CLTI System Response (4A)

-

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

ODE's and Causal LTI Systems

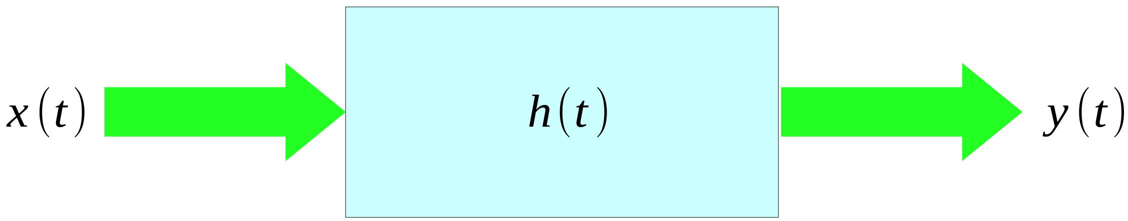
$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0 y(t) \;=\; b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0 x(t)$$

N: the highest order of derivatives of the output y(t) (LHS)

M: the highest order of derivatives of the input x(t) (RHS)

N < M : (M-N) differentiator – magnify high frequency components of noise (seldom used)

N > M : (N-M) Integrator



Different Indexing Schemes

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\,y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_1 \frac{d\,x(t)}{dt} + b_0 x(t)$$

$[N > M]$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,x(t)}{dt} + b_M x(t)$$

$[N = M]$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$[N > M]$ $\quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{M-1} D + b_M)\, x(t)$

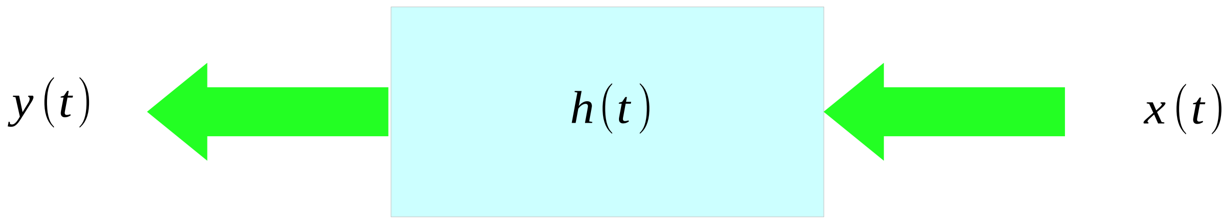
$[N = M]$ $\quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$

ODE Solutions and System Responses

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^N x(t)}{dt^N} + b_1 \frac{d^{N-1} x(t)}{dt^{N-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

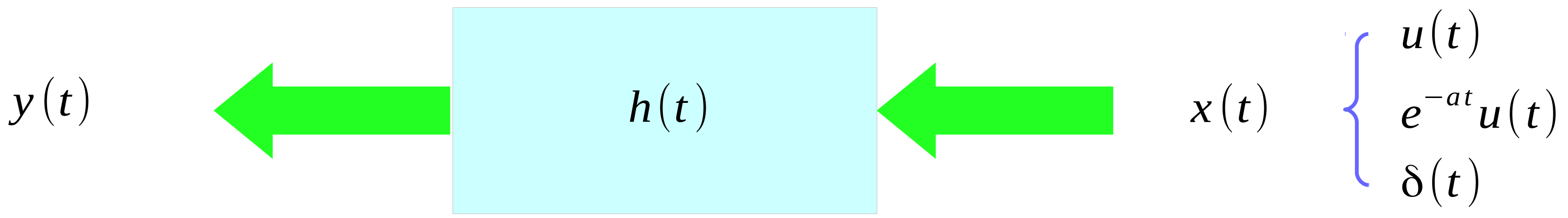
$$Q(D)\, y(t) = P(D)\, x(t)$$



- Zero Input Response
- Zero State Response (Convolution with h(t))

- Natural Response (Homogeneous Solution)
- Forced Response (Particular Solution)

ODEs with u(t) & δ(t)



Causal Output

$y(t) = 0 \quad (t \le 0)$

Causal System

$h(t) = 0 \quad (t \le 0)$

Causal Input

$x(t) = 0 \quad (t \le 0)$

Associated ODE

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_h(t) = 0$$

Homogeneous Solution

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_p(t) = 1 \Leftarrow u(t)$$

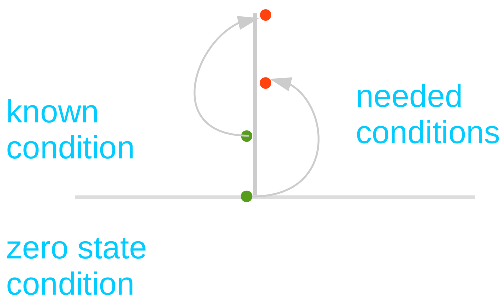
$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_p(t) = 1 \Leftarrow e^{-at}u(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_p(t) = 0 \Leftarrow \delta(t)$$

Particular Solution

Responses to a causal input

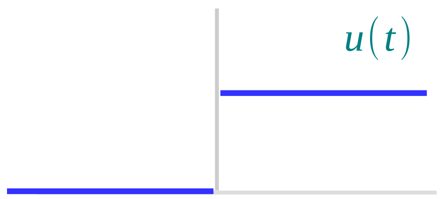
OUTPUT
Initial Conditions



possible jumps in
initial conditions at t = 0

INPUT

$$\begin{cases} u(t) \\ e^{-at}u(t) \\ \delta(t) \end{cases}$$



Direct Inspection	finding any jumps in initial conditions
Balancing Singularities	matching derivatives of singularity functions in the both sides

Discontinuous Initial Conditions

$$y'' + 3y' + 2y = x(t)$$

$$y(0^-) = y(0^+)$$

$$y'(0^-) = y'(0^+)$$

continuous initial conditions (the same)

$$y'' + 3y' + 2y = b_1 x'(t) + b_2 x(t)$$

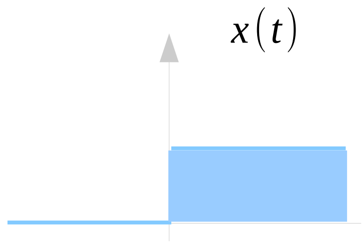
$$y'' + 3y' + 2y = b_0 x''(t) + b_1 x'(t) + b_2 x(t)$$

$$y(0^-) \neq y(0^+)$$

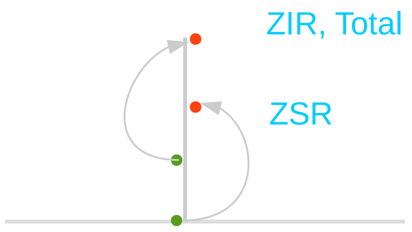
$$y'(0^-) \neq y'(0^+)$$

possible discontinuity

Direct Inspection



causal x(t)

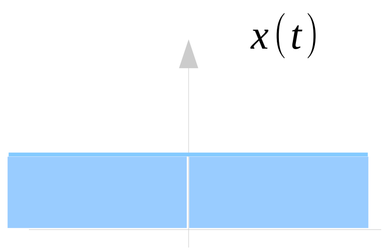


possible jumps in
nitial conditions at t = 0

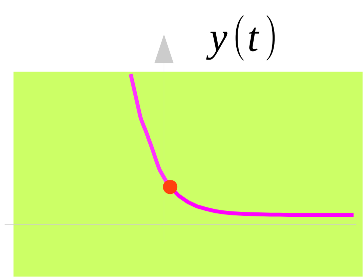
Direct Inspection

$y(0^-)$	$y(0^+)$
$\dot{y}(0^-)$	$\dot{y}(0^+)$
$\ddot{y}(0^-)$	$\ddot{y}(0^+)$

finding any jumps in
initial conditions



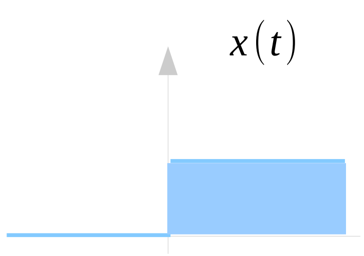
non-causal x(t)



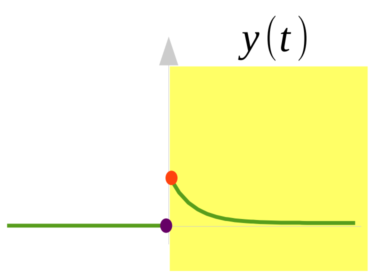
non-causal y(t)

Get the solution for
the IVP with new
initial conditions

a linear ode with
constant coefficients has
a continuous solution



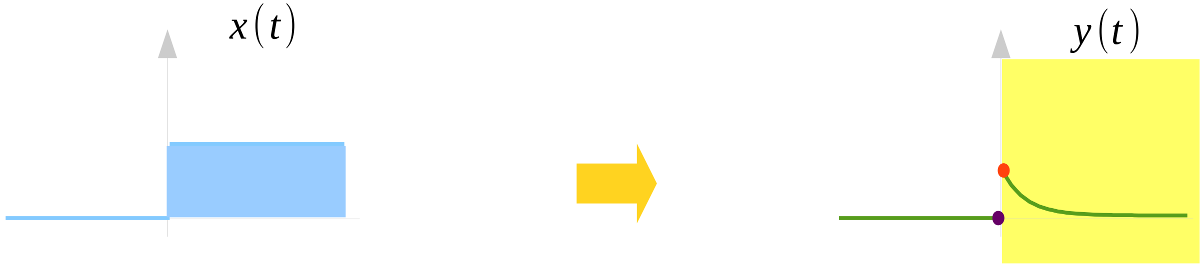
causal x(t)



causal y(t)

Taking only the part
where t > 0.

Balancing Singularities



causal x(t) causal y(t)

Not explicitly converting initial conditions

$$(y_h + y_p)\,u(t)$$

$$u(t) \qquad e^{-at}u(t) \qquad \delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\,y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\,u(t)$$

matching derivatives of singularity functions in the both sides

ODE Solutions and System Responses

A general solution of an ODE

Linear ODE with constant coefficients

$y_h(t) = \sum_i c_i e^{\lambda}$ — Characteristic Mode Terms — $-\infty < t < +\infty$

$y_p(t) \Leftarrow x(t)$ — Similar form as the input x(t) — $-\infty < t < +\infty$

A system response

lumped characteristic mode terms

$y_n(t)$ — Natural response: $y_h(t)$ with all coefficients determined

$y_p(t)$ — Forced response

different coefficients

independent of the causal input x(t)

$y_{zi}(t)$ — Zero input response: $y_h(t)$ with all coefficients determined

$y_{zs}(t)$ — Zero state response

System Responses and Causality

$y_n(t)$	Natural response	~~$y_n(t) = 0 \quad (t < 0)$~~	not explicitly assumed
$y_p(t)$	Forced response	~~$y_p(t) = 0 \quad (t < 0)$~~	
$y_{zi}(t)$	Zero input response	$y_{zi}(t) = 0 \quad (t < 0)$	Causal Input Causal System Causal Output
$y_{zs}(t)$	Zero state response	$y_{zs}(t) = 0 \quad (t < 0)$	

$$y_{zi}(t) + y_{zs}(t)$$

$$y_n(t) + y_p(t)$$

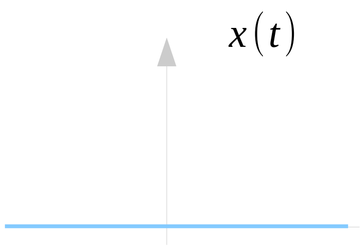
Only considering
responses after $t = 0$ $(t > 0)$
to the applied input x(t) $(t \geq 0)$

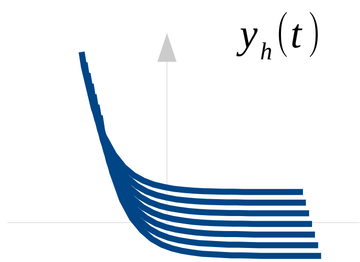
Not considering
responses before $t = 0$ $(t < 0)$

Natural & Forced Responses in terms of y_h & y_p

Natural

assumed solution form

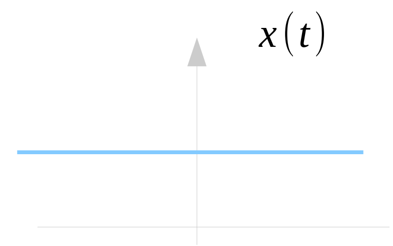


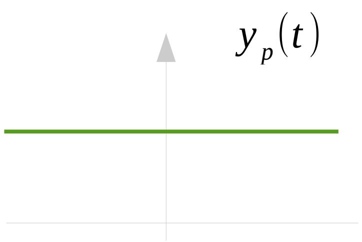


zero input x(t)

Forced

assumed solution form



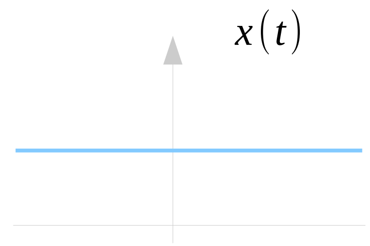


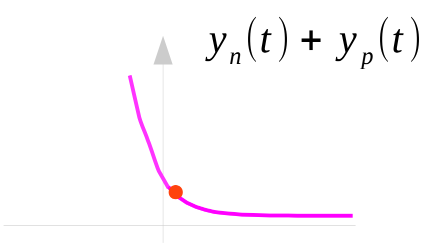
non-causal x(t)

non-causal y(t)

causal x(t)	may create discontinuous initial conditions at t=0+
natural response	is obtained by these initial conditions at t=0+
	however, the obtained y_n and y_p are implicitly assumed non-causal

Total





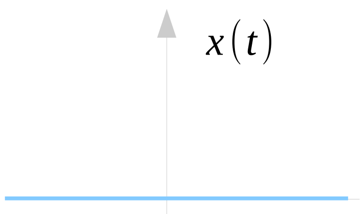
non-causal x(t)

non-causal y(t)

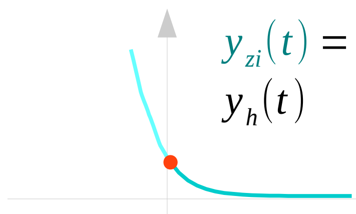
$y(0^-)$	$y(0^+)$
$\dot{y}(0^-)$	$\dot{y}(0^+)$
$\ddot{y}(0^-)$	$\ddot{y}(0^+)$

ZIR & ZSR in terms of y_h & y_p

ZIR

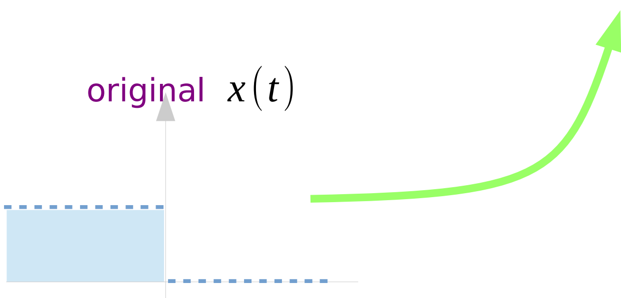


assumed solution form



$y_{zi}(t) =$
$y_h(t)$

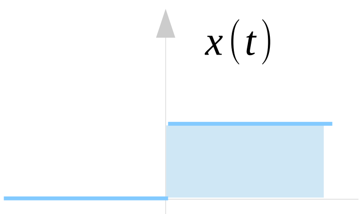
original $x(t)$



The effect of $x(t)$ $(t < 0)$ is summarized as a ZIR with initial conditions at t=0+

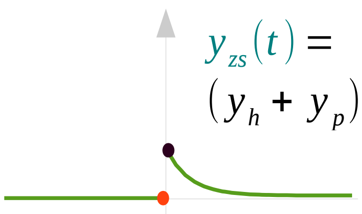
$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$$

ZSR



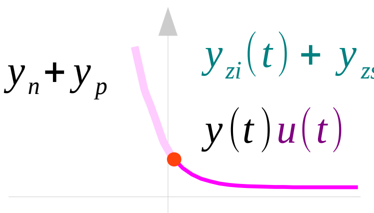
causal x(t)

assumed solution form



$y_{zs}(t) =$
$(y_h + y_p) \cdot u(t)$

original $y_n + y_p$



$y_{zi}(t) + y_{zs}(t) =$
$y(t)u(t)$

causal x(t)

$$h(t) * x(t) = \left(b_0 \delta(t) + \sum_i d_i e^{\lambda_i t} \right) * x(t)$$

$$u(t) \cdot (y_h + y_p) = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

* Valid Intervals

Valid Interval of Laplace Transform

$f(t) \iff F(s) = \int_{0^-}^{\infty} f(t) e^{st}\, dt$

$f'(t) \iff sF(s) - f(0^-)$

$\int_{0^-}^{\infty}$

$0^- < t < \infty$

$f(0^-)$

Valid Intervals of ZIR & ZSR Laplace Transform

$$y'' + 3y' + 2y = x(t) \qquad y(0^-) = k_1, \quad y'(0^-) = k_2$$

$$0^- < t < \infty$$

$$\left[s^2 Y(s) - s\,y(0^-) - y'(0^-)\right] + 3\left[s\,Y(s) - y(0^-)\right] + 2\,Y(s) = X(s)$$

Zero Input Response

$$x(t) = 0$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)}$$

$$= +4\frac{1}{(s+1)} - 3\frac{1}{(s+2)}$$

$$\leftrightarrow \quad y = 4e^{-t} - 3e^{-2t}$$

Zero State Response

$$y(0^-) = 0, \quad y'(0^-) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)} \qquad x(t) = e^{+t}$$

$$= +\frac{1}{3}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\leftrightarrow \quad y = -\frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^{+t}$$

Valid Interval of ZIR & ZSR IVPs

$$y'' + a\,y' + b\,y = f(x)$$
$$y(0^-) = y_0$$
$$y'(0^-) = y_1$$

$$y = y_h + y_p$$

$$0^- < t < \infty$$

$$y(0^-) = y_h(0^-) + y_p(0^-) = y_0 + 0 = y_0$$
$$y'(0^-) = y_h'(0^-) + y_p'(0^-) = y_1 + 0 = y_1$$

$$y'' + a\,y' + b\,y = 0$$
$$y(0^-) = y_0$$
$$y'(0^-) = y_1$$

$$y_h$$

Nonzero Initial Conditions

Zero Input Response

Response due to the initial conditions

$$y'' + a\,y' + b\,y = f(x)$$
$$y(0^-) = 0$$
$$y'(0^-) = 0$$

$$y_h + y_p$$

Zero Initial Conditions

Zero State Response

Response due to the forcing function f

Valid Intervals of Convolution Method

$x(t) \rightarrow$ [$h(t)$] $\rightarrow y(t)$

causal system h(t)

non-causal input x(t)
is also permissible

$$\int_{-\infty}^{t} x(v)\,h(t-v)\,dv \;=\; y(t)$$

$$t - v \;\geq\; 0$$

causal system restriction

$$\int_{-\infty}^{t} h(v)\,x(t-v)\,dv \;=\; y(t)$$

$$t - v \;\geq\; 0$$

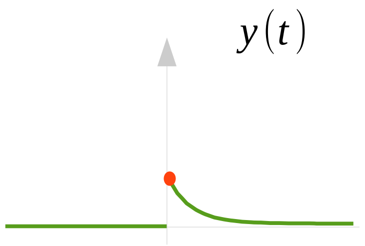
$h(t) \rightarrow$ [$x(t)$] $\rightarrow y(t)$

causal system h(t)

non-causal input h(t)
is also permissible

Valid Intervals of System Responses

ZIR + ZSR



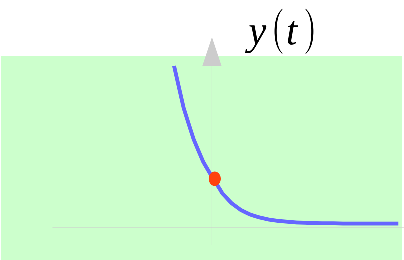
$$0^- < t < \infty$$

General Assumption

$x(t) = 0 \quad (t < 0)$	Causal Input
$h(t) = 0 \quad (t < 0)$	Causal System
$y(t) = 0 \quad (t < 0)$	Causal Output

x(t) * h(t)

non-causal x(t)



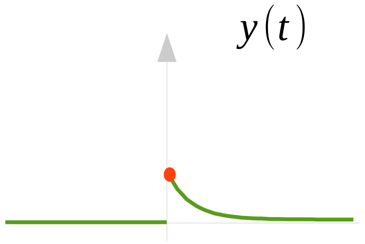
$$-\infty < t < \infty$$

General Assumption

$h(t) = 0 \quad (t < 0)$ Causal System

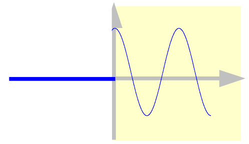
Causal and Everlasting Exponential Inputs

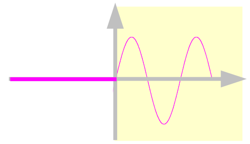
ZIR + ZSR



$$0^- < t < \infty$$

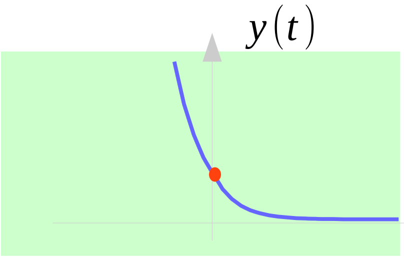
Suitable for causal exponential inputs





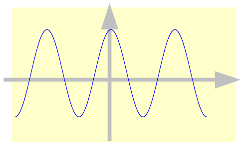
x(t) * h(t)

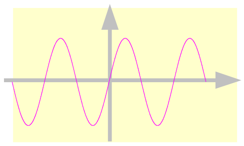
non-causal x(t)



$$-\infty < t < \infty$$

Suitable for everlasting exponential inputs





The effect of an input for t < 0

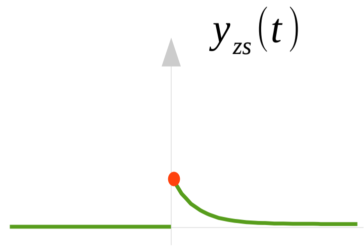
ZIR + ZSR

causal x(t)

$x(t) = 0 \quad (t < 0)$

$h(t) = 0 \quad (t < 0)$

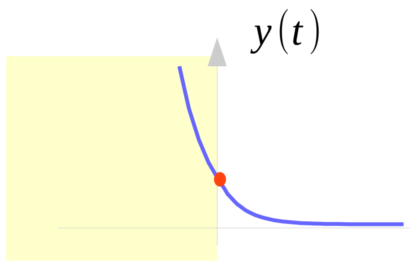
$y(t) = 0 \quad (t < 0)$

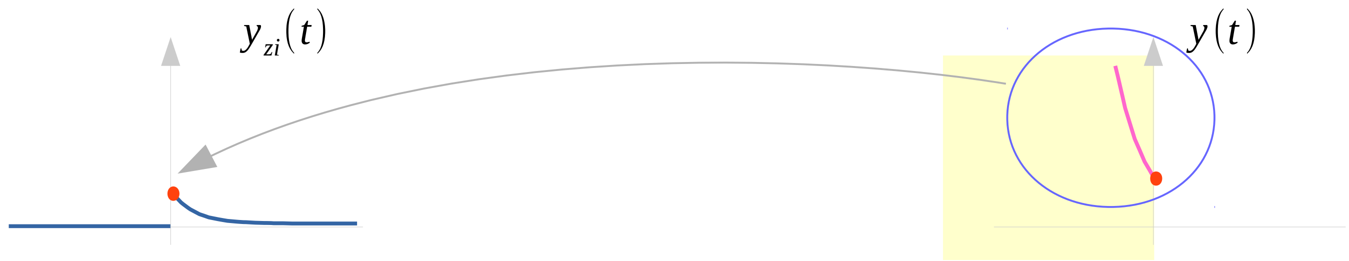


x(t) * h(t)

non-causal x(t)

$h(t) = 0 \quad (t < 0)$





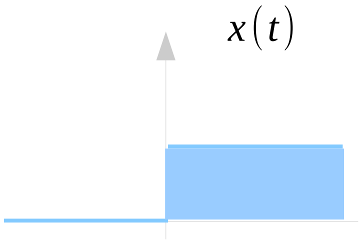
The effect of $x(t) \quad (t < 0)$

is summarized as a ZIR with initial conditions at t=0+

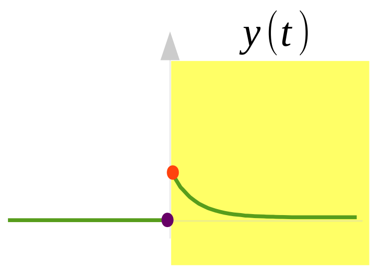
Causal and Non-causal Signals

Total = ZSR + ZIR

together with Laplace transform,
the time interval [0, ∞) is assumed



causal x(t)

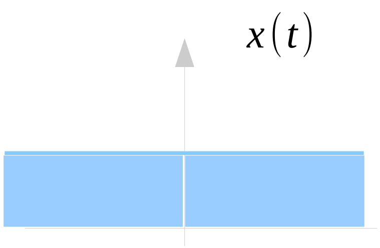


causal y(t)

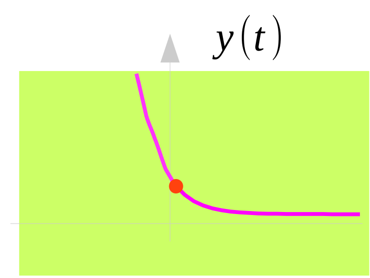
$$0^- < t < \infty$$

x(t) * h(t)

No restriction in the time interval
But by considering only [0, ∞) time interval, this will be the same as the above



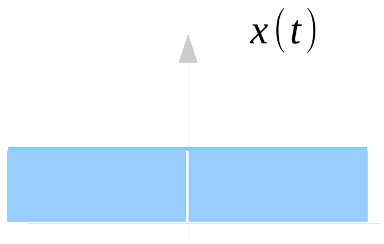
non-causal x(t)



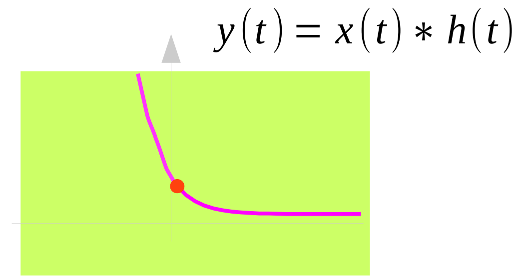
non-causal y(t)

$$-\infty < t < \infty$$

Getting causal signals

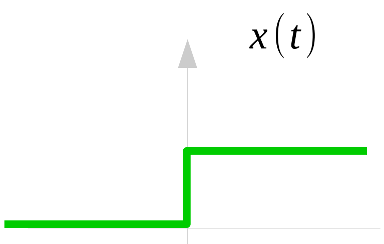


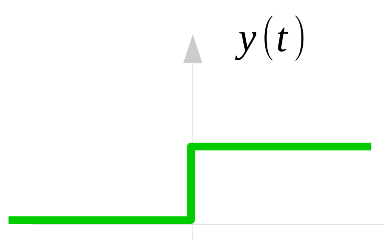
non-causal x(t)

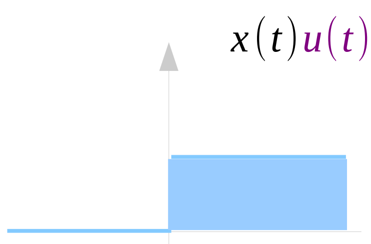


non-causal y(t)

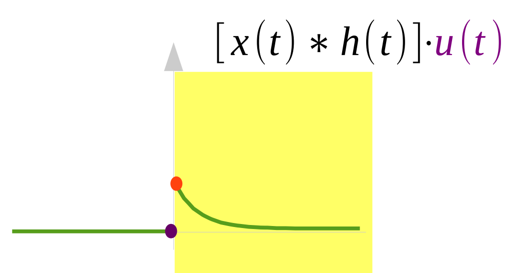
x(t) * h(t)







causal input

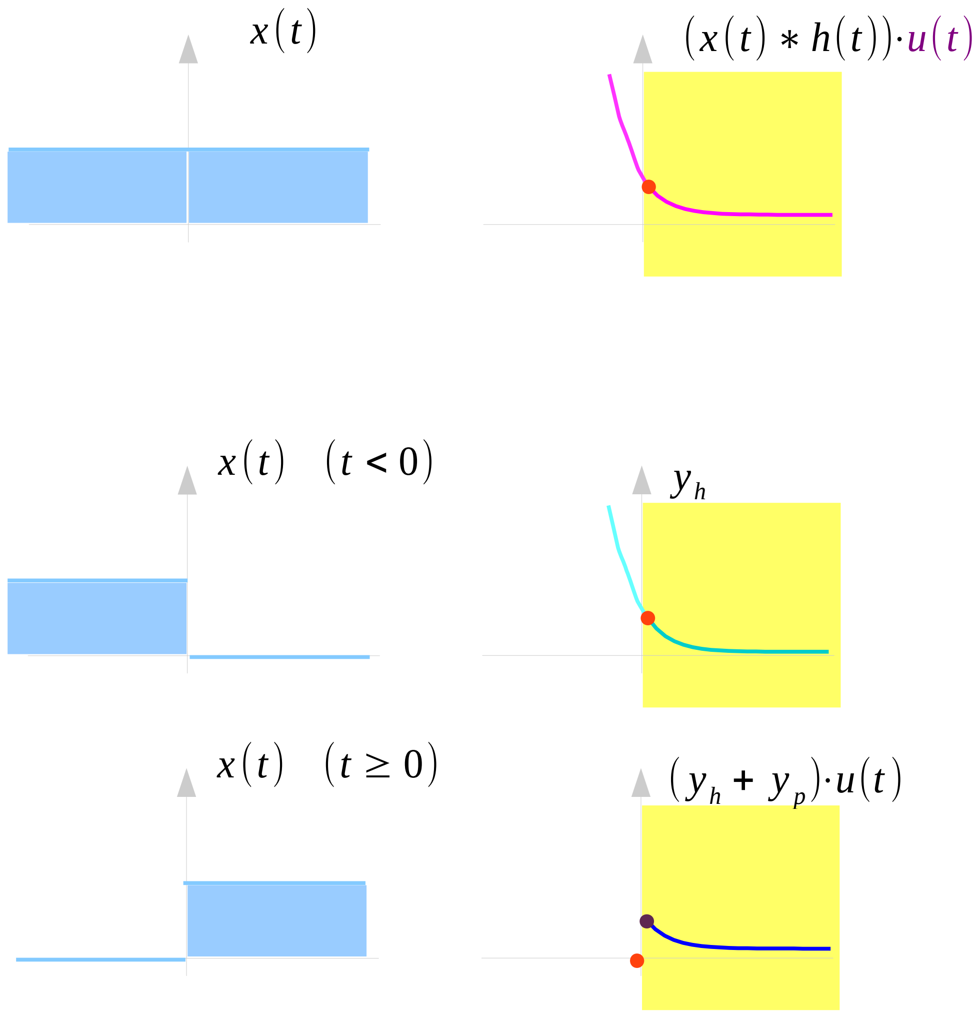


causal output

Total = (x(t) * h(t)) u(t)

non-causal x(t)

Partitioning a non-causal input



Total = (x(t) * h(t)) u(t)

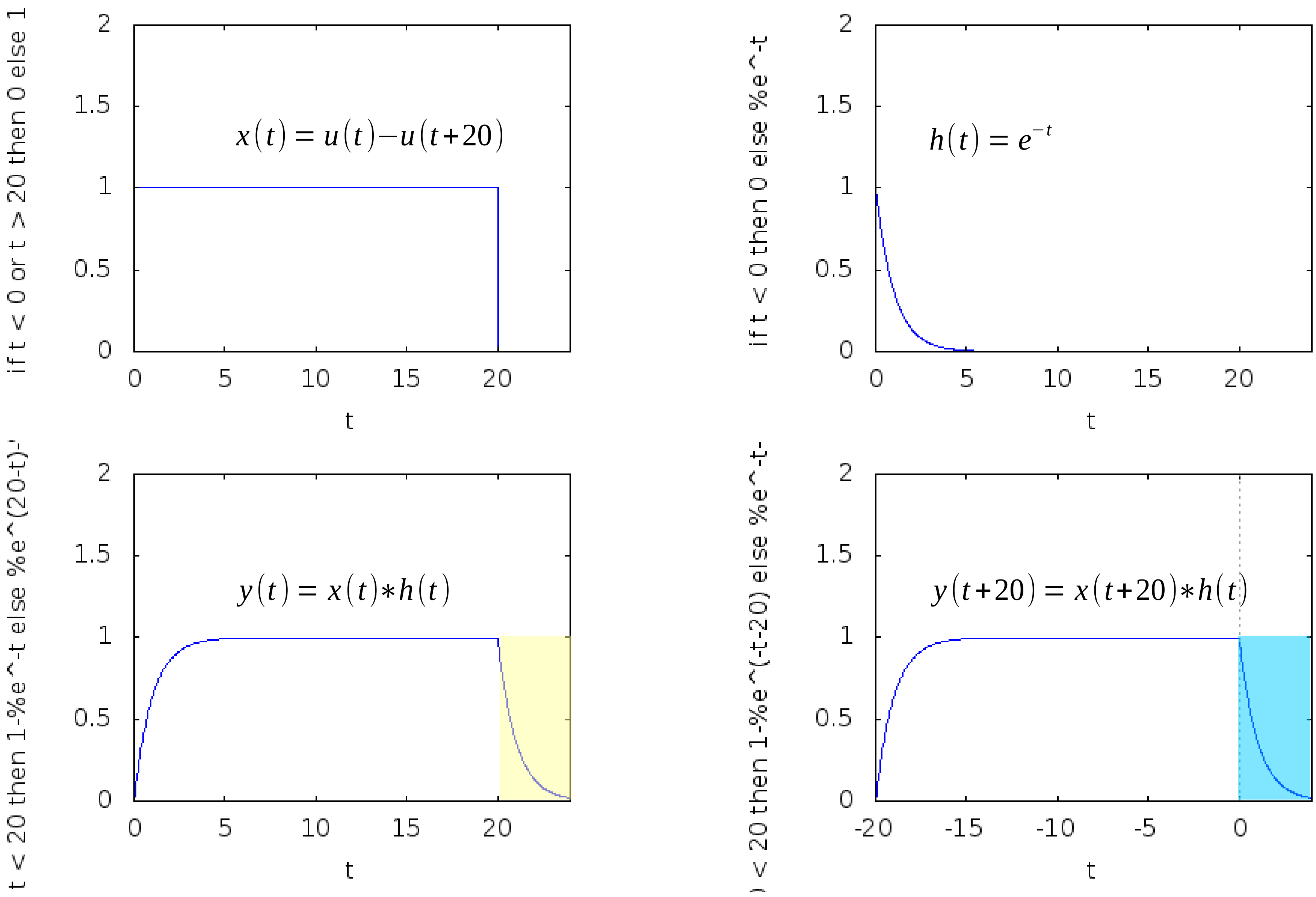
non-causal x(t)

ZIR

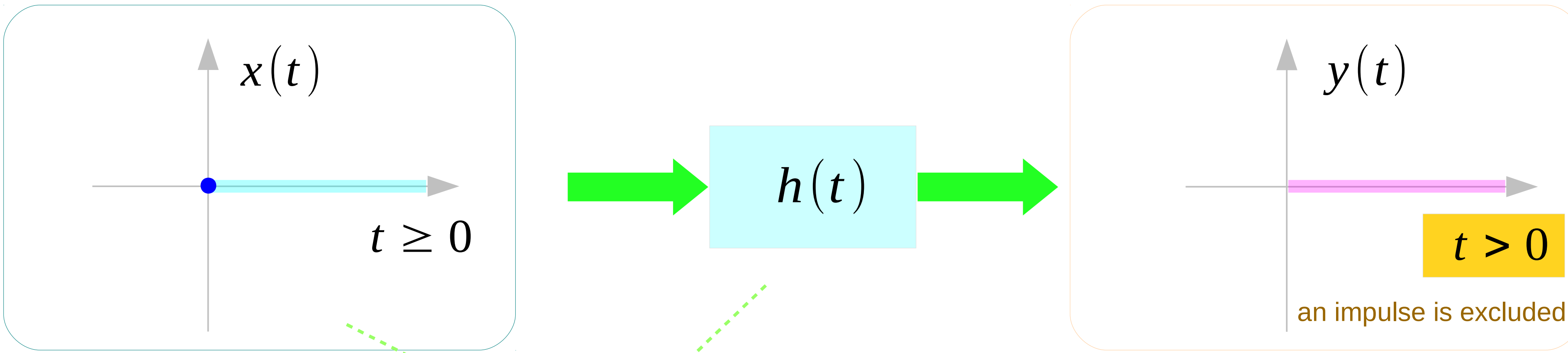
The effect of $x(t) \quad (t < 0)$ is summarized as a ZIR with initial conditions at t=0+

ZSR

Effects of past input



Interval of Validity of a System Response



causal x(t)

causal y(t)

an impulse can be present in *x(t)* and *h(t)* at *t = 0*

$\delta(t)$

discontinuity at *t = 0*
creates initial conditions

$$x^{(N-1)}(0^-) \quad x^{(N-2)}(0^-) \quad \vdots \quad x^{(1)}(0^-) \quad x(0^-)$$

The initial conditions in the term “**Zero State**” refers this initial conditions

$$y^{(N-1)}(0^-) \quad y^{(N-2)}(0^-) \quad \vdots \quad y^{(1)}(0^-) \quad y(0^-)$$

$$y^{(N-1)}(0) \quad y^{(N-2)}(0) \quad \vdots \quad y^{(1)}(0) \quad y(0)$$

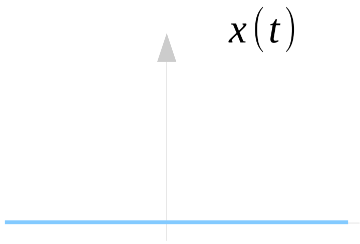
$$y^{(N-1)}(0^+) \quad y^{(N-2)}(0^+) \quad \vdots \quad y^{(1)}(0^+) \quad y(0^+)$$

Usually known IC

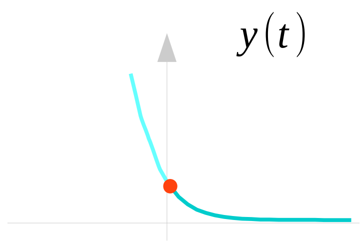
Needed IC

Initial Conditions for ZIR & ZSR

ZIR



zero input x(t)

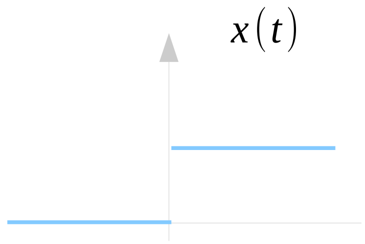


non-causal y(t)

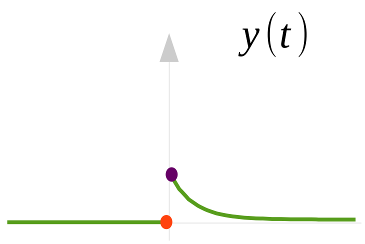
$$\begin{aligned} y(0^-) &= y(0^+) \\ \dot{y}(0^-) &= \dot{y}(0^+) \\ \ddot{y}(0^-) &= \ddot{y}(0^+) \end{aligned}$$

continuous, but
not all zero

ZSR



causal x(t)



causal y(t)

$$\begin{matrix} y(0^-) & \quad & y(0^+) \\ \dot{y}(0^-) & & \dot{y}(0^+) \\ \ddot{y}(0^-) & & \ddot{y}(0^+) \end{matrix}$$

all zero

not all zero

possible
jumps

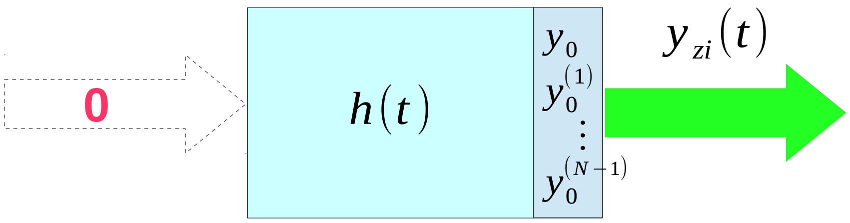
- System Response

(a) Zero Input Response
(b) Zero State Response
(c) Natural Response
(d) Forced Response

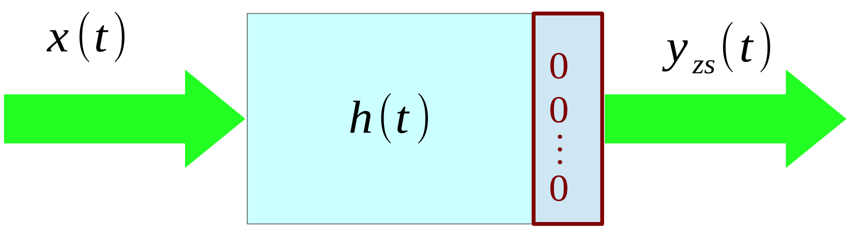
$x(t) = 0 \quad (t < 0)$	Causal Input
$h(t) = 0 \quad (t < 0)$	Causal System
$y(t) = 0 \quad (t < 0)$	Causal Output

Types of System Responses

- **Zero Input Response** State only



- **Zero State Response** Input only



- **Natural Response** Homogeneous

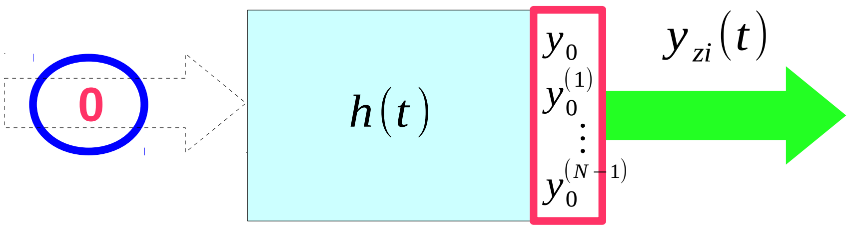
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = 0$$

- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)$$

Comparison of System Responses (1)

- **Zero Input Response** State only



Response of a system when the input x(t) is zero (no input)

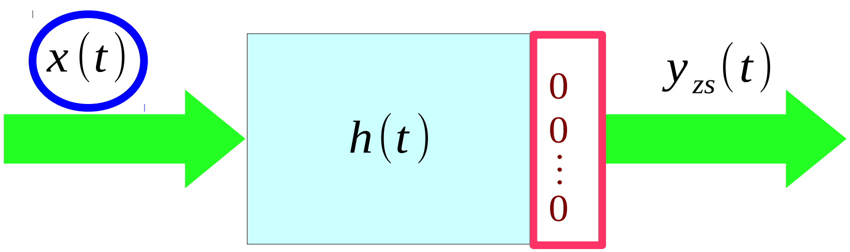
≠

- **Natural Response** Homogeneous

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = 0$$

Solution due to characteristic modes only

- **Zero State Response** Input only



Response of a system when system is at rest initially

≠

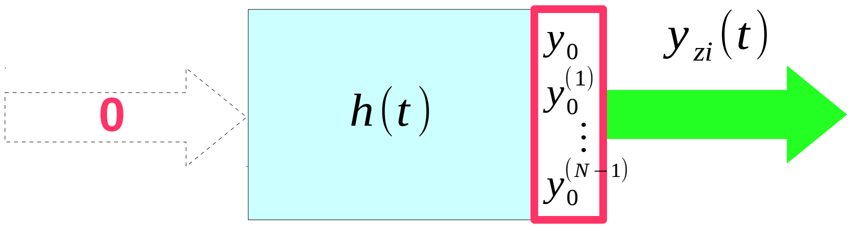
- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)$$

Solution <u>excluding</u> the effect of characteristic modes

Comparison of System Responses (2)

- **Zero Input Response** — State only



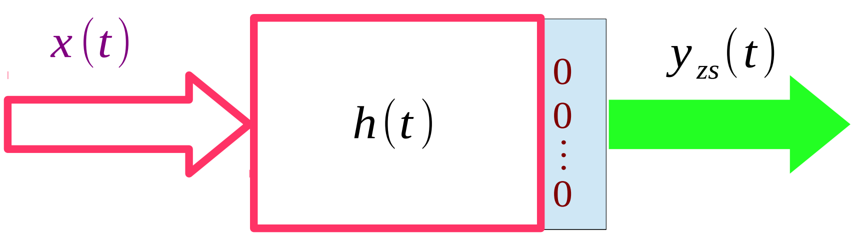
response to the initial conditions only

- **Natural Response** — Homogeneous

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = 0$$

all characteristic modes response

- **Zero State Response** — Input only



response to the input only

- **Forced Response** — Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)$$

non-characteristic mode response

Comparison of System Responses (3)

- **Zero Input Response** State only

response to the *initial conditions only*

$$\left\{ y^{(N-1)}(0^-),\ \cdots,\ y^{(1)}(0^-),\ y(0^-) \right\}$$

- **Natural Response** Homogeneous

all characteristic *modes* response

$$y_n(t) \ = \ \sum_i K_i e^{\lambda_i t}$$

- **Zero State Response** Input only

response to the *input* only causal x(t)

$$h(t) * x(t) \ = \ \left(b_0 \delta(t) + \sum_i d_i e^{\lambda_i t} \right) * x(t)$$

$$u(t) \cdot (y_h + y_p) \ = \ u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

- **Forced Response** Particular

non-characteristic mode response

$$y_p(t)$$

Forms of System Responses

- **Zero Input Response** State only

$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t} \quad \{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$$

determines the coefficients

- **Zero State Response** Input only

convolution form

Impulse matching

$$y_{zs}(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t)\right)$$

step function form

direct inspection
balancing singularities

$$y_{zs}(t) = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

$y_p(t)$ ◄ $x(t)$

- **Natural Response** Homogeneous

$$y_n(t) = \sum_i K_i e^{\lambda_i t}$$

the coefficients K_i's are determined by the initial conditions.

$$y_n(t) + y_p(t) \quad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$$

determines the coefficients

- **Forced Response** Particular

$$y_p(t) = \begin{cases} \beta e^{\xi t} & \text{or} \\ (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) \end{cases}$$

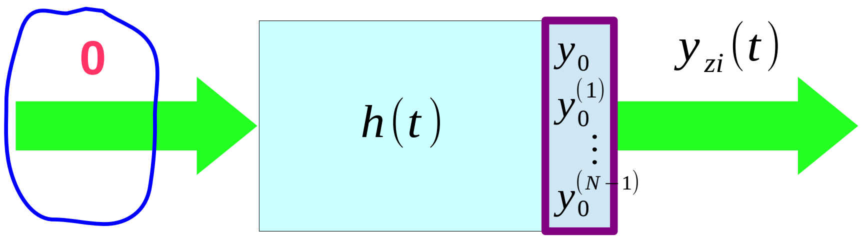
$y_p(t)$ similar to the input, with the coefficients determined by equating the similar terms

Young Won Lim
5/16/15

* Zero Input Response

Zero Input Response : $y_{zi}(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$



$x(t) = 0$

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = 0$$

$x(t) = 0$

$Q(D)\, y_{zi}(t) = 0$ ⇒ $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_{zi}(t) = 0$

⇒ linear combination of {y_{zi}(t) and its derivatives} = 0

⇒ $ce^{\lambda t}$ only this form can be the solution of $y_{zi}(t)$

$Q(\lambda) = 0$ ⇔ $(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)\; ce^{\lambda t} = 0$

$(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$, $ce^{\lambda t} \neq 0$

Young Won Lim
5/16/15

Characteristic Modes

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

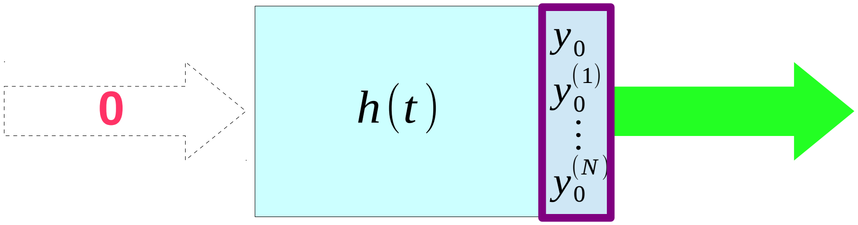
$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = \sum_i c_i e^{\lambda_i t}$$

λ_i characteristic roots

$e^{\lambda_i t}$ characteristic modes

ZIR: a linear combination of the characteristic modes of the system

$(1, a_1, \cdots, a_{N-1}, a_N)$



$(b_0, b_1, \cdots, b_{N-1}, b_N)$

$x(t) = 0$

$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$

the initial condition **before** t=0 is used

$$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$$

$$= \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$$

any input is applied at time t=0, but in the ZIR: the initial condition does not change before and after time t=0 since no input is applied

Young Won Lim
5/16/15

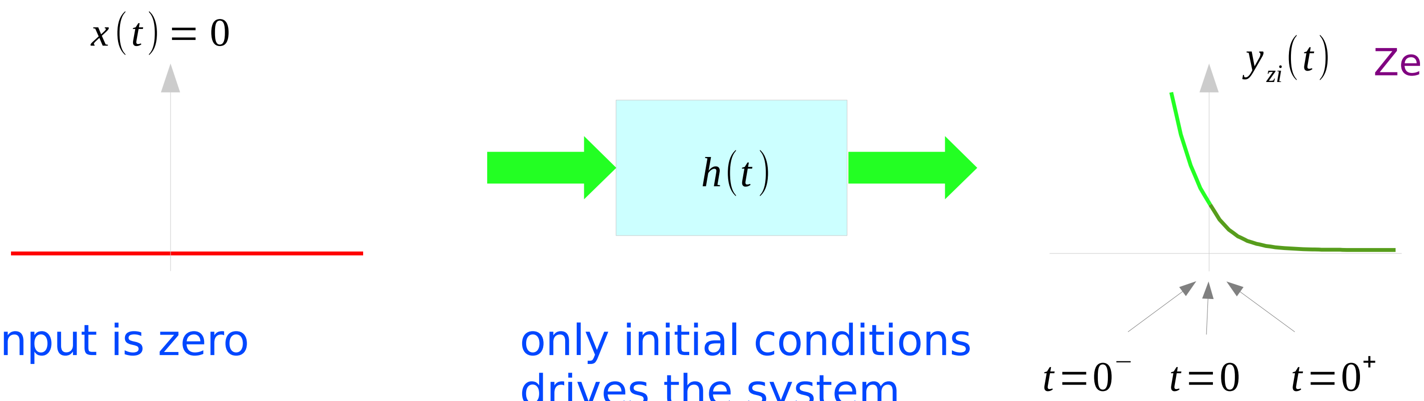
Zero Input Response IVP

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N\, y(t) = 0$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = 0$$

$$\begin{aligned}
y^{(N-1)}(0^-) &= y^{(N-1)}(0) = y^{(N-1)}(0^+) = k_{N-1} \\
y^{(N-2)}(0^-) &= y^{(N-2)}(0) = y^{(N-2)}(0^+) = k_{N-2} \\
&\vdots \\
y^{(1)}(0^-) &= y^{(1)}(0) = y^{(1)}(0^+) = k_1 \\
y(0^-) &= y(0) = y(0^+) = k_0
\end{aligned}$$

ZIR Initial Value Problem (IVP)



Zero Input Response

input is zero

only initial conditions drives the system

$$\begin{aligned}
y_{zi}(0^-) &= y_{zi}(0) = y_{zi}(0^+) \\
\dot{y}_{zi}(0^-) &= \dot{y}_{zi}(0) = \dot{y}_{zi}(0^+) \\
\ddot{y}_{zi}(0^-) &= \ddot{y}_{zi}(0) = \ddot{y}_{zi}(0^+)
\end{aligned}$$

- Zero State Response

Zero State Response y(t)

Zero State Response $y(t) = x(t)*h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$ Convolution

Impulse response $h(t)$

causal system *h(t)*: *response cannot begin before the input*

$h(t-\tau) = 0 \quad t-\tau < 0$

causal input *x(t)*: *the input starts at t=0*

$x(\tau) = 0 \quad \tau < 0$

Causality $y(t) = \int_{0^-}^{t} x(\tau)h(t-\tau)\,d\tau \quad , \quad t \geq 0$

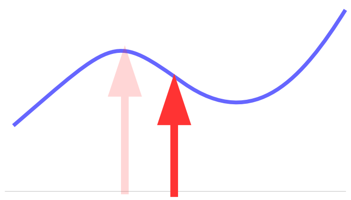
Delayed Impulse Response

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) \;=\; (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

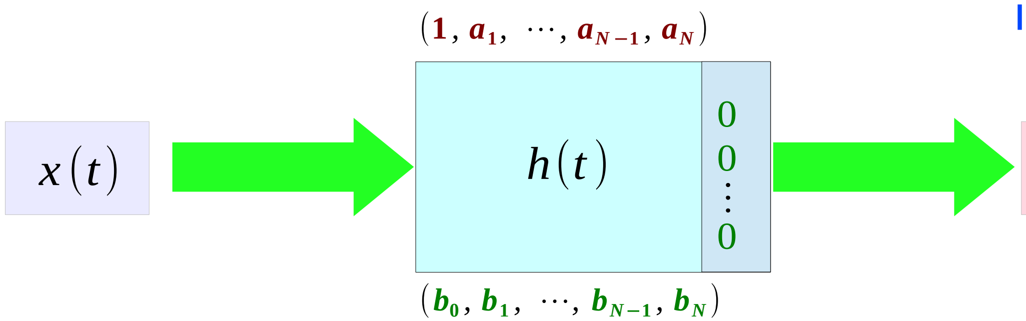
All initial conditions are zero

$$y^{(N-1)}(0^-) \;=\; \cdots \;=\; y^{(1)}(0^-) \;=\; y^{(0)}(0^-) \;=\; 0$$

superposition of inputs
– delayed impulse



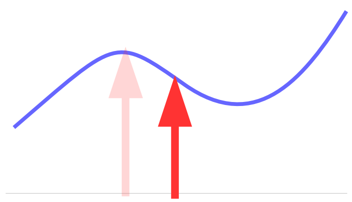
the sum of delayed
impulse responses

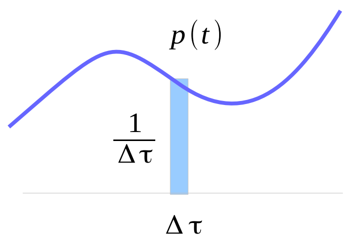


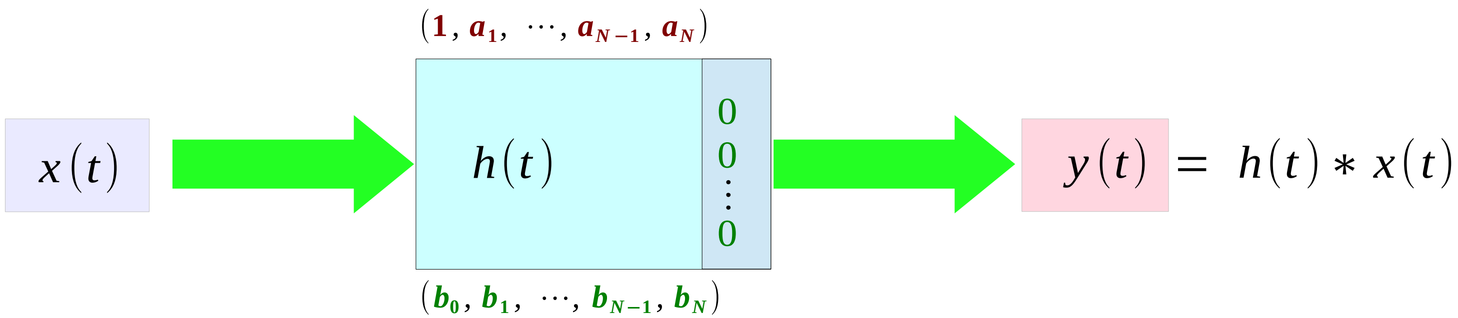
$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

scaling — delayed impulse response







$$x(t) = \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\, p(t - n\Delta\tau)$$

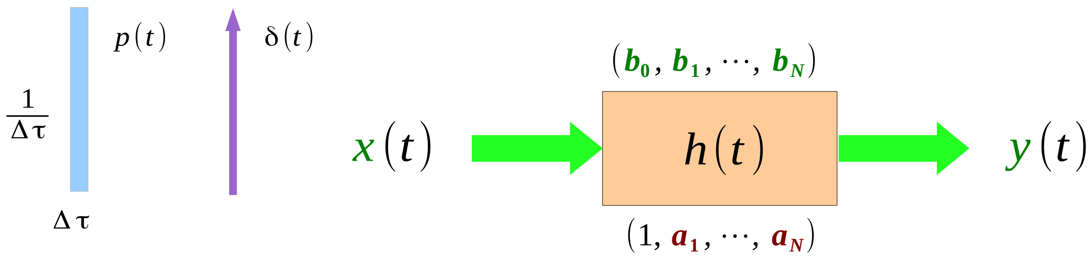
$$= \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau) \frac{p(t - n\Delta\tau)}{\Delta\tau} \Delta\tau$$

$$= \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\, \delta(t - n\Delta\tau) \Delta\tau$$

$$y(t) = \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\, h(t - n\Delta\tau) \Delta\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau$$

Pulse and Impulse Response



$$\delta(t) \quad \longrightarrow \quad h(t)$$

$$\delta(t - n\Delta\tau) \quad \longrightarrow \quad h(t - n\Delta\tau)$$

$$x(n\Delta\tau)\,\delta(t - n\Delta\tau)\,\Delta\tau \quad \longrightarrow \quad x(n\Delta\tau)\,h(t - n\Delta\tau)\,\Delta\tau$$

$$\lim_{\Delta\tau \to 0} x(n\Delta\tau)\,\delta(t - n\Delta\tau)\,\Delta\tau \quad \longrightarrow \quad \lim_{\Delta\tau \to 0} x(n\Delta\tau)\,h(t - n\Delta\tau)\,\Delta\tau$$

$$\lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\,\delta(t - n\Delta\tau)\,\Delta\tau \quad \longrightarrow \quad \lim_{\Delta\tau \to 0} \sum_n x(n\Delta\tau)\,h(t - n\Delta\tau)\,\Delta\tau$$

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t - \tau)\,d\tau \qquad \qquad \int_{-\infty}^{+\infty} x(\tau)\,h(t - \tau)\,d\tau$$

$$x(t) \quad \longrightarrow \quad y(t)$$

Zero State Response IVP

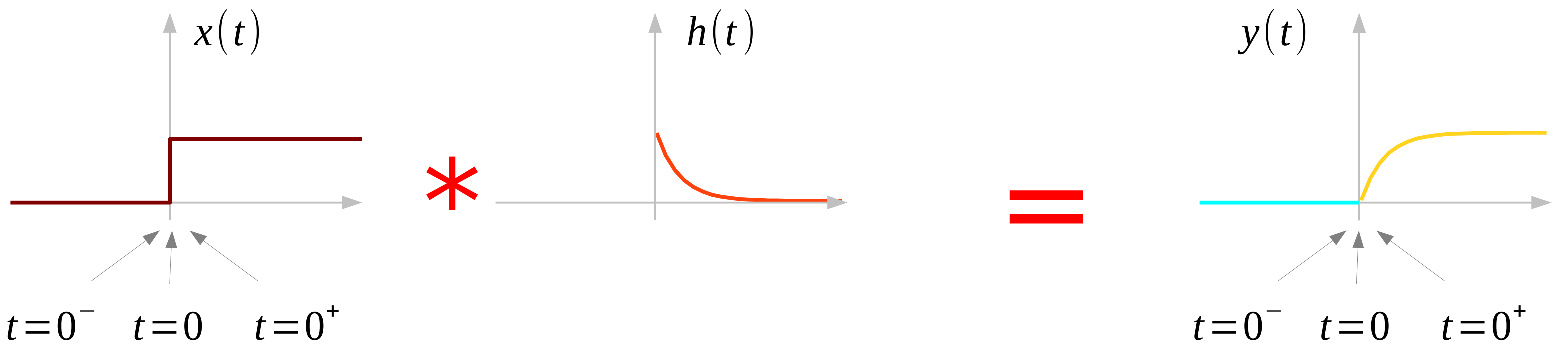
$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

all initial conditions are zero

$$y(0^-) = y^{(1)}(0^-) = y^{(2)}(0^-) \cdots = y^{(N-2)}(0^-) = y^{(N-1)}(0^-) = 0$$

ZSR Initial Value Problem (IVP)



* an impulse in x(t) & h(t) at t = 0
creates non-zero initial conditions

$$y(0^+) = k_0, \quad y^{(1)}(0^+) = k_1, \quad \cdots \quad y^{(N-2)}(0^+) = k_{N-2}, \quad y^{(N-1)}(0^+) = k_{N-1}$$

- Classical Solution

Linear Equations with Constant Coefficients

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = 0$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = 0$$

the system's natural response $y_n(t)$
(homogeneous, complementary solution)
consists of all the characteristic mode terms of the total system response

$$Q(D)\, y_n(t) = 0$$

$$Q(D)\, y_p(t) = P(D)\, x(t)$$

The remaining portion of non-characteristic mode terms form the system's forced response (particular solution) $y_p(t)$

$$Q(D)\left[y_n(t) + y_p(t)\right] = P(D)\, x(t)$$

Linear Equation with Constant Coefficients

→ interval of validity $(-\infty,\ +\infty)$

$$y(t) = y_n(t) + y_p(t)$$

Classical Solution

- **Natural Response**

Homogeneous Solution

$$\frac{d^2 y_n(t)}{dt^2} + a_1 \frac{d\, y_n(t)}{dt} + a_2 y_n(t) = 0$$

Homogeneous Solution

$$Q(D)\, y_n(t) = 0$$

characteristic modes response

- **Forced Response**

Particular Solution

$$\frac{d^2 y_p(t)}{dt^2} + a_1 \frac{d\, y_p(t)}{dt} + a_2 y_p(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d\, x(t)}{dt} + b_2 x(t)$$

Particular Solution

$$Q(D)\, y_p(t) = P(D)\, x(t)$$

non-characteristic mode response

- **Total Response**

$$Q(D)\, [y_n(t) + y_p(t)] = P(D)\, x(t)$$

$$y(t) = y_n(t) + y_p(t)$$

Limits of the Classical Method

cannot separate the internal conditions and the external input

cannot express system response y(t) in terms of explicit function of x(t)

Restricted to a certain class of inputs

only for inputs with the finite derivatives

The auxiliary conditions must be on the total response which exists only for $t \geq 0^+$

In practices, only the initial conditions at $t = 0^-$ is given,

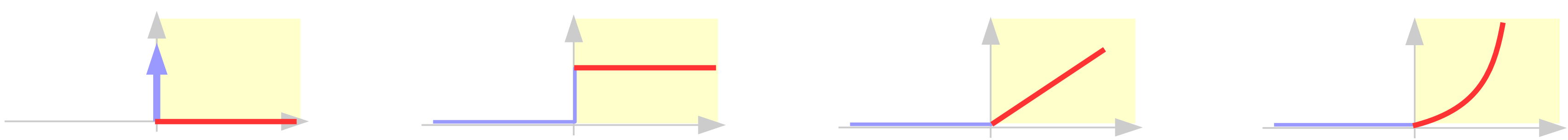
We must drive the initial conditions at $t = 0^+$

Finding Particular Solutions

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + \cdots + a_{N-1} y^{(1)}(t) + a_N h(t) \;=\; b_0 x^{(N)}(t) + b_1 x^{(N-1)}(t) + \cdots + b_{N-1} x^{(1)}(t) + b_N x(t)$$

Particular Solution $y_p(t)$ Depends on the form of the input $x(t)$ for $t > 0$

$x(t) = t^2$	$(t \geq 0)$	➡ $x(t) = t^2$	$(t > 0)$	➡	$y_p(t) = A t^2 + B t + C$	$(t > 0)$
$x(t) = e^t$	$(t \geq 0)$	➡ $x(t) = e^t$	$(t > 0)$	➡	$y_p(t) = A e^t$	$(t > 0)$
$x(t) = u(t)$	$(t \geq 0)$	➡ $x(t) = 1$	$(t > 0)$	➡	$y_p(t) = A$	$(t > 0)$
$x(t) = \delta(t)$	$(t \geq 0)$	➡ $x(t) = 0$	$(t > 0)$	➡	$y_p(t) = 0$	$(t > 0)$



Find $y_h(t) + y_p(t)$ **for the associated ODE (t > 0)**

Finding Total Responses

Direct Inspection

$$y(t) = (y_h + y_p)$$

$$y_n^{(N-1)}(0^+),\ \cdots,\ y_n^{(1)}(0^+),\ y_n(0^+)$$

conversion

$$y_n^{(N-1)}(0^-),\ \cdots,\ y_n^{(1)}(0^-),\ y_n(0^-)$$

Balancing Singularities

For ZSR : all zero initial conditions at t= 0-

$$y^{(N-1)}(0^-) = \cdots = y^{(1)}(0^-) = y(0^-) = 0$$

$$y_{zs}(t) = (y_h + y_p)\cdot u(t)$$

For ZIR : use the initial conditions at t= 0-

$$y_{zi}(t) = (y_h)$$

$$y^{(N-1)}(0^+) = y^{(N-1)}(0^-),\ \cdots,\ y^{(1)}(0^+) = y^{(1)}(0^-),\ y(0^+) = y(0^-)$$

* Total Response

Partitioning a Total Response

$$y(t) = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{Zero Input Response}} + \underbrace{x(t) * h(t)}_{\text{Zero State Response}}$$

$$y(t) = \underbrace{y_n(t)}_{\text{Natural Response}} + \underbrace{y_p(t)}_{\text{Forced Response}}$$

Classical Approach

Total Response and Initial Conditions

zero input response
+
zero state response

ZIR $\left[0^-, \quad 0^+\right]$

$$y(t) = y_{zi}(t) \Leftarrow t \le 0^-$$

because the input has not started yet

continuous at $t = 0$

$$y(0^-) = y_{zi}(0^-) = y_{zi}(0^+)$$
$$\dot{y}(0^-) = \dot{y}_{zi}(0^-) = \dot{y}_{zi}(0^+)$$

$\left[0^+, \quad +\infty\right]$ ZSR + ZIR

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$y(0^+) \neq y(0^-)$$

possible discontinuity at $t = 0$

$$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$$
$$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$$

natural response
+
forced response

$$y_n(0^-) \neq y_{zi}(0^-)$$
$$\dot{y}_n(0^-) \neq \dot{y}_{zi}(0^-)$$

$$y_p(0^-) \neq y_{zi}(0^-)$$
$$\dot{y}_p(0^-) \neq \dot{y}_{zi}(0^-)$$

$\left[0^+, \quad +\infty\right]$ $y_n + y_f$

$$y(t) = y_n(t) + y_f(t)$$

$$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+)$$
$$\dot{y}(0^+) = \dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)$$
$$y(0^+) = y_n(0^+) + y_p(0^+)$$
$$\dot{y}(0^+) = \dot{y}_n(0^+) + \dot{y}_n(0^+)$$

taking the part $t > 0$

Total Response = ZIR + ZSR

$$y(t) = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{ZIR } y_{zi}(t)} + \underbrace{x(t) * h(t)}_{\text{ZSR } y_{zs}(t)}$$

ZIR $y_{zi}(t)$

ZSR $y_{zs}(t)$

$y(t) = y_{zi}(t)$ ⬅ $t \leq 0^-$

because the input has not started yet

$y_{zi}(0^-) = y_{zi}(0^+)$

$\dot{y}_{zi}(0^-) = \dot{y}_{zi}(0^+)$

continuous ZIR initial conditions

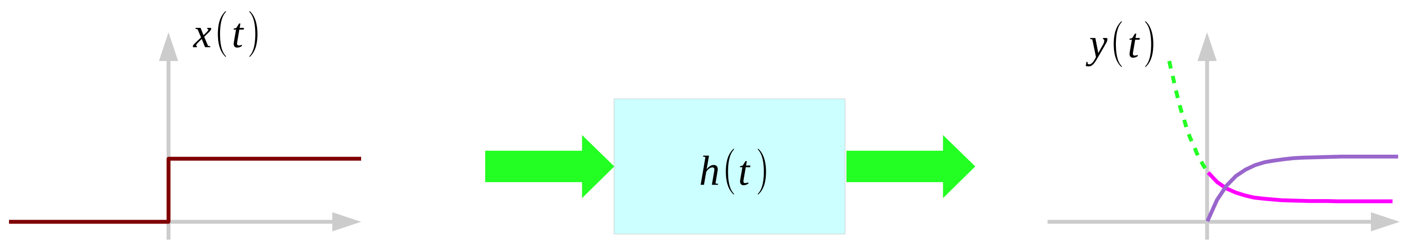
possible discontinuity at t = 0

in general, the total response

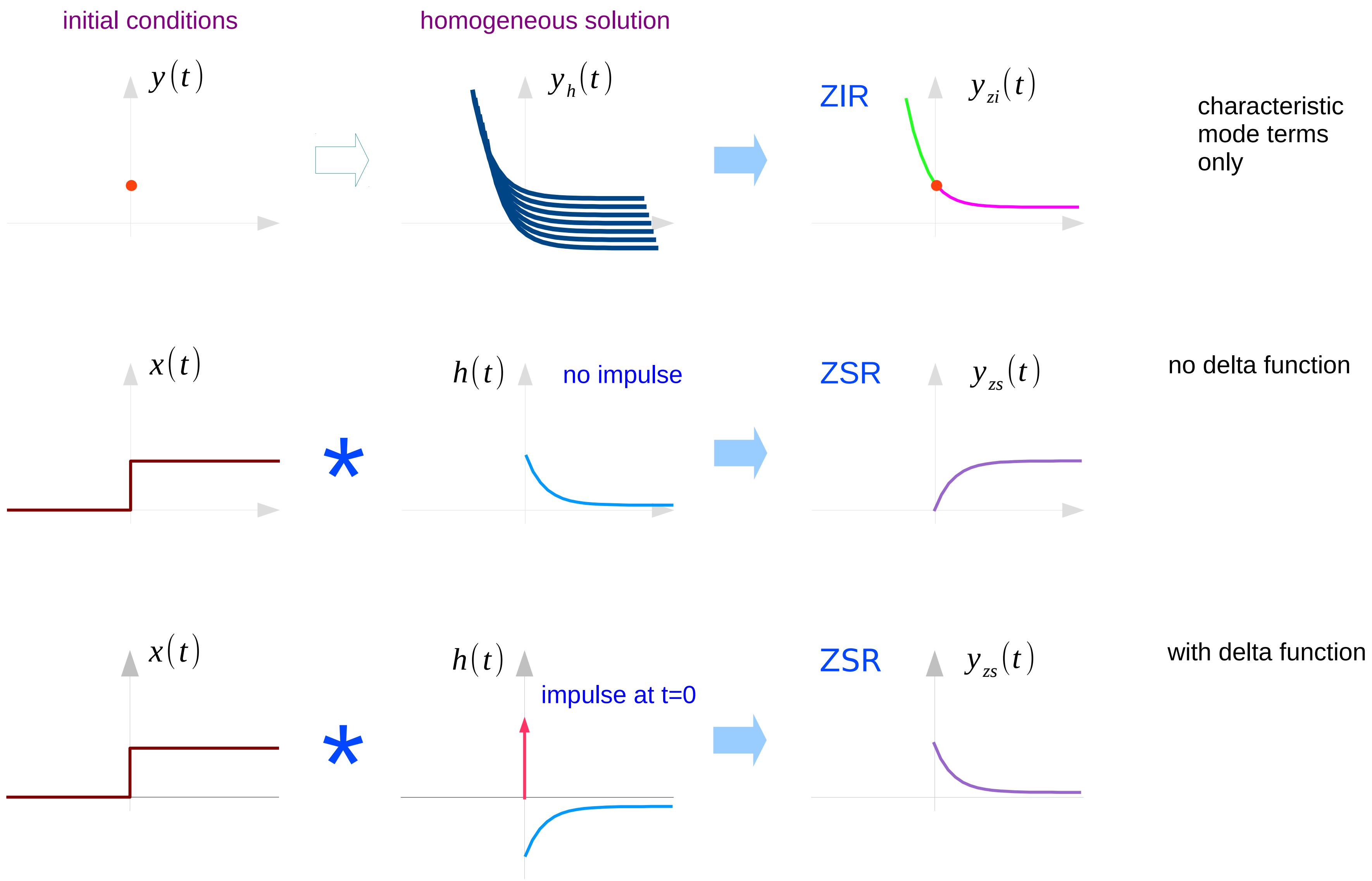
~~$y_{zs}(0^-) = y_{zs}(0^+)$~~

~~$\dot{y}_{zs}(0^-) = \dot{y}_{zs}(0^+)$~~

discontinuous ZSR initial conditions



Total Response = ZIR + ZSR



Total Response = $y_n + y_p$

causal x(t)

$$y(t) = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{Zero Input Response}} + \underbrace{x(t) * h(t)}_{\text{Zero State Response}}$$

Zero Input Response

Zero State Response

$y_p(t)$ has a similar form as the input $x(t)$

$y_p(t)$ is included in ZSR

$$x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$ convolution form

$$u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$ step function form

$$y(t) = \underbrace{y_n(t)}_{\text{Natural Response}} + \underbrace{y_p(t)}_{\text{Forced Response}}$$

Natural Response

Forced Response

Classical Approach

Forced Response y_p in ZSR

$$x(t) * h(t)$$

Zero State Response

$$x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$

convolution form

$$= \int_0^t x(\tau) \left(\sum_i d_i e^{\lambda_i (t-\tau)} + b_0 \delta(t-\tau) \right) d\tau$$

$$= \sum_i d_i e^{\lambda_i t} \left(\int_0^t x(\tau) e^{-\lambda_i \tau} \, d\tau \right) + b_0 \int_0^t x(\tau) \delta(t-\tau) \, d\tau$$

example $x(t) = t$ → $t e^{-\lambda_i t}, \; e^{-\lambda_i t}$ → forced response $y_p(t) = A t + B$

$e^{-\lambda_i 0}$ → $\sum_i k_i e^{\lambda_i t}$

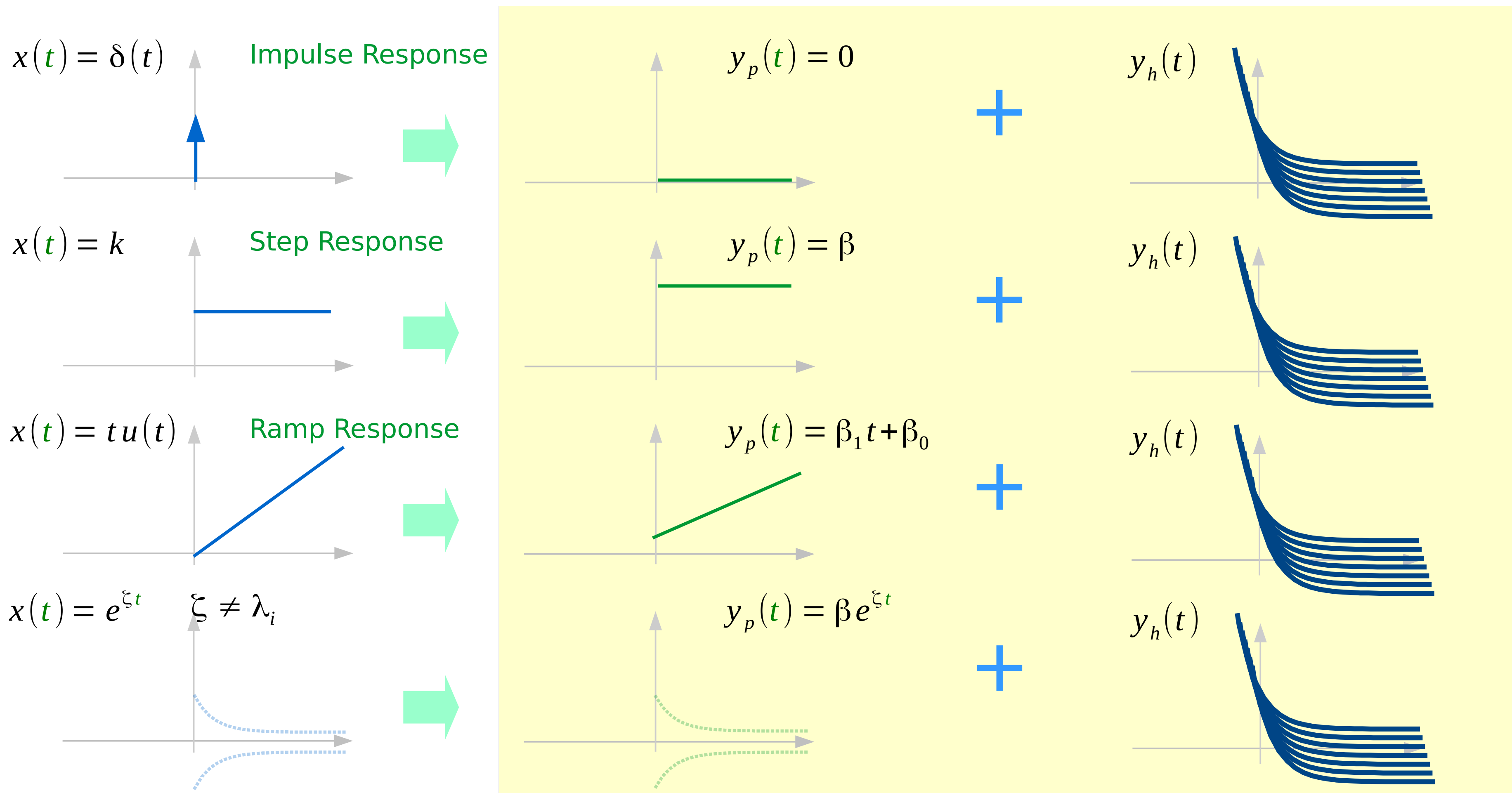
$$u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

step function form

System Responses

$$y(t) = y_p(t) + y_n(t)$$

Forced Response ($y_p(t)$) + Natural Response ($y_n(t)$)



Total Response and Characteristic Modes

ZIR	ZSR
$\sum_i c_i e^{\lambda_i t}$	
$\sum_i k_i e^{\lambda_i t}$	$y_p(t)$

$\equiv$

ZIR	ZSR
$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$	
$y_{zs}(t) = h(t) * x(t)$ causal x(t)	

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$

Natural Response	Forced Response
$y_n(t) = \sum_i K_i e^{\lambda_i t}$	$y_p(t)$

$y_p(t) = 0 \quad \Leftarrow \quad x(t) = \delta(t)$

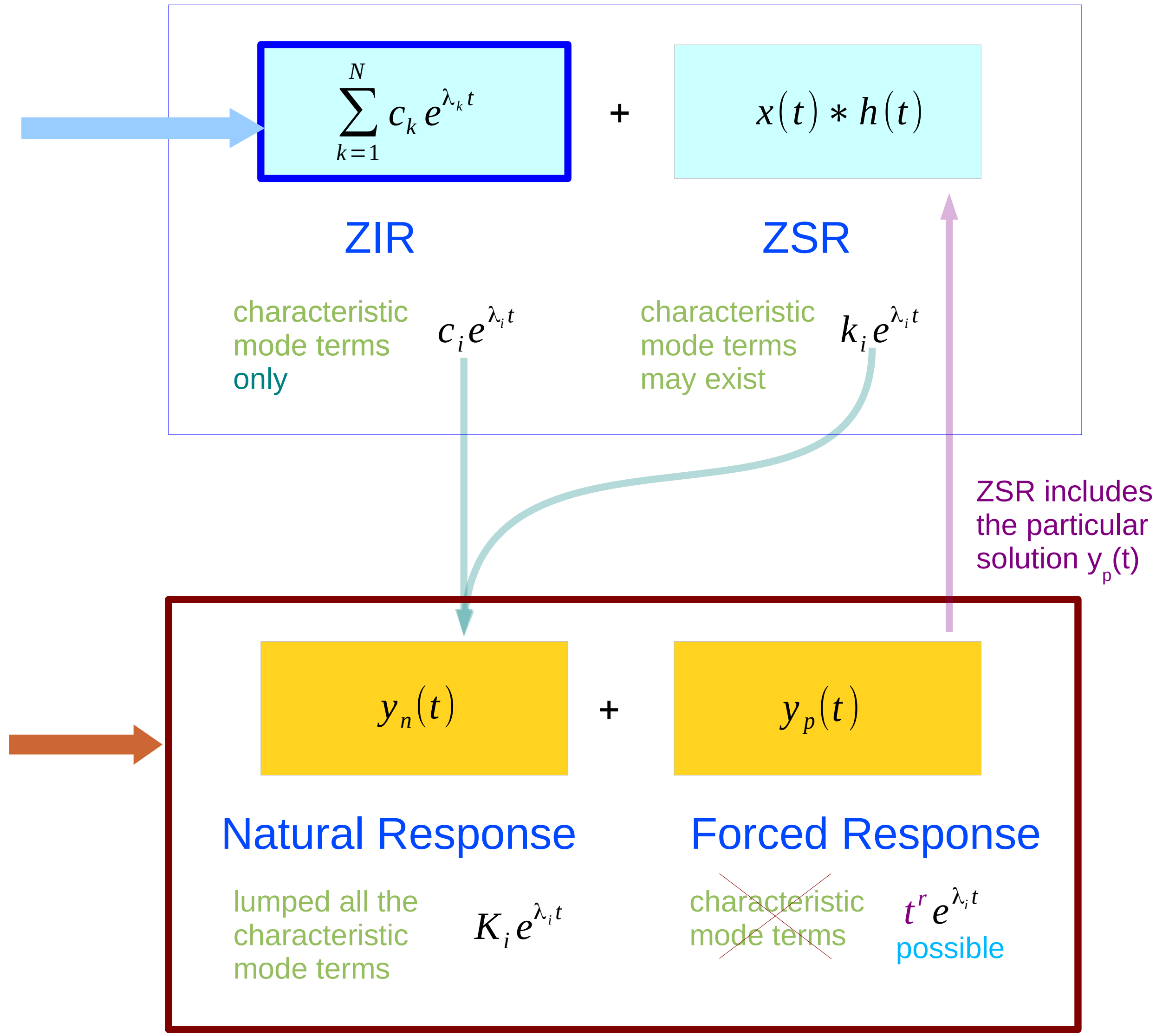
$y_p(t) = \beta \quad \Leftarrow \quad x(t) = k$

$y_p(t) = \beta_1 t + \beta_0 \quad \Leftarrow \quad x(t) = t\,u(t)$

$y_p(t) = \beta e^{\zeta t} \quad \Leftarrow \quad x(t) = e^{\zeta t} \quad \zeta \neq \lambda_i$

Young Won Lim
5/16/15

Characteristic Mode Coefficients



Determining Characteristic Mode Coefficients

the initial condition **after** t=0 is used

$$\{y^{(N-1)}(0^+),\ \cdots,\ y^{(1)}(0^+),\ y(0^+)\}$$

the same initial condition **before** t=0

$$\{y^{(N-1)}(0^-),\ \cdots,\ y^{(1)}(0^-),\ y(0^-)\}$$

any input is applied at time 0, but in the ZIR: the initial condition does not change before and after time 0 since no input is applied

$$\sum_{k=1}^{N} c_k e^{\lambda_k t} + x(t) * h(t)$$

ZIR ZSR

the initial condition **after** t=0 is used

$$\{y^{(N-1)}(0^+),\ \cdots,\ y^{(1)}(0^+),\ y(0^+)\}$$

So the effects of the char. modes of ZSR are included.

$$y_n(t) + y_p(t)$$

Natural Response Forced Response

Finding the Necessary Initial Conditions

ZIR Coefficients

$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$$

initial conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

continuous initial conditions (the same)

$t > 0$

Natural Response Coefficients

$$y(t) = \sum_i K_i e^{\lambda_i t} + y_p(t)$$

initial conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

conversion needed

- **direct inspection**

$t > 0$

ZSR Coefficients

$$y_{zs}(t) = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

zero conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

$t > 0$

- **direct inspection**
- **balancing singularities**

  (*use* $(y_h+y_p)u(t)$ *without converting i.c.*)

$$y_{zs}(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$

zero conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

- **impulse matching**

$t > 0$

Finding the Necessary Initial Conditions

ZIR Coefficients

$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$$

$$y^{(i)}(0^+)$$

$$t > 0$$

Natural Response Coefficients

$$y(t) = \sum_i K_i e^{\lambda_i t} + y_p(t)$$

$$y^{(i)}(0^+)$$

- **direct inspection**

$$t > 0$$

ZSR Coefficients

$$y_{zs}(t) = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

$$y^{(i)}(0^+)$$

- **direct inspection**
- **balancing singularities**

  (*use* $(y_h + y_p)u(t)$ *without converting i.c.*)

$$t > 0$$

$$y_{zs}(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$

$$h^{(i)}(0^+)$$

- **impulse matching**

$$t > 0$$

Finding Causal LTI System Responses

- **Zero Input Response**

$$y_{zp}(t) = \sum_i c_i e^{\lambda_i t}$$

$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$

=

$\{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$

- **Zero State Response**

(1) $y_{zs}(t) = h(t) * x(t)$ $\quad h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$

$\{0, \cdots, 0, 0\}$ → $\{h^{(N-1)}(0^+), \cdots, h^{(1)}(0^+), h(0^+)\}$

(2) $y_{zs}(t) = \left(\sum_i b_i e^{\lambda_i t} + y_p(t)\right) \cdot u(t)$

$\{0, \cdots, 0, 0\}$ → $\{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$

- **Natural Response** Homogeneous

$$y_n(t) = \sum_i K_i e^{\lambda_i t}$$

- **Forced Response** Particular

$$y_p(t) = \begin{cases} \beta e^{\zeta t} \quad \text{or} \\ (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) \end{cases}$$

$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$

↓

$y_n(t) + y_p(t)$ $\quad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M. J. Roberts, Fundamentals of Signals and Systems
[4] S. J. Orfanidis, Introduction to Signal Processing
[5] B. P. Lathi, Signals and Systems