Relationship between Power Spectrum and Autocorrelation Function

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = A[R_{XX}(t,t+\tau)]$$

$$S_{XX}(\omega) = \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{+T} X(t_1)e^{+j\omega t_1}dt_1 \int_{-T}^{+T} X(t_2)e^{-j\omega t_2}dt_2\right]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T} E\left[X(t_1)X(t_2)\right]e^{-j\omega(t_2-t_1)}dt_2 dt_1$$

Definition

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} E\left[X(t_1)X(t_2)\right] e^{-j\omega(t_2-t_1)} dt_2 dt_1$$

$$E\left[X(t_1)X(t_2)\right] = R_{XX}(t_1, t_2)$$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_2 dt_1$$

Definition

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \right\} e^{+j\omega\tau} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1$$

Definition

$$\int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T}\int_{-T}^{+T} \{R_{XX}(t_1, t_2)\delta(t_1 - t_2 - \tau)\} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t+\tau) dt \qquad -T < t + \tau <$$

Definition

$$2\pi\delta(\tau - t_1 + t_2) = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{+j\omega t}d\omega$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\int_{-T}^{+T}\{R_{XX}(t_2,t_1)\delta(t_1 - t_2 - \tau)\}\,dt_2dt_1$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{+j\omega t}d\omega = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} R_{XX}(t,t+\tau)dt \qquad -T < t+\tau <$$

Definition

$$A[R_{XX}(t,t+\tau)] = \lim_{T\to\infty} \int_{-T}^{+T} R_{XX}(t,t+\tau)dt$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t,t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t,t+\tau)] \Longleftrightarrow S_{XX}(\omega)$$

Definition

$$A[R_{XX}(t, t+\tau)] = \lim_{T \to \infty} \int_{-T}^{+T} R_{XX}(t, t+\tau) dt = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$$