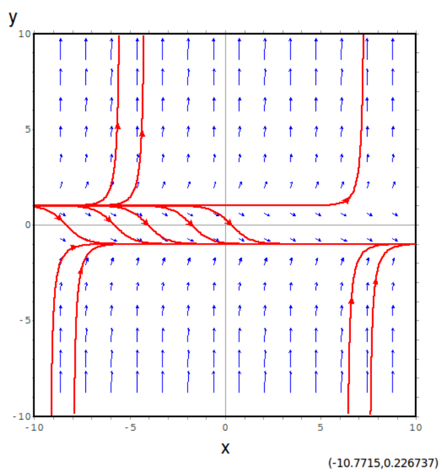
Examples of Autonomous ODEs (2)

```
wxplot2d([(1+%e^(2*x))/(1-%e^(2*x))], [x, -5, 5], [y, -5, 5]);
```






```
wxplot2d([(1-%e^(2*x))/(1+%e^(2*x))], [x, -5, 5], [y, -5, 5]);
```



Linear First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

$$a_1(x)\,y' + a_0(x)\,y = g(x)$$

$$y = f(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y' + P(x)\,y = Q(x)$$

Standard Form of First Order ODEs

Homogeneous and Particular Solutions

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

total solution

$$y = y_h + y_p$$

The Homogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

the common part of the solutions of many different differential equations whose homogeneous DE's are the same

The Nonhomogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

particular solution

$$y_p = f_p(x)$$

the particular solution of a specific differential equation, excluding common part of the solution

Three Different Linear ODEs (2)

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \leftarrow y_1 \rightarrow \quad y' + P(x)\,y = Q(x) \qquad \textbf{EQ 1}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \leftarrow y_h \rightarrow \quad y' + P(x)\,y = 0$$

$$\frac{d[y_1+y_h]}{dx} + P(x)[y_1+y_h] = Q(x) \qquad [y_1+y_h]' + P(x)[y_1+y_h] = Q(x) \qquad y_1 + y_h \quad \textit{solution of EQ 1}$$

$$\frac{dy}{dx} + P(x)\,y = R(x) \quad \leftarrow y_2 \rightarrow \quad y' + P(x)\,y = R(x) \qquad \textbf{EQ 2}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \leftarrow y_h \rightarrow \quad y' + P(x)\,y = 0$$

$$\frac{d[y_2+y_h]}{dx} + P(x)[y_2+y_h] = R(x) \qquad [y_2+y_h]' + P(x)[y_2+y_h] = R(x) \qquad y_2 + y_h \quad \textit{solution of EQ 2}$$

$$\frac{dy}{dx} + P(x)\,y = S(x) \quad \leftarrow y_3 \rightarrow \quad y' + P(x)\,y = S(x) \qquad \textbf{EQ 3}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \leftarrow y_h \rightarrow \quad y' + P(x)\,y = 0$$

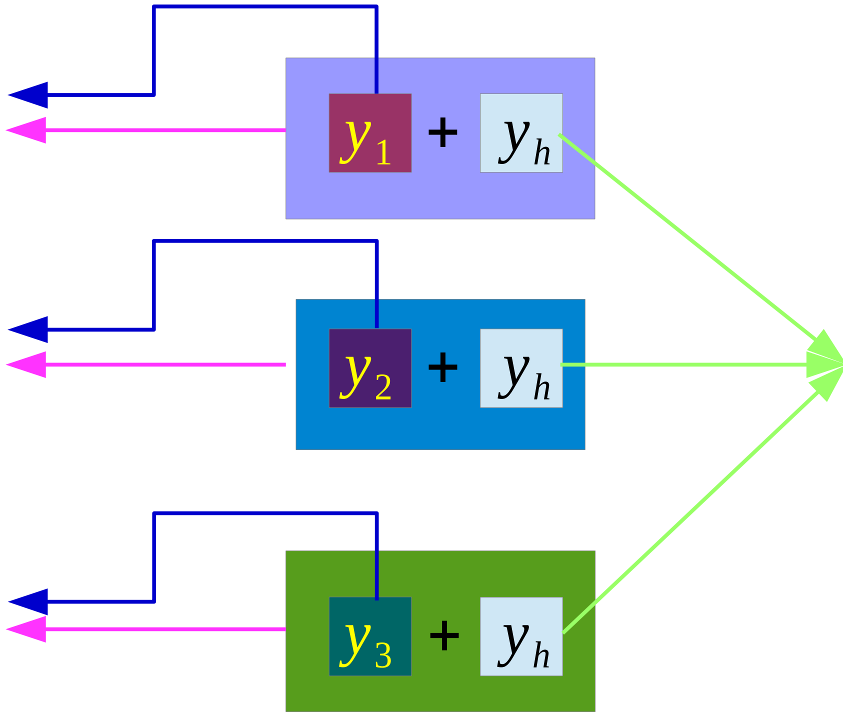
$$\frac{d[y_3+y_h]}{dx} + P(x)[y_3+y_h] = S(x) \qquad [y_3+y_h]' + P(x)[y_3+y_h] = S(x) \qquad y_3 + y_h \quad \textit{solution of EQ 3}$$

Three Different Linear ODEs (2)

EQ 1 $\frac{dy}{dx} + P(x)y = Q(x)$

EQ 2 $\frac{dy}{dx} + P(x)y = R(x)$

EQ 3 $\frac{dy}{dx} + P(x)y = S(x)$



$\frac{dy}{dx} + P(x)y = 0$

similarly

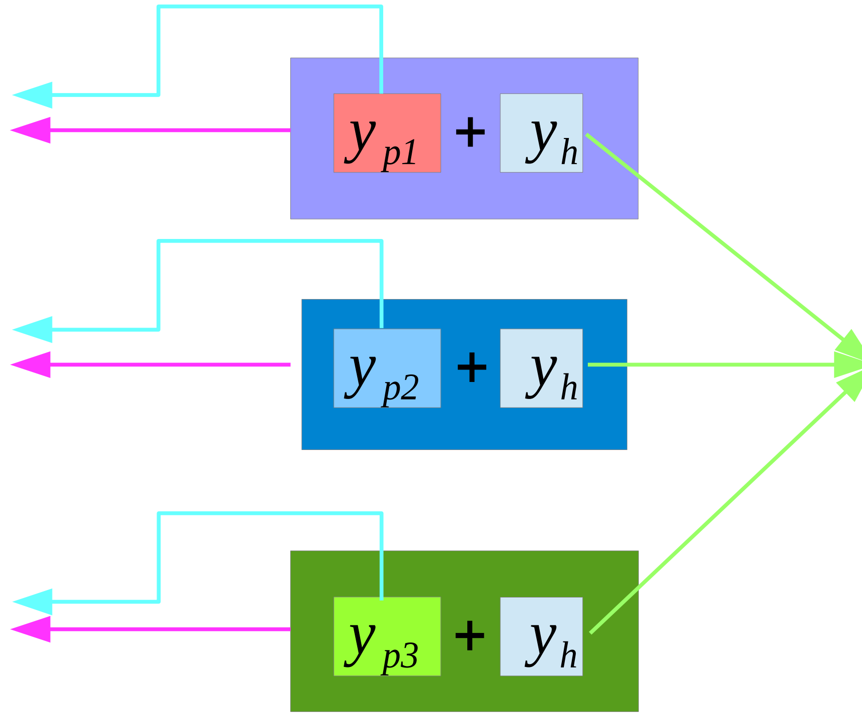
$$\begin{cases} y_1 + k \cdot y_h \\ y_2 + k \cdot y_h \\ y_3 + k \cdot y_h \end{cases}$$

Three Different Linear ODEs (2)

EQ 1 $\frac{dy}{dx} + P(x)y = Q(x)$

EQ 2 $\frac{dy}{dx} + P(x)y = R(x)$

EQ 3 $\frac{dy}{dx} + P(x)y = S(x)$



$$\frac{dy}{dx} + P(x)y = \mathbf{0}$$

y_p ***particular solution***

excluding any homogeneous solution

y_h ***homogeneous solution***

Total Solution

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y' + P(x)\,y = Q(x)$$

$$\begin{cases} y = y_p \\ y = y_h + y_p \end{cases}$$

$$\begin{cases} y = y_p \\ y = y_h + y_p \end{cases}$$

total solution

$$y = y_h + y_p$$

$$\frac{dy_h}{dx} + P(x)\,y_h = 0$$

$$y_h' + P(x)\,y_h = 0$$

homogeneous solution

$$+$$

$$\frac{dy_p}{dx} + P(x)\,y_p = Q(x)$$

$$y_p' + P(x)\,y_p = Q(x)$$

particular solution

$$=$$

$$\frac{d[y_h + y_p]}{dx} + P(x)[y_h + y_p] = Q(x)$$

$$[y_h + y_p]' + P(x)[y_h + y_p] = Q(x)$$

Solving the Homogeneous DE

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

$$\frac{dy}{dx} = -P(x)y \qquad y' = -P(x)y$$

$$\frac{1}{y}\frac{dy}{dx} = -P(x) \qquad \frac{1}{y}y' = -P(x)$$

$$\int \frac{1}{y}\frac{dy}{dx}dx = -\int P(x)dx \qquad \int \frac{1}{y}y'dx = -\int P(x)dx$$

not a ratio

$$dy = \frac{df}{dx}dx$$

$$\int \frac{1}{y}dy = -\int P(x)dx \qquad \int \frac{1}{y}dy = -\int P(x)dx$$

$$\ln|y| = -\int P(x)dx + C \qquad \ln|y| = -\int P(x)dx + C$$

$$|y| = e^{-\int P(x)dx + C} \qquad |y| = e^{-\int P(x)dx + C}$$

$$y = \pm e^{-\int P(x)dx} \cdot e^{C} = \pm e^{C} \cdot e^{-\int P(x)dx}$$

$$y = c\,e^{-\int P(x)dx} \qquad y = c\,e^{-\int P(x)dx}$$

Integrating Factor

$$\frac{dy}{dx} + P(x)\,y = 0$$

$$y' + P(x)\,y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

$$y = c\,e^{-\int P(x)dx}$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y_h = c\,y_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_1 = e^{-\int P(x)dx}$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

$$y_p = u(x)\,y_1$$

$$y_p = u(x)\,y_1$$

$$\frac{dy_p}{dx} + P(x)\,y_p = Q(x)$$

$$y_p' + P(x)\,y_p = Q(x)$$

particular solution

$$y_p = f_p(x)$$

Solving the Non-homogeneous DE (1)

$$\frac{dy_p}{dx} + P(x)y_p = Q(x)$$

$$y_p = u(x)\,y_1$$

$$\left\{\frac{du}{dx}\cdot y_1 + u\cdot\frac{dy_1}{dx}\right\} + P(x)u\,y_1 = Q(x)$$

$$u\cdot\left\{\frac{dy_1}{dx} + P(x)y_1\right\} + \frac{du}{dx}\cdot y_1 = Q(x)$$

$$\frac{du}{dx}dx = \frac{Q(x)}{y_1(x)}dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\,dx + c\right]\cdot y_1$$

$$y_p' + P(x)y_p = Q(x)$$

$$y_p = u(x)\,y_1$$

$$\{u'\cdot y_1 + u\cdot y_1'\} + P(x)u\,y_1 = Q(x)$$

$$u\cdot\{y_1' + P(x)y_1\} + u'\cdot y_1 = Q(x)$$

$$u'dx = \frac{Q(x)}{y_1(x)}dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\,dx + c\right]\cdot y_1$$

$$y_p = f_p(x)$$

$$y_h(x) = c\cdot y_1$$

$$\left\{\frac{dy_h}{dx} + P(x)y_h\right\} = 0$$

$$c\left\{\frac{dy_1}{dx} + P(x)y_1\right\} = 0$$

$$y_p(x) = u\cdot y_1$$

$$y_h(x) = c\cdot y_1$$

excluding homogeneous parts of the solution

Solving the Non-homogeneous DE (2)

$$\frac{d y_p}{d x} + P(x) y_p = Q(x)$$

$$y_p' + P(x) y_p = Q(x)$$

$$y_p = u(x)\, y_1$$

$$\frac{d}{dx}\{u \cdot y_1\} + P(x) u \cdot y_1 = Q(x)$$

$$(u \cdot y_1)' + P(x) u \cdot y_1 = Q(x)$$

$$\frac{du}{dx} \cdot y_1 = Q(x)$$

$$u' \cdot y_1 = Q(x)$$

$$y_1 = e^{-\int P(x) dx}$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\, dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\, dx$$

Integrating factor

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\, dx\right] \cdot y_1$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\, dx\right] \cdot y_1$$

$$\frac{1}{y_1} = e^{+\int P(x) dx}$$

$$y_p(x) = \left[\int \left\{Q(x) \cdot e^{+\int P(x) dx}\right\} dx\right] \cdot y_1 = \left[\int \left\{Q(x) \cdot e^{+\int P(x) dx}\right\} dx\right] \cdot e^{-\int P(x) dx}$$

$$y(x) = y_h(x) + y_p(x) = c\, e^{-\int P(x) dx} + e^{-\int P(x) dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x) dx}\right\} dx\right]$$

Solving the Non-homogeneous DE (3)

$$\frac{d\,y_p}{d\,x} + 2\,y_p = x$$

$$y_1 = e^{-\int 2\,dx} = e^{-2x}$$

$$y_h = c\,e^{-2x}$$

$$\frac{d\,y_h}{d\,x} + 2\,y_h = -2\,c\,e^{-2x} + 2\,c\,e^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx = \int \frac{x}{e^{-2x}}\,dx = \int x\,e^{+2x}\,dx$$

$$= \frac{1}{2}\left(x\,e^{+2x} - e^{2x}\right) = \frac{1}{2}(x-1)\,e^{+2x}$$

$$y_p(x) = u(x)\,e^{-2x}$$

$$= \frac{1}{2}(x-1)\,e^{-2x}\,e^{-2x} = \frac{1}{2}(x-1)$$

$$\frac{d\,y_p}{d\,x} + 2\,y_p = \frac{1}{2} + 2\frac{1}{2}(x-1) = x$$

Total Solution

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y_1 = e^{-\int P(x)\,dx}$$

total solution

$c\,y_1 + u(x)\,y_1$

$u(x)\,y_1$

homogeneous solution

particular solution

$$c\,e^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x)\,dx}\right\} dx\right]$$

Integrating factor

$c\,y_1$

$$\frac{dy}{dx} + P(x)\,y = 0$$

General Solution

if there is any solution, it is in the form of B

A $$\frac{dy}{dx} + P(x)y = Q(x)$$

B $$y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x)dx}\right\} dx\right]$$

B is the solution of A ODE only

$$e^{+\int P(x)dx} \times \quad y(x) = \left(c\,e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x)dx}\right\} dx\right]\right) \quad \times e^{+\int P(x)dx}$$

$$\left\{e^{+\int P(x)dx}\right\} \cdot y(x) = c + \left[\int \left\{Q(x) \cdot e^{+\int P(x)dx}\right\} dx\right]$$

$$\frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\} \cdot y(x)\right] = Q(x) \cdot \left\{e^{+\int P(x)dx}\right\}$$

$$\left[\left\{e^{+\int P(x)dx}\right\} \cdot \frac{dy}{dx} + \left\{e^{+\int P(x)dx}\right\} P(x) \cdot y(x)\right] = Q(x) \cdot \left\{e^{+\int P(x)dx}\right\}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y' + P(x)\,y = Q(x)$$

$$y = f(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\cdot\left[\frac{dy}{dx} + P(x)\,y\right] = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$y_1 = e^{-\int P(x)dx} \qquad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \left[\left\{e^{+\int P(x)dx}\right\}P(x)\right]y = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\cdot P(x) = \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\} \qquad f'(g(x))\,g'(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\}y = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\}y = \frac{d}{dx}\left[e^{+\int P(x)dx}\cdot y\right]$$

$$f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right] = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\int \frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right]dx = \int \left\{e^{+\int P(x)dx}\right\}Q(x)\,dx + c$$

$$\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right] = \int \left\{e^{+\int P(x)dx}\right\}Q(x)\,dx + c$$

$$y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int \left\{Q(x)\right\}\cdot\left\{e^{+\int P(x)dx}\right\}dx\right]$$

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3/9/15

Linear First Order ODEs – Summary

Linear First-Order ODE

$$y' + P y = Q \qquad P(x),\ Q(x)$$

$$y' + P y = 0 \qquad P(x),\ Q(x) = 0$$

$$\frac{dy}{dx} = -P \cdot y \;\Rightarrow\; \frac{dy}{y} = -P \cdot dx \;\Rightarrow\; \ln y = -\int P\,dx + C \;\Rightarrow\; y = e^{-\int P\,dx + C}$$

$$y = A e^{-I} \qquad \left(I = e^{-\int P\,dx},\ A = e^{+C} \right) \;\Rightarrow\; y e^{I} = A$$

$$y' + P y = Q \qquad P(x),\ Q(x) \neq 0$$

$$\frac{d}{dx}(y e^{I}) = y' e^{I} + y e^{I} \frac{dI}{dx} = y' e^{I} + y e^{I} P = e^{I}(y' + P y) = e^{I} Q \qquad (\neq 0)$$

$$\frac{d}{dx}(y e^{I}) = e^{I} Q \;\Rightarrow\; y e^{I} = \int e^{I} Q\,dx + c \;\Rightarrow\; y = e^{-I}\left(\int e^{I} Q\,dx + c\right)$$

$$y = e^{-I}\left(\int e^{I} Q\,dx + c\right) \qquad \left(I = e^{-\int P\,dx} \right)$$

Standard Form

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_c' + P y_c = 0 \qquad P(x),\ f(x) = 0$$

$$y_g = y_c + y_p$$

general solution = homogeneous solution + particular solution

$$\frac{d}{dx}\left[y_c + y_p\right] + P(x)\left[y_c + y_p\right] = f(x)$$

$$\left[y_c' + P(x)y_c\right] + \left[y_p' + P(x)y_p\right] = f(x)$$

$$y_c' + P y_c = 0 \qquad y_p' + P y_p = f(x)$$

The Homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_c' + P y_c = 0 \qquad P(x),\ f(x) = 0$$

$$\frac{dy_c}{dx} = -P\cdot y_c \Rightarrow \frac{dy_c}{y_c} = -P\cdot dx \Rightarrow \ln y_c = -\int P\,dx + C \Rightarrow y_c = e^{-\int P\,dx + C}$$

$$y_c = c\,y_1 \qquad \left(y_1 = e^{-\int P\,dx},\ c = e^{+C} \right)$$

homogeneous solution

$$\frac{dy_c}{dx} + P\cdot y_c = c\left(\frac{d}{dx}\left(e^{-\int P\,dx}\right) + P\cdot e^{-\int P\,dx}\right) = c\left(-P\cdot e^{-\int P\,dx} + P\cdot e^{-\int P\,dx}\right) = 0$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_p' + P y_p = f(x) \qquad P(x),\ f(x) \neq 0$$

$$y_c = c\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ c = e^{+C}\ )$$ homogeneous solution

$$y_p = u\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ u(x) \neq const\ )$$ particular solution

$$\frac{d}{dx}(y_1 u) + P \cdot y_1 u = f(x) \qquad \left[\frac{dy_1}{dx} + P \cdot y_1\right]u + y_1\frac{du}{dx} = f(x)$$

$$y_1\frac{du}{dx} = f(x) \qquad du = \frac{f(x)}{y_1(x)}dx \qquad u = \int \frac{f(x)}{y_1(x)}\,dx$$

$$y_p = u\,y_1 \qquad y_p = y_1\int \frac{f(x)}{y_1(x)}\,dx = e^{-\int P\,dx}\int f(x)e^{+\int P\,dx}\,dx$$

$$y_p = u\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ u = \int f(x)e^{+\int P\,dx}\,dx\ )$$ particular solution

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_p' + P y_p = f(x) \qquad P(x),\ f(x) \neq 0$$

$$y_p = u y_1 \qquad \left(y_1 = e^{-\int P\,dx},\ u = \int f(x) e^{+\int P\,dx}\,dx \right)$$

particular solution

$$y_g = y_c + y_p = c e^{-\int P\,dx} + e^{-\int P\,dx} \int f(x) e^{+\int P\,dx}\,dx$$

general solution

$$y_g e^{+\int P\,dx} = c + \int f(x) e^{+\int P\,dx}\,dx \qquad \frac{d}{dx}\left(y_g e^{+\int P\,dx} \right) = f(x) e^{+\int P\,dx}$$

$$\left(\frac{dy_g}{dx} \right) e^{+\int P\,dx} + \left(P y_g \right) e^{+\int P\,dx} = f(x) e^{+\int P\,dx}$$

$$\frac{dy_g}{dx} + P y_g = f(x)$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$\left(\frac{dy_g}{dx}\right)e^{+\int P\,dx} + \left(P y_g\right)e^{+\int P\,dx} = f(x)e^{+\int P\,dx}$$

integration factor $e^{+\int P\,dx}$

$$\frac{d}{dx}\left(y_g e^{+\int P\,dx}\right) = f(x)e^{+\int P\,dx}$$

$$y_g e^{+\int P\,dx} = c + \int f(x)e^{+\int P\,dx}\,dx$$

$$y_g = y_c + y_p = c e^{-\int P\,dx} + e^{-\int P\,dx}\int f(x)e^{+\int P\,dx}\,dx$$

Exact First Order ODEs

Differential Form & Equation

A **differential form**

$$P(x,y)dx + Q(x,y)dy$$

A first order **differential equation**

$$P(x,y)dx + Q(x,y)dy = 0$$

this differential form is **exact**
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

exact equation
in a region R if there is a function
f(x,y) such that

$$P(x,y)dx + Q(x,y)dy$$
$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \boxed{df = 0}$$

$$P(x,y)dx + Q(x,y)dy$$

is an exact differential in a region R
if it corresponds to
the total differential of
some function f(x,y)

$$df(x,y) = 0$$

$$f(x,y) = c$$

To be exact

$$P(x,y)\,dx + Q(x,y)\,dy = df \quad \xrightarrow{\text{to be exact}} \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$P(x,y) = \frac{\partial f}{\partial x} \qquad Q(x,y) = \frac{\partial f}{\partial y} \qquad \frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ all defined and continuous $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P(x,y)dx + Q(x,y)dy$ is an exact (total) differential $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Path Independence

A differential *df* of the following form

$$P(x,y)dx + Q(x,y)dy = df$$

is **exact,** if $\int df$ is path independent

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ all defined and continuous $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P(x,y)dx + Q(x,y)dy$ is an exact (total) differential $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

path independent $\int df$ $\Leftrightarrow$ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Exact and Inexact Differential Examples

$$df = 2xy^3\,dx + 3x^2y^2\,dy$$

Is there a function f=f(x,y)
such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2xy^3 \qquad \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$f(x,y) = x^2y^3$ exact

$$\int_{(x_1, y_1)}^{(x_2, y_2)} df = f(x_2, y_2) - f(x_1, y_1)$$

Only initial & final points
Path independent

$$df = 2x^2y^3\,dx + 3x^3y^2\,dy$$

Is there a function f=f(x,y)
such that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$\frac{\partial f}{\partial x} = 2x^2y^3 \qquad \frac{\partial f}{\partial y} = 3x^3y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6x^2y^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 9x^2y^2$$

no $f(x,y)$ inexact

Integration result depends
on the path also,
in addition to initial & final points

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$z = f(x,y)$$

$$\frac{df}{dx} = M(x) \Rightarrow f(x) = \int M(x)dx + c$$

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \frac{\partial f}{\partial y} = N(x,y)$$

$$\int \frac{\partial f}{\partial x}dx = \int M(x,y)dx + c \qquad \int \frac{\partial f}{\partial y}dy = \int N(x,y)\,dy + c$$

$$f(x,y) = \int M(x,y)dx + g(y) \qquad f(x,y) = \int N(x,y)dy + h(x)$$

$$\frac{\partial f}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y)$$

$$\frac{\partial f}{\partial x} = M(x,y) = \frac{\partial}{\partial x}\int N(x,y)dy + h'(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \qquad h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy$$

Exact Equations (2)

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$$f(x,y) = \int N(x,y)\,dy + h(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

$$g(y) = \int g'(y)\,dy$$

$$= \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

$$h(x) = \int h'(x)\,dx$$

$$= \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

$$f(x,y) = \int M(x,y)\,dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

$$f(x,y) = \int N(x,y)\,dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

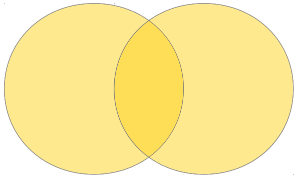
Exact Equations (3)

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$f(x,y) = \int M(x,y)dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \right\} dy$$

$$f(x,y) = \int N(x,y)dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy \right\} dx$$



$$f(x,y) = \int M(x,y)dx + \int N(x,y)\,dy - \int \frac{\partial}{\partial y}\int M(x,y)dx\,dy \qquad \int \frac{\partial}{\partial y}\int \frac{\partial f}{\partial x}dx\,dy$$

$$f(x,y) = \int N(x,y)dy + \int M(x,y)\,dx - \int \frac{\partial}{\partial x}\int N(x,y)dy\,dx \qquad \int \frac{\partial}{\partial x}\int \frac{\partial f}{\partial y}dy\,dx$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$f(x,y) = \int M(x,y)dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \right\} dy$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int M(x,y)dx = \frac{\partial M}{\partial y}$$

$$\frac{\partial f}{\partial x} = M(x,y) + \int \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dy = M(x,y)$$

$$\frac{\partial f}{\partial y} = \int \frac{\partial M}{\partial y}dx + N(x,y) - \int \frac{\partial M}{\partial y}dx = N(x,y)$$

$$f(x,y) = \int N(x,y)dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy \right\} dx$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int N(x,y)dy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x} = \int \frac{\partial N}{\partial x}dy + M(x,y) - \int \frac{\partial N}{\partial x}dy = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y) + \int \left\{ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right\} dx = N(x,y)$$

NonExact First Order ODEs

NonExact Equations (1)

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x,y)$

Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$ or $\mu(y)$

$\mu(y)$ $\mu(y)$

Multiplying NonExact Equations by $\mu(x, y)$

***NonExact* Equations**

$$M(x, y)dx + N(x, y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

***Exact* Equations**

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x, y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \longleftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu M_y - \mu N_x = \mu_x N - \mu_y M$$

Partial Differential Equation

difficult to find $\mu(x, y)$

Multiplying NonExact Equations by $\mu(x)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \longleftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$\mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

generally $P(x,y)$

sometimes $P(x)$ $\longrightarrow$ $$\frac{d\mu}{dx} - P(x)\mu = 0$$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \longleftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$

$$M\frac{d\mu}{dy} = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$

$$M\frac{d\mu}{dy} = \mu N_x - \mu M_y$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

generally $P(x,y)$

sometimes $P(y)$ $\longrightarrow$ $\frac{d\mu}{dy} - P(y)\mu = 0$

Solving NonExact Equations

$$\frac{\partial}{\partial y}[\mu(x) M(x,y)] = \frac{\partial}{\partial x}[\mu(x) N(x,y)]$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x) dx}$$

$$\mu(x) = c e^{\int \left(\frac{M_y - N_x}{N}\right) dx}$$

$$\mu(x) M(x,y) dx + \mu(x) N(x,y) dy = 0$$

$$\frac{\partial}{\partial y}[\mu(x) M(x,y)] = \frac{\partial}{\partial x}[\mu(x) N(x,y)]$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

$$\mu(y) = c e^{\int P(y) dy}$$

$$\mu(y) = c e^{\int \left(\frac{N_x - M_y}{M}\right) dy}$$

$$\mu(y) M(x,y) dx + \mu(y) N(x,y) dy = 0$$

Verifying Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\mu(x) = c\,e^{\int \left(\frac{M_y - N_x}{N}\right)dx}$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \mu_y M + \mu M_y$$

$$\frac{\partial}{\partial x}[\mu(x)N(x,y)] = \mu_x N + \mu N_x$$

$$\mu_x N + \mu N_x = \left(\frac{M_y - N_x}{N}\right)\mu N + \mu N_x$$

$$\mu_x N + \mu N_x = \mu M_y$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \frac{\partial}{\partial x}[\mu(x)N(x,y)]$$

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(x)\mu$$

$$\mu(y) = c\,e^{\int \left(\frac{N_x - M_y}{M}\right)dy}$$

$$\frac{\partial}{\partial y}[\mu(y)M(x,y)] = \mu_y M + \mu M_y$$

$$\frac{\partial}{\partial x}[\mu(y)N(x,y)] = \mu_x N + \mu N_x$$

$$\mu_y M + \mu M_y = \left(\frac{N_x - M_y}{M}\right)\mu M + \mu M_y$$

$$\mu_y M + \mu M_y = \mu N_x$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \frac{\partial}{\partial x}[\mu(x)N(x,y)]$$

Substitution Method

Substitution Method (1)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$z = f(x(t),\ y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = h_x(x,u) + h_u(x,u)\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x, u)) = h_x(x, y) + h_u(x, u)\frac{du}{dx}$$

Substitution Method (2)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

$$y' = s\left(\frac{y}{x}\right)$$

$$y = h(x,u) \qquad u = \Phi(x)$$

$$y = h(x,u) \leftarrow u = \Phi(x)$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y = u\,x \qquad u = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y' = u + x u'$$

$$\frac{dy}{dx} \leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x,y) = g(x, h(x,u))$$

$$y' = g(x, ux) = s(u)$$

$$y \leftarrow h(x,u)$$

$$s(u) = u + x u'$$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$s(u) - u = x u'$$

$$\frac{du}{s(u) - u} = \frac{dx}{x}$$

Substitution Method (3)

a new literal a function of x

$$u = \Phi(x)$$

contains x and y literals
(y is also a function of x)

a new literal u is introduced
using old literals x and y :
a new function of x

$$u = \frac{y}{x}$$

a old literal a function of x and u

$$y = h(x, u)$$

the old literal y can be replaced by
the new literal u and the old literal x:
a new function of u and x

$$y = u\,x$$

Substitution Method (4)

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y' = u + xu' \Leftarrow y = ux \Leftarrow u = \frac{y}{x}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

$$y' = g(x, ux) = s(u) \Leftarrow \frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

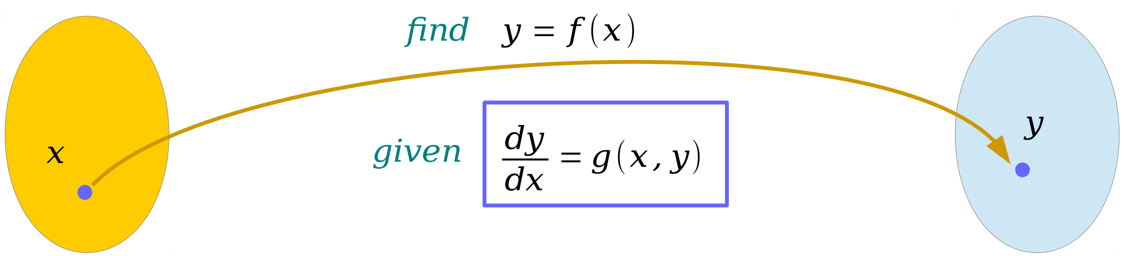
Find $y = f(x)$ **in**

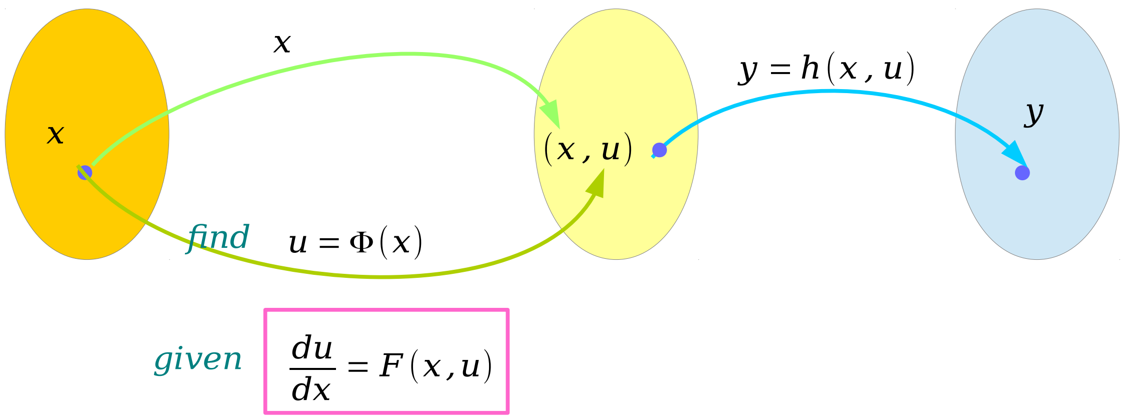
$$\frac{dy}{dx} = g(x, y)$$

Find $u = \Phi(x)$ **in**

$$\frac{du}{dx} = F(x, u)$$

Substitution Method (5)





$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

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Substitution Method (6)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, \Phi(x))$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$g(x, h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x,u)) = h_x(x,u) + h_u(x,u)\frac{du}{dx}$$

$$\frac{du}{dx} = \left\{ g(x, h(x,u)) - \frac{\partial h}{\partial x} \right\} / \frac{\partial h}{\partial u}$$

$$\frac{du}{dx} = \{ g(x, h(x,u)) - h_x(x,u) \} / h_u(x,u)$$

$$\frac{du}{dx} = F(x, u)$$

$$\frac{du}{dx} = F(x, u)$$

$$u = \Phi(x)$$

Homogeneous First Order ODEs

Homogeneous Functions

A homogeneous function of degree α

$$f(tx, ty) = t^{\alpha} f(x, y)$$

$$f(x, y) = x^2 + y^2$$

$$f(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2 (x^2 + y^2)$$

$$= t^2 f(x, y)$$

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(tx, ty) = t^{\alpha} M(x, y)$$

$$N(tx, ty) = t^{\alpha} N(x, y)$$

$$M(x, y) = M(x, x \cdot y/x) = x^{\alpha} M(1, y/x)$$

$$M(x, y) = M(y \cdot x/y, y) = y^{\alpha} M(x/y, 1)$$

$$N(x, y) = N(x, x \cdot y/x) = x^{\alpha} N(1, y/x)$$

$$N(x, y) = N(y \cdot x/y, y) = y^{\alpha} N(x/y, 1)$$

Homogeneous Equations (1)

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = y/x \qquad y = u\,x$$

$$x^{\alpha} M(1, y/x)dx + x^{\alpha} N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$dy = \frac{\partial}{\partial x}(u\,x)dx + \frac{\partial}{\partial u}(u\,x)du$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = y^{\alpha} M(x/y, 1)$$

$$N(x,y) = y^{\alpha} N(x/y, 1)$$

$$v = x/y \qquad x = v\,y$$

$$y^{\alpha} M(x/y, 1)dx + y^{\alpha} N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$dx = \frac{\partial}{\partial y}(v\,y)dy + \frac{\partial}{\partial v}(vy)dv$$

$$dx = v\,dy + y\,dv$$

$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

Homogeneous Equations (2)

A *homogeneous* Equations of *degree α*

$$M(x,y)dx + N(x,y)dy = 0$$

$$x^{\alpha} M(1, y/x)dx + x^{\alpha} N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$u = y/x \qquad y = ux$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

$$[M(1,u) + u\,N(1,u)]dx + x\,N(1,u)du = 0$$

$$\frac{dx}{x} + \frac{N(1,u)du}{[M(1,u) + u\,N(1,u)]} = 0$$

A *homogeneous* Equations of *degree α*

$$M(x,y)dx + N(x,y)dy = 0$$

$$y^{\alpha} M(x/y, 1)dx + y^{\alpha} N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$v = x/y \qquad x = vy$$

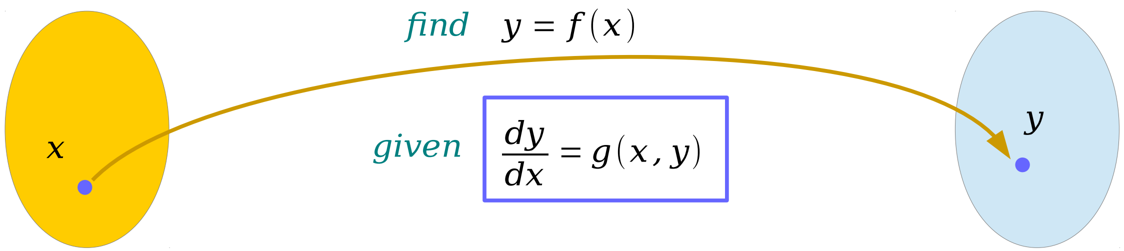
$$dx = v\,dy + y\,dv$$

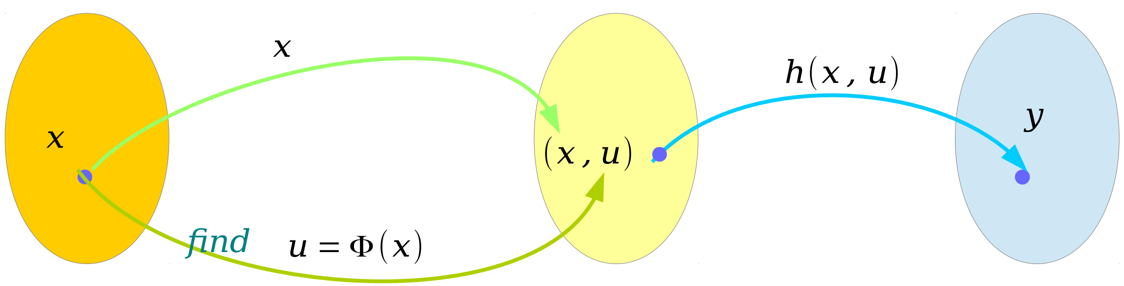
$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

$$[v\,M(v,1) + N(v,1)]dy + y\,M(v,1)dv = 0$$

$$\frac{dy}{y} + \frac{M(v,1)dv}{[v\,M(v,1) + N(v,1)]} = 0$$

Homogeneous Equations (3)





given $\frac{du}{dx} = F(x, u)$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx}dx = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial u}\frac{du}{dx}dx$$

$$dy = u\,dx + x\,du$$

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = x^{\alpha} M(1, y/x)$$

$$N(x, y) = x^{\alpha} N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(x, u) = u\,x$$

$$dy = u\,dx + x\,du$$

$$\frac{dx}{x} + \frac{N(1, u)\,du}{[M(1, u) + u\,N(1, u)]} = 0$$

Homogeneous Equations (4)

all are functions of x

$$y = f(x) \Rightarrow y(x)$$

$$u = \Phi(x) \Rightarrow u(x)$$

$$u = y/x \Rightarrow u(x) = y(x)/x$$

$$y = ux \Rightarrow y(x) = u(x)x$$

$$y = h(x,u) = h(x,\Phi(x))$$

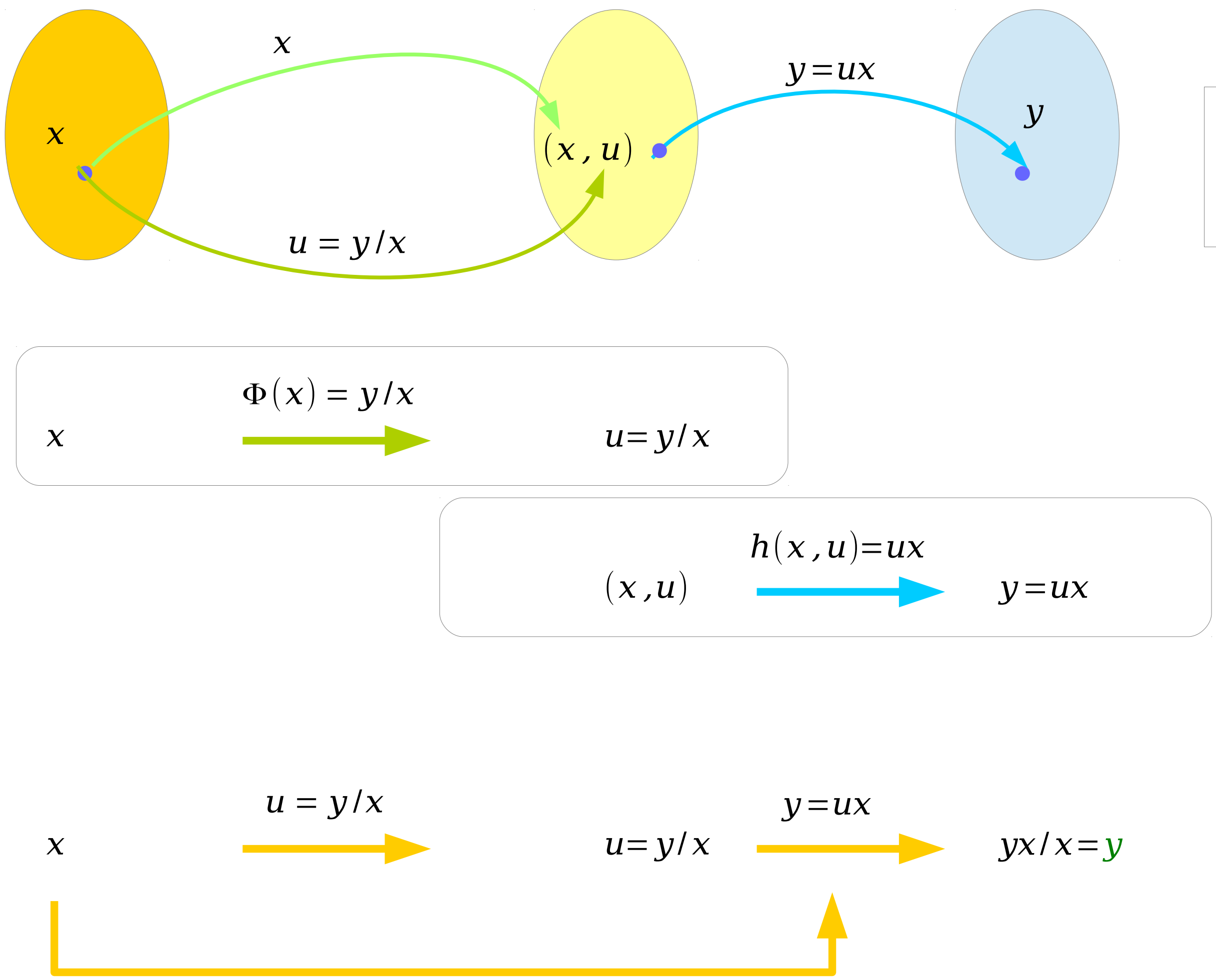
$$= ux \qquad = \Phi(x)x$$

$$= \frac{y}{x}x$$

$$y = ux$$

$$dy = u\,dx + x\,du$$

Homogeneous Equations (5)



$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(u,x) = u\,x$$

$$dy = u\,dx + x\,du$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{[M(1,u) + u\,N(1,u)]} = 0$$

Bernoulli's First Order ODEs

Bernoulli's Equations (1)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{dy}{dx} + P(x)y = Q(x)y^0$$

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad n = 0$$

***Linear* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^1$$

$$\frac{dy}{dx} + [P(x) - Q(x)]y = 0 \qquad n = 1$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)y = Q(x)y^n$$

$$y' + P(x)y = Q(x)y^0$$

$$y' + P(x)y = Q(x) \qquad n = 0$$

***Linear* Equation**

$$y' + P(x)y = Q(x)y^1$$

$$y' + [P(x) - Q(x)]y = 0 \qquad n = 1$$

***Linear* Equation**

Bernoulli's Equations (2)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} = Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \qquad \frac{du}{dx} = (1-n)\boxed{y^{-n}\frac{dy}{dx}}$$

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)u = Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n}y' + P(x)\frac{y}{y^n} = Q(x)$$

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \qquad u' = (1-n)\boxed{y^{-n}y'}$$

$$\frac{1}{(1-n)}u' + P(x)u = Q(x)$$

$$u' + (1-n)P(x)u = (1-n)Q(x)$$

***Linear* Equation**

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