Z Transform (H.1)
Definition

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Based on
Complex Analysis for Mathematics and Engineering
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Z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k] \, z^{-k}$$

$$z = |r| e^{j2\pi F} = |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z - transform

$$X(z) = \sum_{k=0}^{+\infty} x[k] \, z^{-k}$$

Inverse z - Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

Admissible Form of z-transform

$$X(z) = \sum_{k=0}^{\infty} x[n] \, z^{-n}$$

admissible z-transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a   polynomial of degree p

$Q(z)$: a   polynomial of degree q

D: Simply connected domain
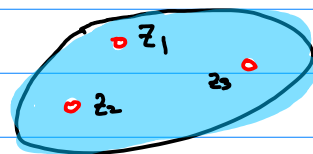
C: Simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
except at the points $z_1, z_2, \ldots, z_k$ in C
then

$$\frac{1}{2\pi i} \int_C f(z)\, dz = \sum_{j=1}^{k} \text{Res}\left(f(z), z_j\right)$$

$$\oint_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\left(f(z), z_k\right)$$

finite number k of
Singular points z_k

$$\oint_c f(z)\,dz = 0$$

if $f(z)$ is continuous in D and
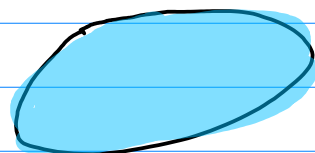$f(z) = F'(z)$; $F(z)$ is an antiderivative of $f(z)$
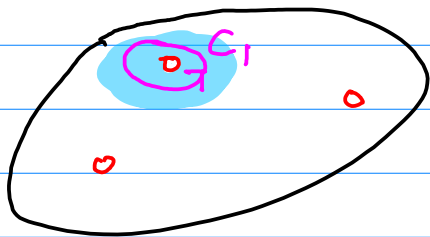fundamental theorem of calculus

$$\oint_c f(z)\,dz = 0$$

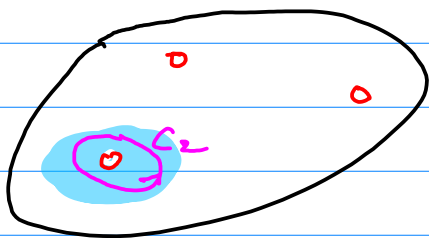if $f(z)$ is analytic Within and on C

no singularity

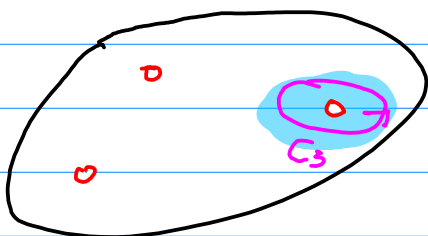$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right) \quad z_k \text{ within } C$$

$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n \geqslant 0$$

$a_n^{\{0\}}$ expansion at z_0

$a_n^{\{1\}}$ expansion at z_1

$a_n^{\{2\}}$ expansion at z_2

$$a_n^{\{m\}} = \text{Res}\left(f(z), z_m\right)$$
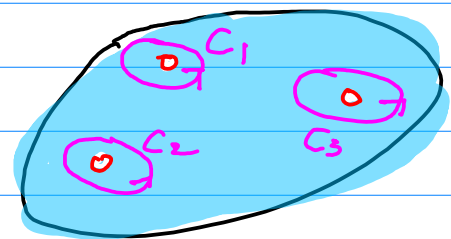
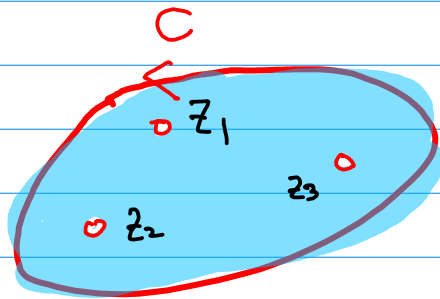the residue of $f(z)$ at z_m using C_m

assumed that

there are several (m) singularities (poles) of $f(z)$ in a region

but that

C is taken to enclose only the pole z_m : C_m

Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \quad \text{valid for} \quad r < |z - z_0| < R$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds, \qquad k = 0, \pm 1, \pm 2, \cdots$$

C : a simple closed curve

that lies entirely within D

that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \quad \Longrightarrow \quad \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = \operatorname{Res}(f(z), z_0)$$

$$= \begin{cases} \lim\limits_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\[4mm] \frac{1}{(n-1)!} \lim\limits_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$
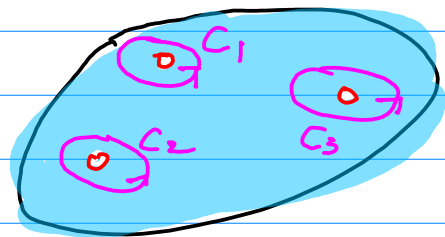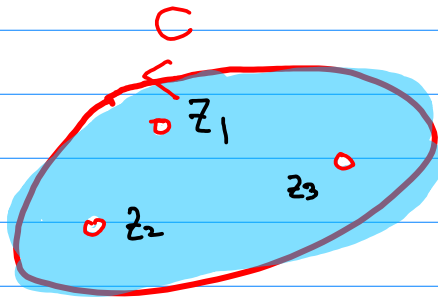
Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

$z_1, z_2, \cdots, z_n$ within C

then

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

D: a simply connected domain

C: a simple closed contour in D

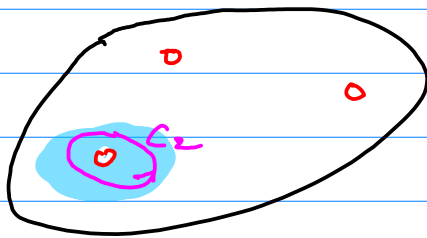|		
$\boxed{z_1}$	$f(z) = \displaystyle\sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$	$a_{-1}^{\{1\}} = \dfrac{1}{2\pi i} \displaystyle\oint_{C_1} f(s)\, ds = \text{Res}(f(z), z_1)$
$\boxed{z_2}$	$f(z) = \displaystyle\sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$	$a_{-1}^{\{2\}} = \dfrac{1}{2\pi i} \displaystyle\oint_{C_2} f(s)\, ds = \text{Res}(f(z), z_2)$
$\boxed{z_3}$	$f(z) = \displaystyle\sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$	$a_{-1}^{\{3\}} = \dfrac{1}{2\pi i} \displaystyle\oint_{C_3} f(s)\, ds = \text{Res}(f(z), z_3)$

$\boxed{z_1}$
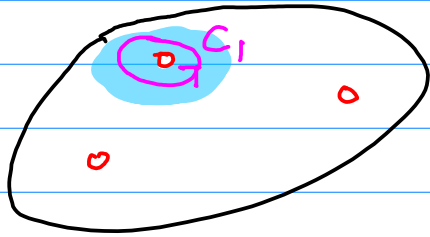
Laurent series expansion at z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{(1)} = \frac{1}{2\pi i} \oint_{C_1} f(s) \, ds = \text{Res}(f(z), z_1)$$
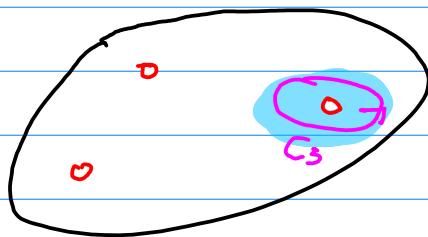
$\boxed{z_2}$

Laurent series expansion at z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1}^{(2)} = \frac{1}{2\pi i} \oint_{C_2} f(s) \, ds = \text{Res}(f(z), z_2)$$
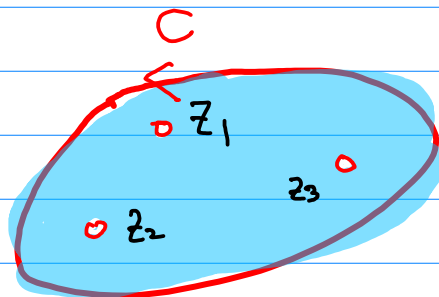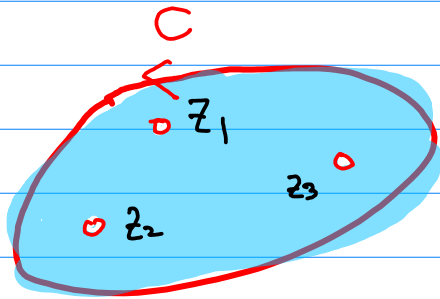
$\boxed{z_3}$

Laurent series expansion at z_3
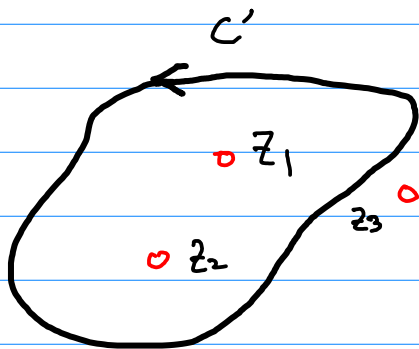
$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

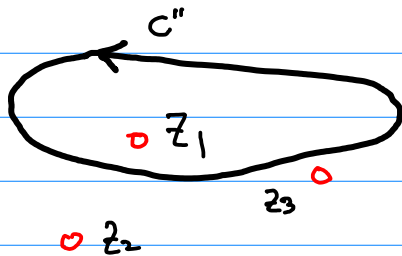$$a_{-1}^{(3)} = \frac{1}{2\pi i} \oint_{C_3} f(s) \, ds = \text{Res}(f(z), z_3)$$

C

z_1

z_3

z_2

$$\int_C f(z) \, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

C

z_1

z_3

z_2

$$\int_c f(z)\,dz = 2\pi i \; \text{Res}(f(z), z_1)$$
$$+ \; 2\pi i \; \text{Res}(f(z), z_2)$$
$$+ \; 2\pi i \; \text{Res}(f(z), z_3)$$

C'

z_1

z_3

z_2

$$\int_{c'} f(z)\,dz = 2\pi i \; \text{Res}(f(z), z_1)$$
$$+ \; 2\pi i \; \text{Res}(f(z), z_2)$$

C''

z_1

z_3

z_2

$$\int_{c''} f(z)\,dz = 2\pi i \; \text{Res}(f(z), z_1)$$

C'''

z_1

z_3

z_2

$$\int_{c''} f(z)\,dz = 0$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$
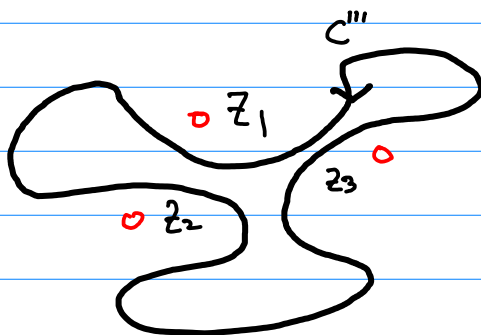
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

C is in the same region of analyticity of $f(z)$
   typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{\{m\}}$ depends on $f(z)$, z_m, region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not
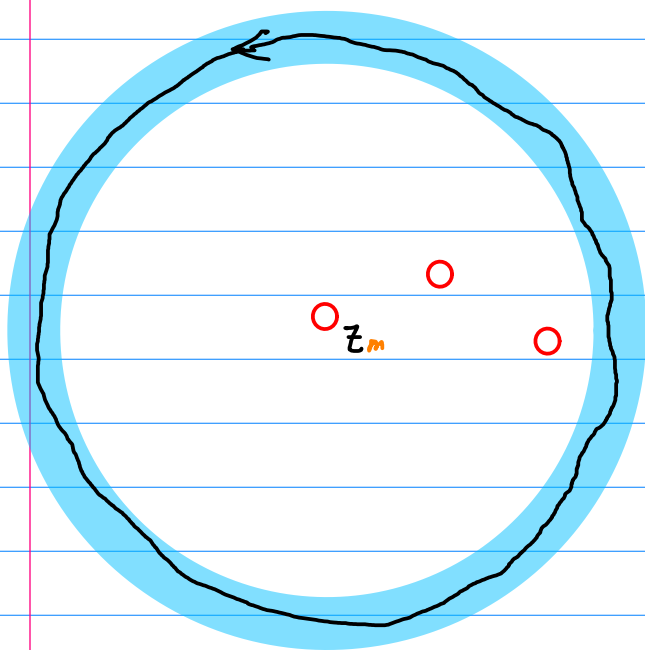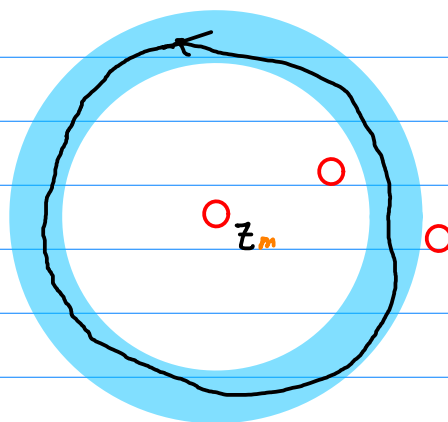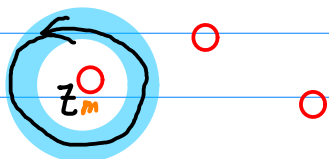   or at other points between z and z_m
We can expand $f(z)$ about any point z_m
         over powers of $(z - z_m)$.

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_n\right)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_n \right)$$

analytic at z_m

$n_1 \geqslant 0$ Taylor Series

general n_1 $z_m = 0$ MacLaurin Series

Singular at z_m

general n_1 Laurent Series

general n_1 $z_m = 0$ z - Transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k\right)$$

$$z_m = 0 \qquad a_{-n}^{\{0\}} = h(n) \qquad n \to -n$$

$$H(z) = \sum_{n=n_1}^{\infty} h(-n) \, z^n$$

$$h(n) = \frac{1}{2\pi i} \oint_C \frac{H(z')}{z'^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{H(z)}{z^{n+1}}, z_k\right)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \, z^{-n}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z') \, z'^{n-1} \, dz'$$

$$= \sum_k \text{Res}\left(H(z) \, z^{n-1}, z_k\right)$$

C is in the same region of analyticity of $f(z)$
typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$
typically a circle centered on z_m

generally a circle centered on the origin
may enclose any or all singularities of $H(z)$
often the unit circle

z_k within C : singularities of $H(z) z^{n-1}$