

# Relationship between Power Spectrum and Autocorrelation Function

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi



# AutoCorrelation Function

$N$  Gaussian random variables

## Definition

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{+j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} E \left[ \frac{1}{2T} \int_{-T}^{+T} X(t_1) e^{+j\omega t_1} dt_1 \int_{-T}^{+T} X(t_2) e^{-j\omega t_2} dt_2 \right]$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} E[X(t_1)X(t_2)] e^{-j\omega(t_2-t_1)} dt_2 dt_1$$

# AutoCorrelation and Expectation

$N$  Gaussian random variables

## Definition

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} E[X(t_1)X(t_2)] e^{-j\omega(t_2-t_1)} dt_2 dt_1$$

$$E[X(t_1)X(t_2)] = R_{XX}(t_1, t_2)$$

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2-t_1)} dt_2 dt_1$$

# Inverse Transform (1)

$N$  Gaussian random variables

## Definition

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) e^{-j\omega(t_2 - t_1)} dt_2 dt_1 \right\} e^{+j\omega \tau} d\omega$$

$$= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau - t_1 - t_2)} d\omega \right\} dt_2 dt_1$$

# Inverse Transform (2)

$N$  Gaussian random variables

## Definition

$$\int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1-t_2)} d\omega = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} R_{XX}(t_1, t_2) \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{+j\omega(\tau-t_1-t_2)} d\omega \right\} dt_2 dt_1$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} \{ R_{XX}(t_1, t_2) \delta(t_1 - t_2 - \tau) \} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t+\tau) dt \quad -T < t+\tau < +T$$

# Impulse Function

$N$  Gaussian random variables

## Definition

$$2\pi\delta(\tau - t_1 + t_2) = 2\pi\delta(t_1 - t_2 - \tau)$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-T}^{+T} \{R_{XX}(t_2, t_1) \delta(t_1 - t_2 - \tau)\} dt_2 dt_1$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt \quad -T < t + \tau < T$$



# Time Average of an Auto-Correlation

$N$  Gaussian random variables

## Definition

$$A[R_{XX}(t, t + \tau)] = \lim_{T \rightarrow \infty} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$$

# Time Average of an Auto-Correlation : a WSS case

$N$  Gaussian random variables

## Definition

$$A[R_{XX}(t, t + \tau)] = \lim_{T \rightarrow \infty} \int_{-T}^{+T} R_{XX}(t, t + \tau) dt = R_{XX}(\tau)$$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

$$R_{XX}(\tau) \iff S_{XX}(\omega)$$



