Implication (6A)

Young Won Lim
3/17/18

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Material Implication & Logical Implication

Given two propositions **A** and **B**,

If **A** $\Rightarrow$ **B** is a **tautology** ← in every interpretation

It is said that **A** **logically implies** **B** (**A** $\Rrightarrow$ **B**)

Material Implication **A** $\Rightarrow$ **B** (not a tautology) **A** $\rightarrow$ **B**

Logical Implication **A** $\Rrightarrow$ **B** (a **tautology**) **A** $\Rightarrow$ **B**

A	B	A⇒B
T	T	T
T	F	F
F	T	T
F	F	T

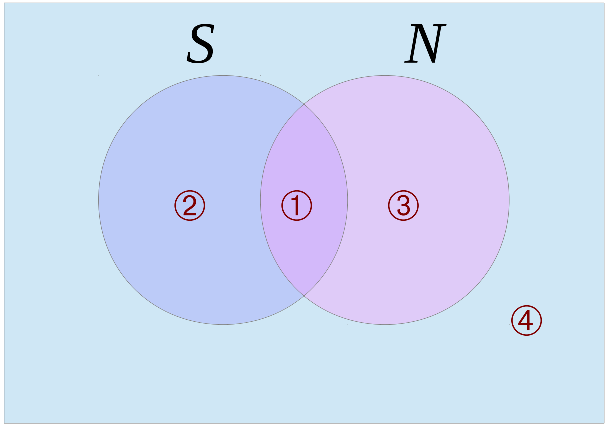
A	B	A∧B	A∧B ⇒ A
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

tautology

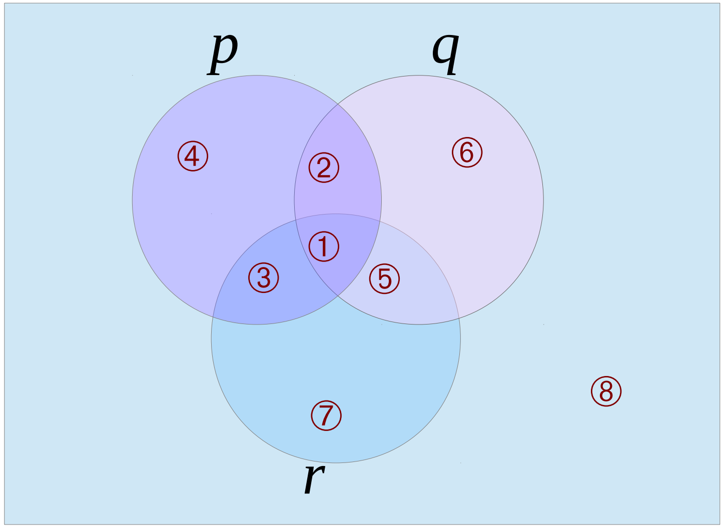
A∧B $\Rrightarrow$ **A**

Logic and Venn diagram (1)

	S	N	S⇒N
case ①	T	T	T
case ②	T	F	F
case ③	F	T	T
case ④	F	F	T



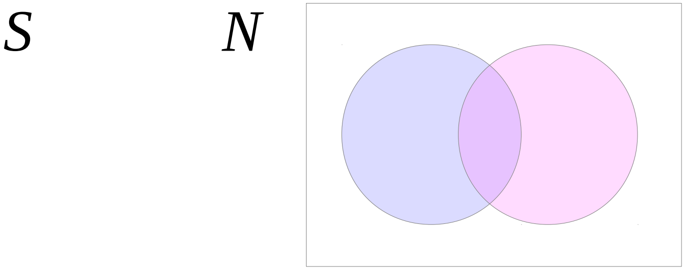
Logic and Venn diagram (2)

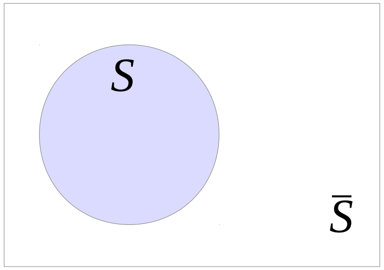


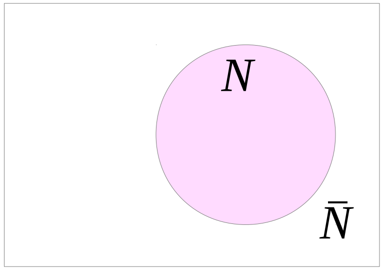
	p	q	r	$p \rightarrow q$	$\cdots$
case ①	T	T	T	T	
case ②	T	T	F	T	
case ③	T	F	T	F	
case ④	T	F	F	F	
case ⑤	F	T	T	T	
case ⑥	F	T	F	T	
case ⑦	F	F	T	T	
case ⑧	F	F	F	T	

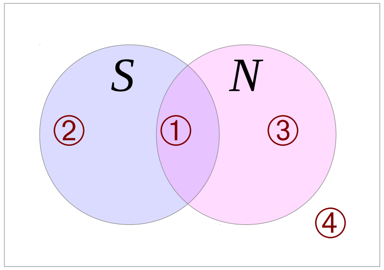
Material Implication and Venn Diagram

	S	N	S⇒N
case ①	T	T	T
case ②	T	F	F
case ③	F	T	T
case ④	F	F	T

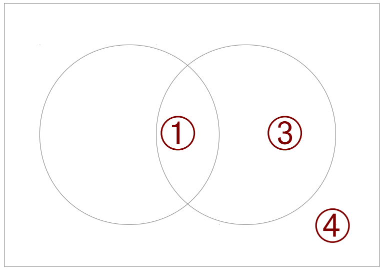






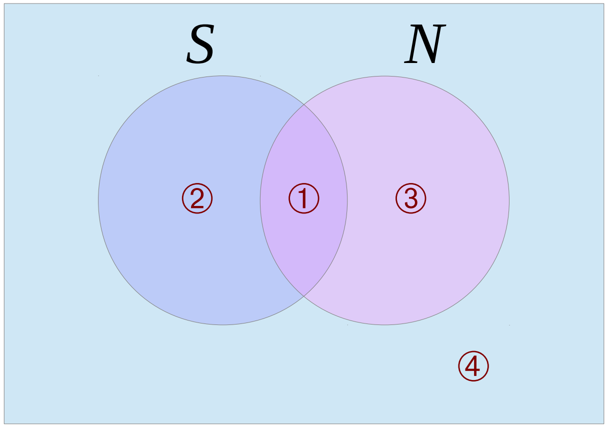


When S⇒N is True



When S⇒N is a true statement

	S	N	S⇒N	
case ①	T	T	T	(1)
case ②	T	F	F	
case ③	F	T	T	
case ④	F	F	T	(2)



S⇒N is True

cases ①+③+④

if the conditional statement (**S⇒N**) is a **true** statement,

(1) then the consequent **N** must be **true** **if** **S** is **true**

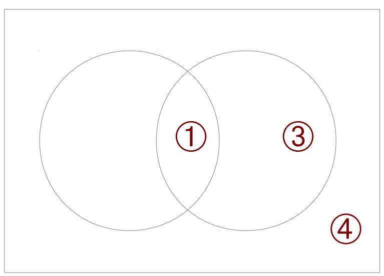
(2) the antecedent **S** can not be **true** **without** **N** being **true**

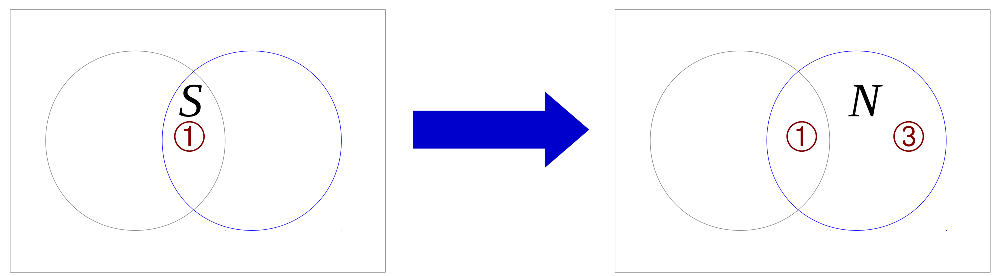
S ⊆ N

if the conditional statement **(S⇒N)** is a **true** statement,

then the consequent **N** must be **true** **if** **S** is **true**

S⇒N





case ①

cases ①+③

S⊆N

x∈**S**

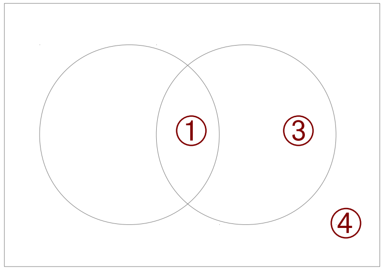
x∈**N**

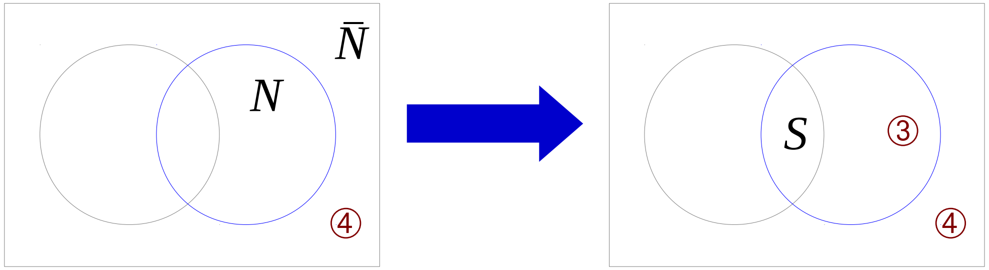
~N ⊆ ~S

if the conditional statement (S⇒N) is a **true** statement,

the antecedent **S** can <u>not</u> be **true** <u>without</u> **N** being **true**

¬N⇒¬S





case ④

cases ③+④

¬N⊆¬S

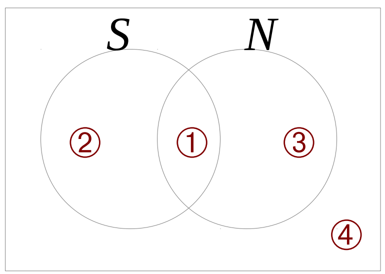
x∈**¬N**

x∈**¬S**

Material Implication vs. Logical Consequence

Material Implication

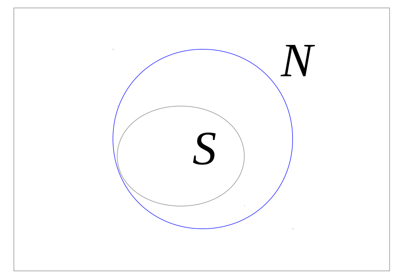
S⇒N



T⇒F exists ②

Logical Consequences

S⇛N



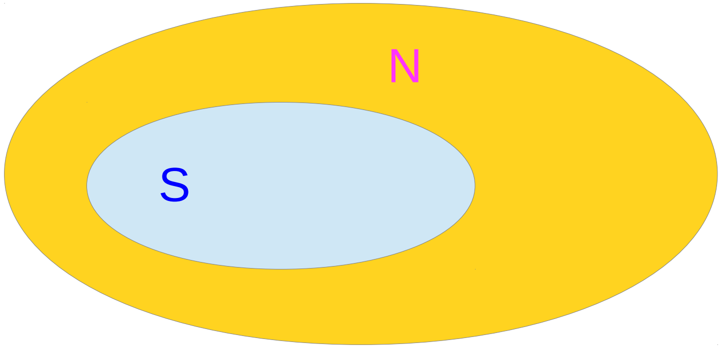
Always True
(Tautology)

entailment

S ⊢ N syntactic (proof)

S ⊨ N semantic (model)

Implication



If **S**, then N.

S implies N.

N whenever **S**.

S is sufficient for N.

S only if **N**.

not S if not **N**.

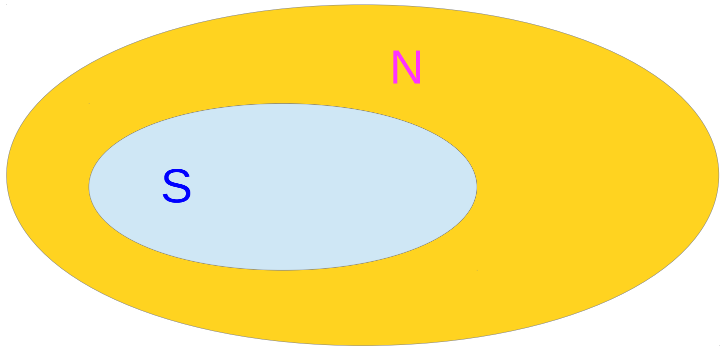
not S without **N**.

N is necessary S.

http://en.wikipedia.org/wiki/

Necessity and Sufficiency

condition that guarantees N	condition that must be satisfied for S
sufficiency for N	necessity for S
S satisfies at least N	without **N**, it can't be S
N if **S**	**S** only if **N**

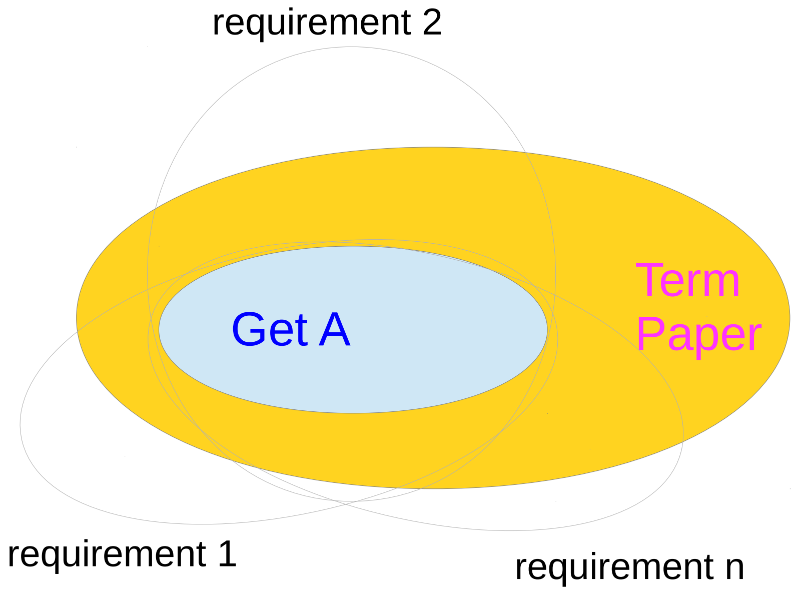


Other Necessary Conditions

S **N**

Get A → Term Paper submitted
- Midterm taken
- Final test taken
- Rank < 15%

necessary conditions of S

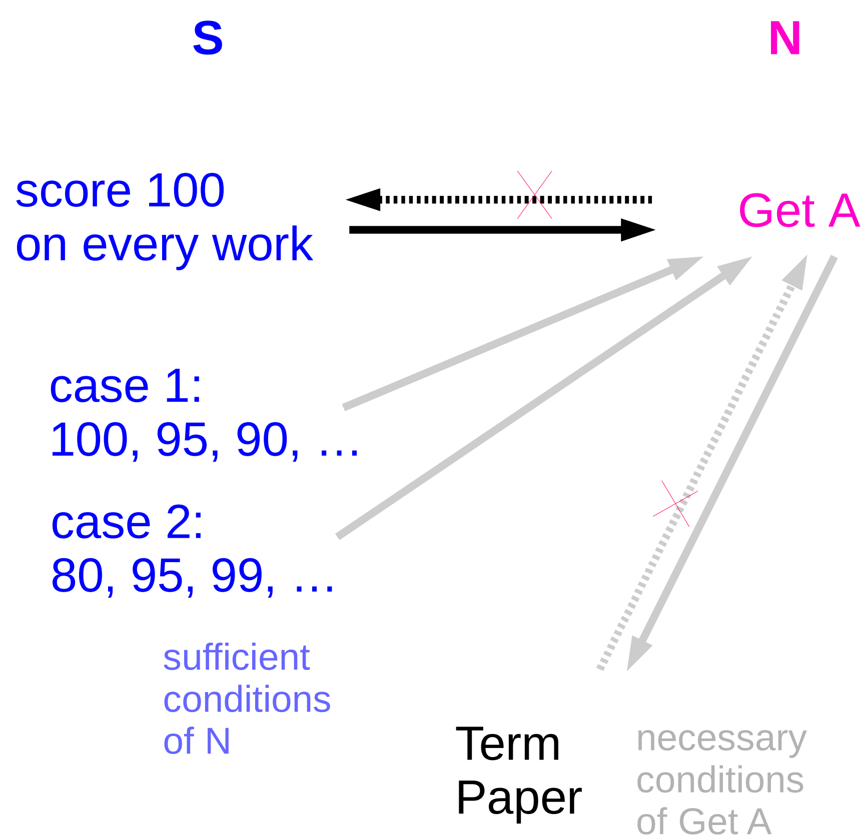


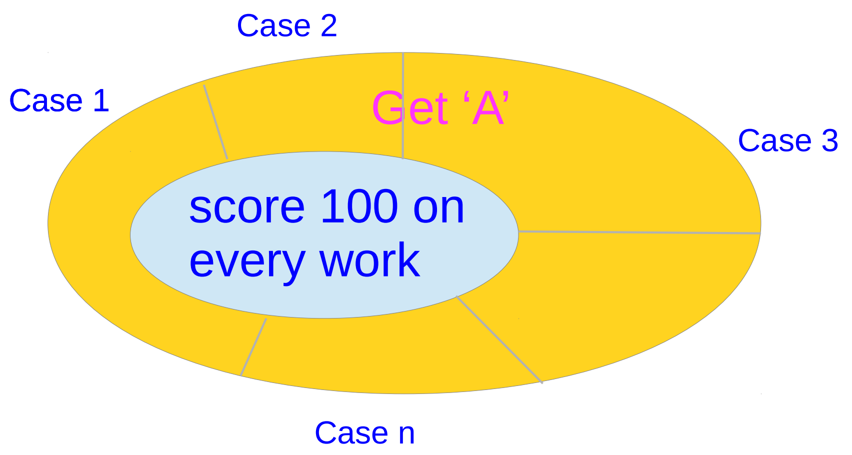
S only if N

not S if not N

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm`

Other Sufficient Conditions





N if S

N whenever S

http://philosophy.wisc.edu/hausman/341/Skill/nec-suf.htm`

Resolution (7A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Argument

$$\frac{\begin{array}{c}(p \vee q) \\ (\neg p \vee r)\end{array}}{q \vee r}$$

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow q \vee r$$

Truth Table

p	q	r	$p \vee q$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

p	q	r	$\neg p$	$\neg p \vee r$
T	T	T	F	T
T	T	F	F	F
T	F	T	F	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

p	q	r	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$
T	T	T	T	T
T	T	F	F	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	F

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow q \vee r$$

Interpretation of this truth table

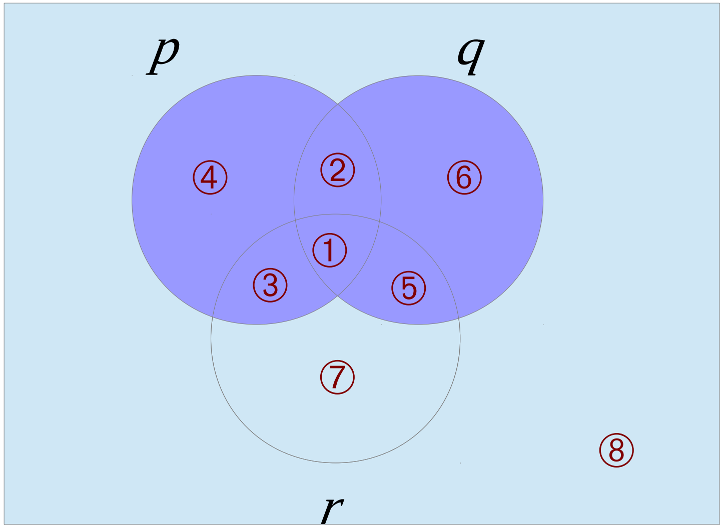
	p	q	r	$p \vee q$	$\neg p \vee r$	A: $(p \vee q) \wedge (\neg p \vee r)$		B: $q \vee r$	$A \rightarrow B$
case ①	T	T	T	T	T	T	→	T	T
case ②	T	T	F	T	F	F	→	T	T
case ③	T	F	T	T	T	T	→	T	T
case ④	T	F	F	T	F	F	→	F	T
case ⑤	F	T	T	T	T	T	→	T	T
case ⑥	F	T	F	T	T	T	→	T	T
case ⑦	F	F	T	F	T	F	→	T	T
case ⑧	F	F	F	F	T	F	→	F	T

Whenever $p \vee q$ and $\neg p \vee r$ are **true**, $q \vee r$ is **true**

$$(p \vee q) \wedge (\neg p \vee r) \rightarrow q \vee r$$

Venn diagram for $p \vee q$

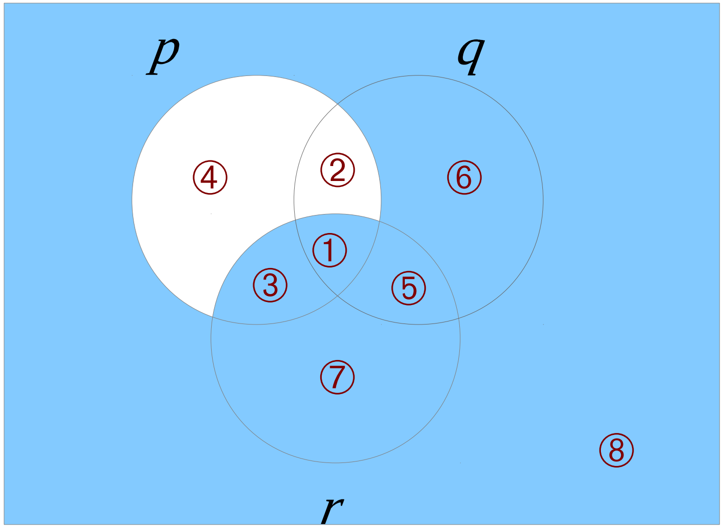
$p \vee q$



	p	q	r	$p \vee q$
case ①	T	T	T	T
case ②	T	T	F	T
case ③	T	F	T	T
case ④	T	F	F	T
case ⑤	F	T	T	T
case ⑥	F	T	F	T
case ⑦	F	F	T	F
case ⑧	F	F	F	F

Venn diagram for $\neg p \lor r$

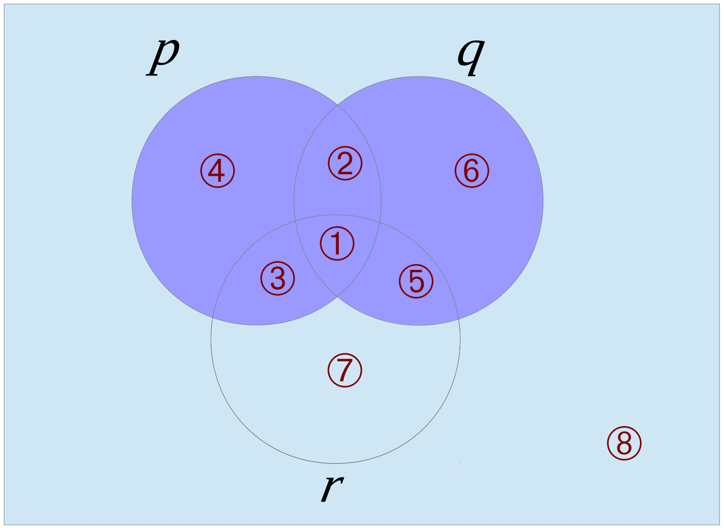
$\neg p \lor r$

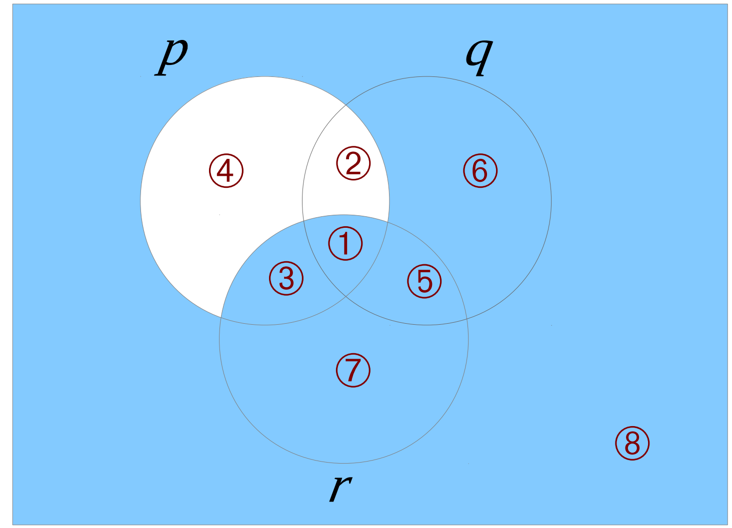


	p	q	r	$\neg p$	$\neg p \lor q$
case ①	T	T	T	F	T
case ②	T	T	F	F	F
case ③	T	F	T	F	T
case ④	T	F	F	F	F
case ⑤	F	T	T	T	T
case ⑥	F	T	F	T	T
case ⑦	F	F	T	T	T
case ⑧	F	F	F	T	T

When $(p \vee q) \wedge (\neg p \vee r)$ is true

$p \vee q$



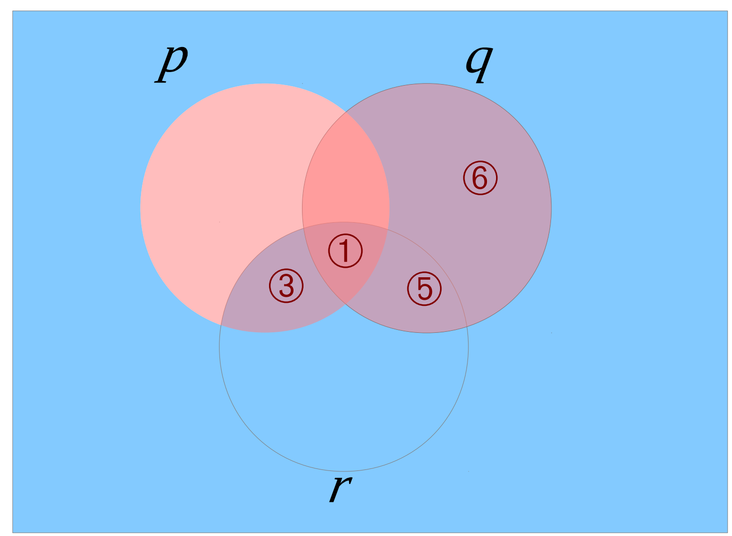


$\neg p \vee r$

When $p \vee q$ is true
and $\neg p \vee r$ is true

$(p \vee q) \wedge (\neg p \vee r)$

cases ①+③+⑤+⑥

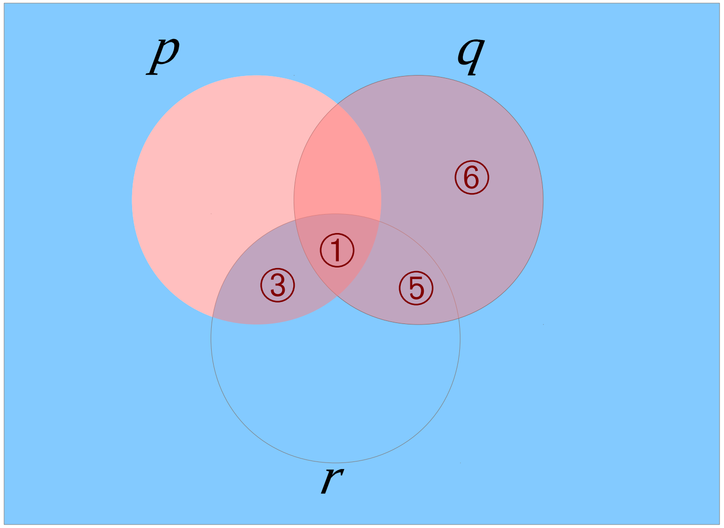


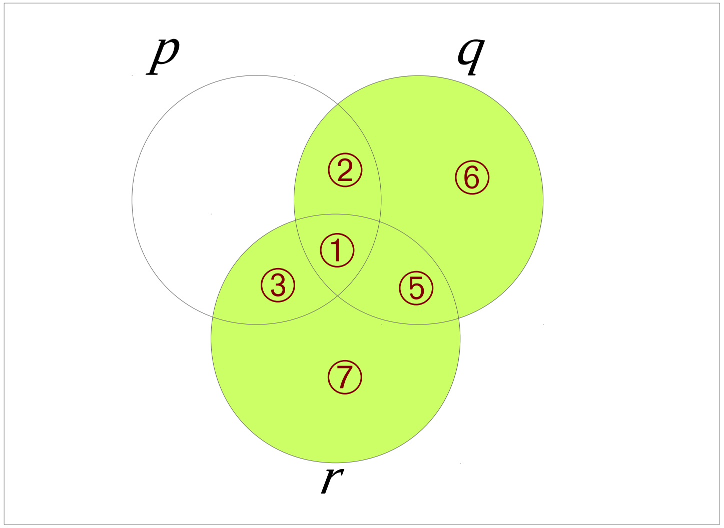
When $(p \vee q) \wedge (\neg p \vee r)$ is true, $q \vee r$ is also true

$p \vee q$

$\neg p \vee r$

$q \vee r$





cases ①+③+⑤+⑥ $\subset$ cases ①+③+⑤+⑥+②+⑦

$(p \vee q) \wedge (\neg p \vee r) \longrightarrow q \vee r$

Argument

$p \vee q$	Case 1: ***p*** is false $F \vee q$	Case 2: ***p*** is true $T \vee q$
$\neg p \vee r$	$T \vee r$	$F \vee r$
$q \vee r$	q	r

$$\frac{p \vee q \qquad \neg p \vee r}{q \vee r}$$

when ***p*** is false,
q must be true.

when ***p*** is true,
r must be true.

Therefore regardless of truth value of ***p***,
If both premises hold,
then the conclusion $q \vee r$ is true

http://en.wikipedia.org/wiki/Derivative

Resolution Examples

$$\begin{array}{c} p \lor q \\ \neg p \lor r \\ \hline q \lor r \end{array} \Rightarrow \begin{array}{c} \cancel{p} \lor q \\ \neg \cancel{p} \lor r \\ \hline \end{array} \Rightarrow \begin{array}{c} \cancel{p} \lor q \\ \neg \cancel{p} \lor r \\ \hline q \lor r \end{array}$$

$$\begin{array}{c} p \lor q \\ \neg p \\ \hline q \end{array} \Rightarrow \begin{array}{c} \cancel{p} \lor q \\ \neg \cancel{p} \\ \hline \end{array} \Rightarrow \begin{array}{c} \cancel{p} \lor q \\ \neg \cancel{p} \\ \hline q \end{array}$$

$$\begin{array}{c} p \\ \neg p \lor r \\ \hline r \end{array} \Rightarrow \begin{array}{c} \cancel{p} \\ \neg \cancel{p} \lor r \\ \hline \end{array} \Rightarrow \begin{array}{c} \cancel{p} \\ \neg \cancel{p} \lor r \\ \hline r \end{array}$$

Resolution in Prolog

Conjunctive Norm Form (CNF) is assumed

$$(\cdots \vee \cdots) \wedge (\cdots \vee \cdots \vee \cdots) \wedge (\cdots \vee \cdots)$$

the variables A, B, C, D, and E are in conjunctive normal form:

¬ A ∧ (B ∨ C)
(A ∨ B) ∧ (¬ B ∨ C ∨ ¬ D) ∧ (D ∨ ¬ E)
A ∨ B
A ∧ B

The following formulas are not in conjunctive normal form:

¬ (B ∨ C)
(A ∧ B) ∨ C
A ∧ (B ∨ (D ∧ E))

Example A

$p \lor q$
$p \lor \neg r$
$\neg p \lor q$
$\neg q \lor r$

---

q
$p \lor \neg r$
$\neg q \lor r$

$p \lor \neg r$
r

---

p

$$(p \lor q) \land (p \lor \neg r) \land (\neg p \lor q) \land (\neg q \lor r)$$
$$\vdash (p \lor q) \land (p \lor \neg r) \land (\neg p \lor q) \land (\neg q \lor r) \land (q)$$
$$\vdash (p \lor q) \land (p \lor \neg r) \land (\neg p \lor q) \land (\neg q \lor r) \land (q) \land (r)$$
$$\vdash (p \lor q) \land (p \lor \neg r) \land (\neg p \lor q) \land (\neg q \lor r) \land (q) \land (r) \land (p)$$

James Aspnes, Notes on Discrete Mathematics, CS 202: Fall 2013

Example B – (1)

$p \rightarrow q \vee r$
$p \vee \neg q$
$r \vee q$

$\neg p \vee q \vee r$
$p \vee \neg q$
$r \vee q$

$q \vee \neg q \vee r$
$r \vee q$

$r \vee q$

$$q \vee \neg q \vee r = T \vee r = T$$

$p \rightarrow q \vee r$
$p \vee \neg q$
$r \vee q$

$\neg p \vee q \vee r$
$p \vee \neg q$
$r \vee q$

$p \vee r$

Discrete Mathematics, Johnsonbough

Example B – (2)

$p \rightarrow q \vee r$
$p \vee \neg q$
$r \vee q$

$p \rightarrow q \vee r$
$\neg p \rightarrow \neg q$
$q \vee r$

$p \rightarrow q \vee r$
$q \rightarrow p$
$q \vee r$

$p \vee r$

Discrete Mathematics, Johnsonbough

Truth Table

p	q	r	$\neg p$	$\neg p \vee q \vee r$
T	T	T	F	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

p	q	r	$\neg q$	$p \vee \neg q$
T	T	T	F	T
T	T	F	F	T
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	T
F	F	F	T	T

p	q	r	$q \vee r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

p	q	r	$\neg p \vee q \vee r$	$p \vee \neg q$	$q \vee r$	$H1 \wedge H2 \wedge H3$	$p \vee r$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	T	F	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	F	F

$H1 = \neg p \vee q \vee r$
$H2 = p \vee \neg q$
$H3 = q \vee r$

$H1 \wedge H2 \wedge H3 \rightarrow H3$
$H1 \wedge H2 \wedge H3 \rightarrow H2$
$H1 \wedge H2 \wedge H3 \rightarrow H1$
$H1 \wedge H2 \wedge H3 \rightarrow (p \vee r)$

References

[1] http://en.wikipedia.org/
[2]

Functions (4A)

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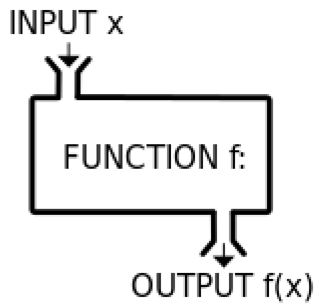
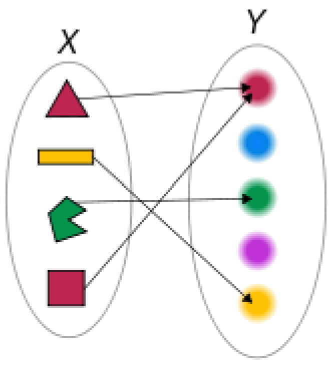
Function



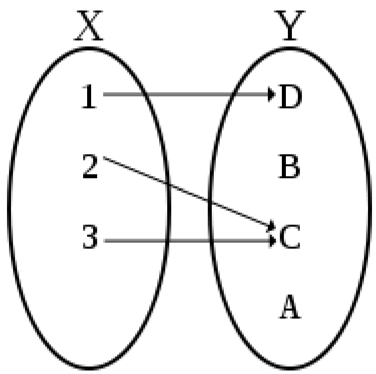
A function f takes an input x, and returns a single output f(x). One metaphor describes the function as a "machine" or "black box" that for each input returns a corresponding output.



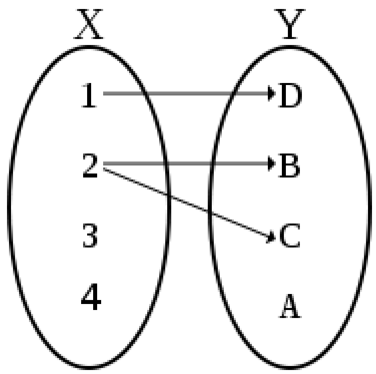
A function that associates any of the four colored shapes to its color.

https://en.wikipedia.org/wiki/Function_(mathematics)

Function



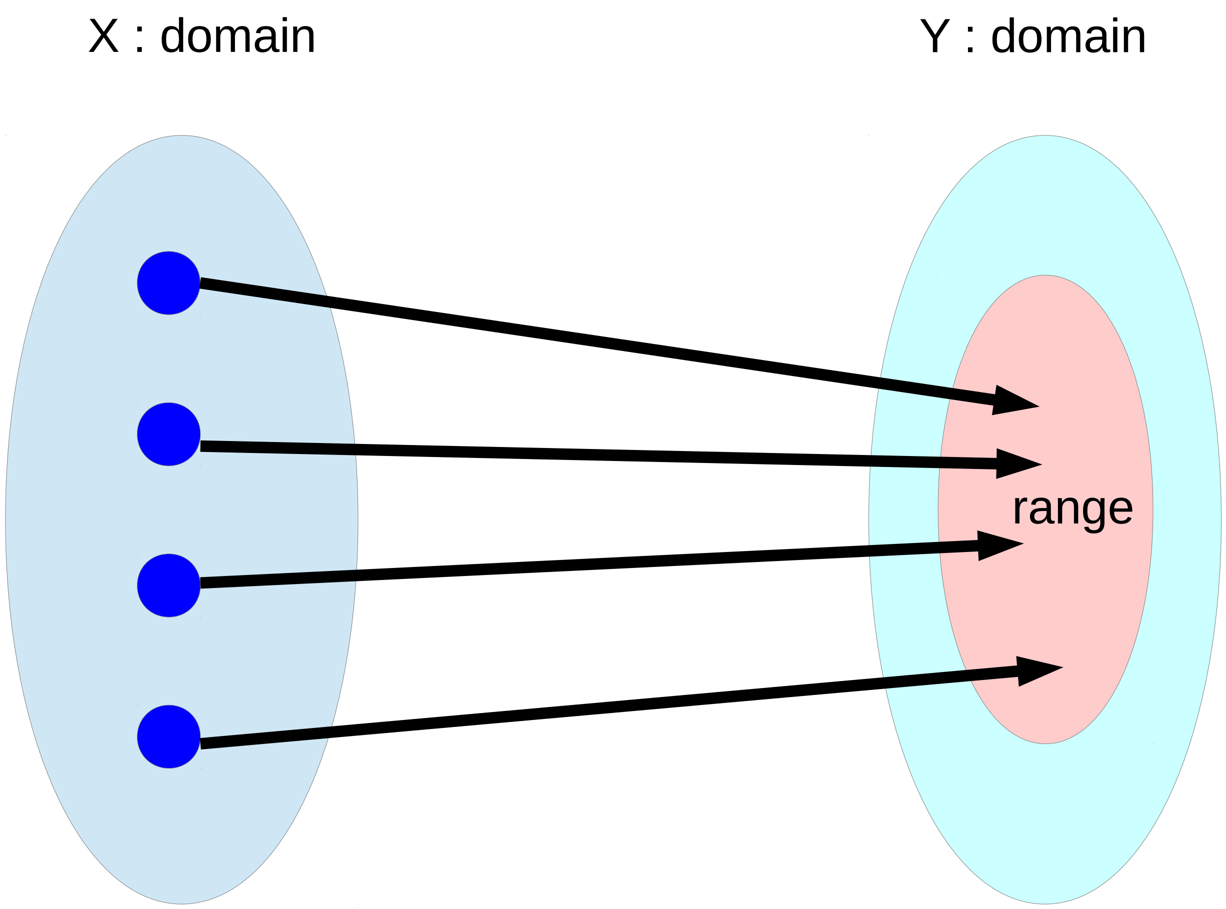
The above diagram represents a function with domain {1, 2, 3}, codomain {A, B, C, D} and set of ordered pairs {(1,D), (2,C), (3,C)}. The image is {C,D}.



However, this second diagram does not represent a function. One reason is that 2 is the first element in more than one ordered pair. In particular, (2, B) and (2, C) are both elements of the set of ordered pairs. Another reason, sufficient by itself, is that 3 is not the first element (input) for any ordered pair. A third reason, likewise, is that 4 is not the first element of any ordered pair.

https://en.wikipedia.org/wiki/Function_(mathematics)

Domain, Codomain, and Range

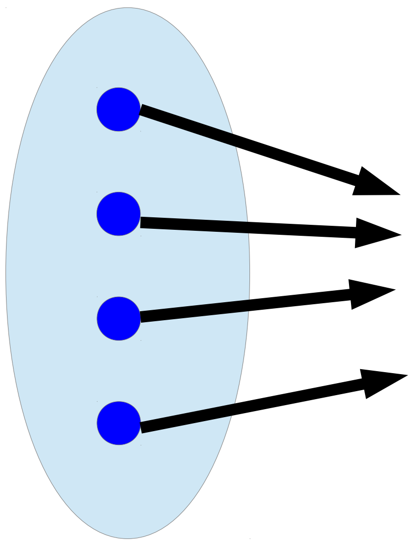


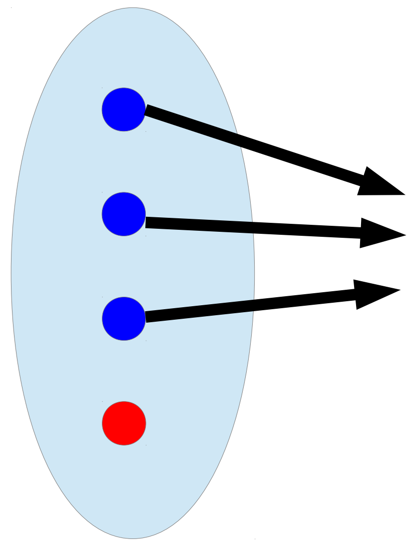
Function condition : One emanating arrow

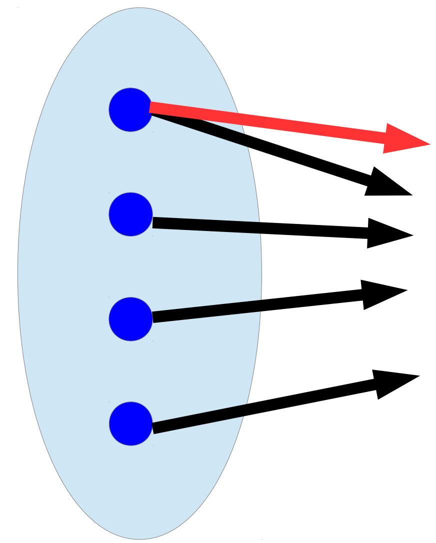
Each, one starting arrow

Some, no starting arrow

Some, two starting arrows







Function (O)

Relation (O)

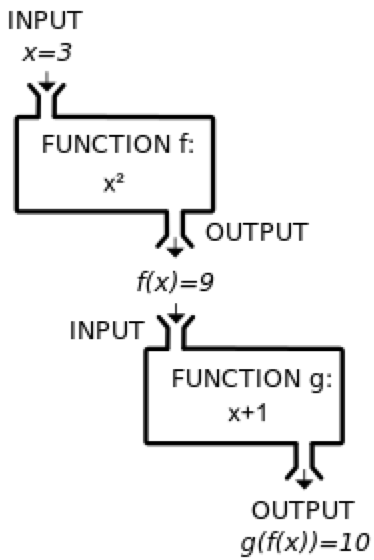
Function (X)

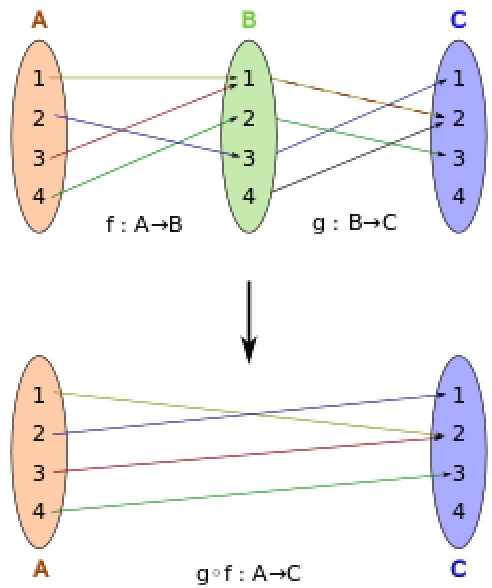
Relation (O)

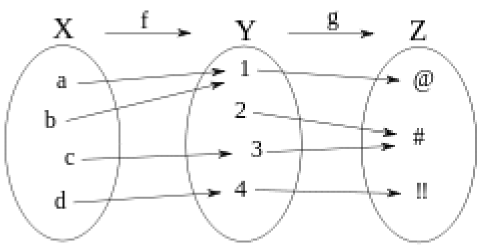
Function (X)

Relation (O)

Composite Function



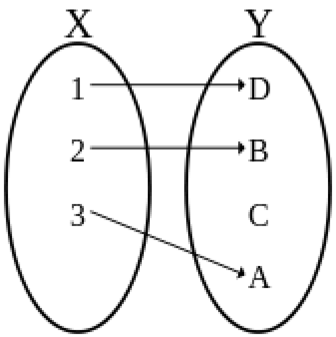




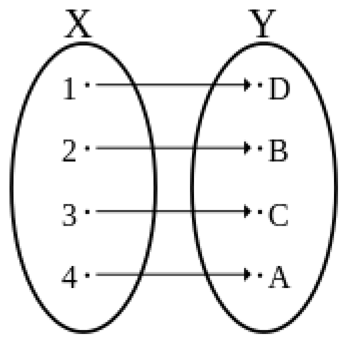
A composite function g(f(x)) can be visualized as the combination of two "machines". The first takes input x and outputs f(x). The second takes as input the value f(x) and outputs g(f(x)).

https://en.wikipedia.org/wiki/Function_(mathematics)

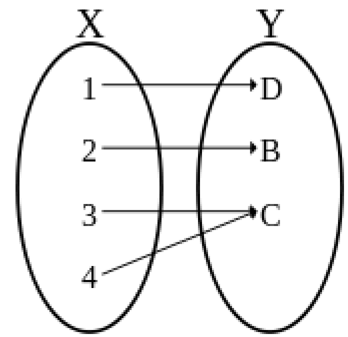
Injective Function



An **injective** non-**surjective** function (injection, not a bijection)



An **injective surjective** function (**bijection**)



A non-**injective surjective** function (surjection, not a bijection)

https://en.wikipedia.org/wiki/Function_(mathematics)

Injective Function

In mathematics, an **injective** function or **injection** or **one-to-one** function is a function that preserves **distinctness**:

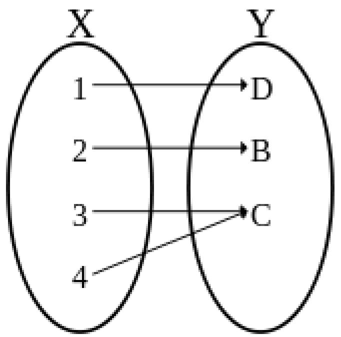
it never maps distinct elements of its **domain** to the same element of its **codomain**.

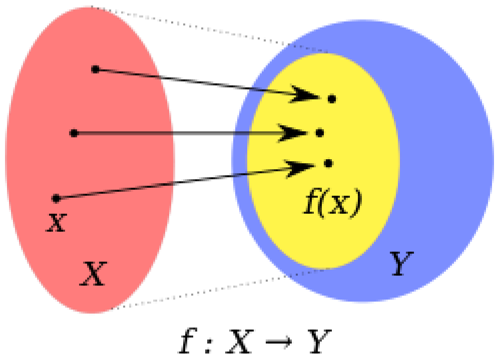
every element of the function's **codomain** is the image of at most one element of its **domain**.

The term one-to-one function must not be confused with one-to-one correspondence (a.k.a. **bijective** function), which uniquely maps all elements in both **domain** and **codomain** to each other.

https://en.wikipedia.org/wiki/Function_(mathematics)

Surjective Function



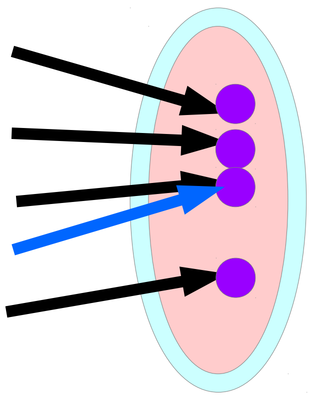


A surjective function from domain X to codomain Y. The function is surjective because every point in the codomain is the value of f(x) for at least one point x in the domain.

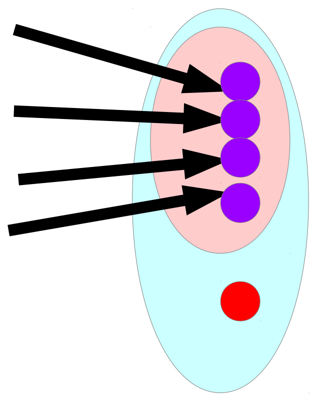
A non-surjective function from domain X to codomain Y. The smaller oval inside Y is the image (also called range) of f. This function is not surjective, because the image does not fill the whole codomain. In other words, Y is colored in a two-step process: First, for every x in X, the point f(x) is colored yellow; Second, all the rest of the points in Y, that are not yellow, are colored blue. The function f is surjective only if there are no blue points.

https://en.wikipedia.org/wiki/Function_(mathematics)

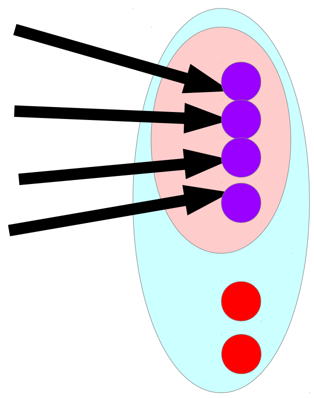
Range = Codomain
Every, arriving arrow(s)



Surjective (O)



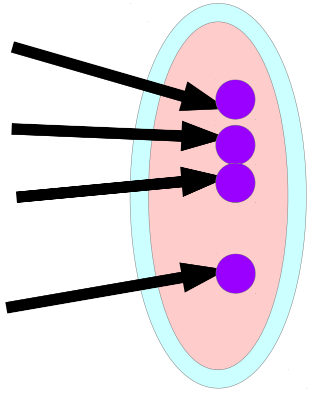
Surjective (X)



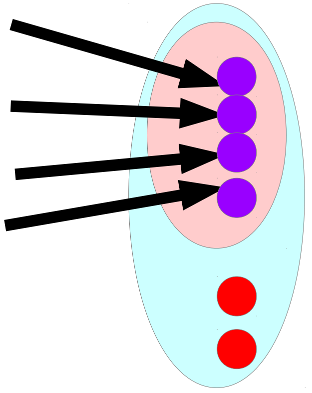
Surjective (X)

Injective Functions

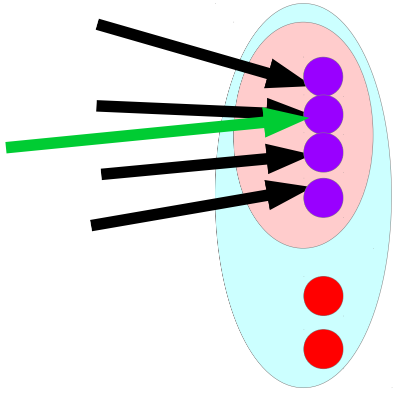
Every, Less than one
arriving arrow



Injective (O)



Injective (O)



Injective (X)

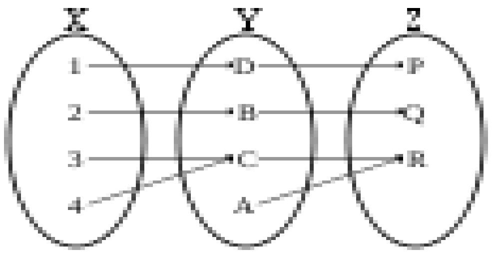
Surjective Function

A **surjective** function is a function
whose **image** is equal to its **codomain**.

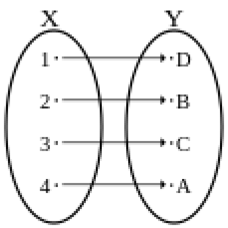
a function f with **domain** X and **codomain** Y is surjective
if for every y in Y there exists at least one x in X with $f(x) = y$.

https://en.wikipedia.org/wiki/Function_(mathematics)

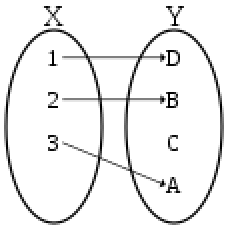
Surjective Functions



Surjective composition: the first function need not be surjective.



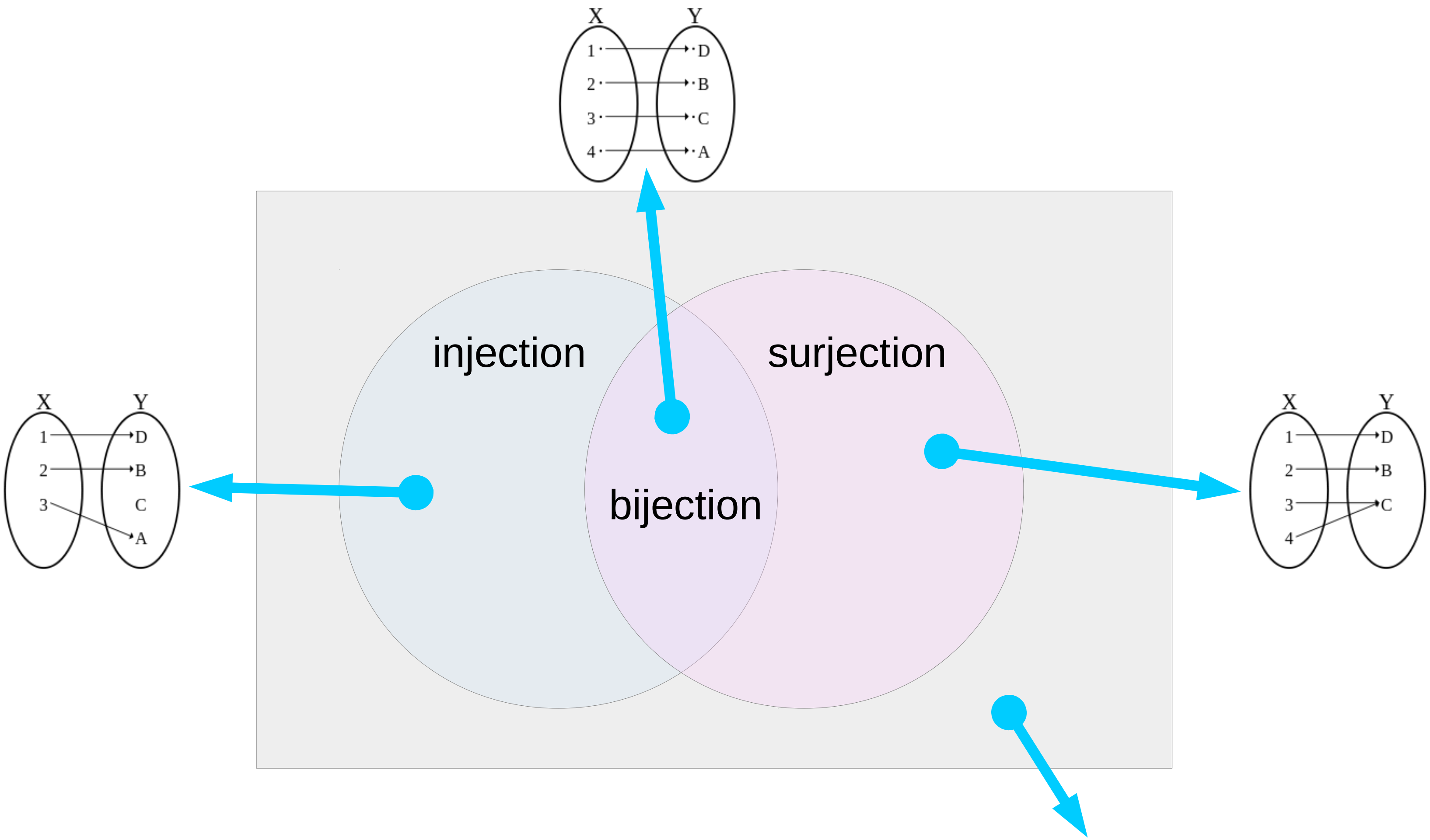
Another surjective function. (This one happens to be a bijection)



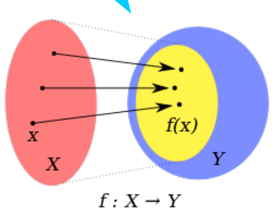
A non-surjective function. (This one happens to be an injection)

https://en.wikipedia.org/wiki/Function_(mathematics)

Types of Functions



https://en.wikipedia.org/wiki/Function_(mathematics)



References

[1] http://en.wikipedia.org/
[2]

Set Operations (15A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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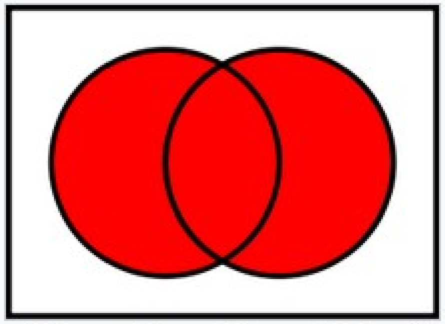
Unions

Two sets can be "added" together. The *union* of *A* and *B*, denoted by *A* ∪ *B*, is the set of all things that are members of either *A* or *B*.

Examples:

- {1, 2} ∪ {1, 2} = {1, 2}.
- {1, 2} ∪ {2, 3} = {1, 2, 3}.
- {1, 2, 3} ∪ {3, 4, 5} = {1, 2, 3, 4, 5}



The **union** of *A* and *B*, denoted *A* ∪ *B*

https://en.wikipedia.org/wiki/Set_(mathematics)

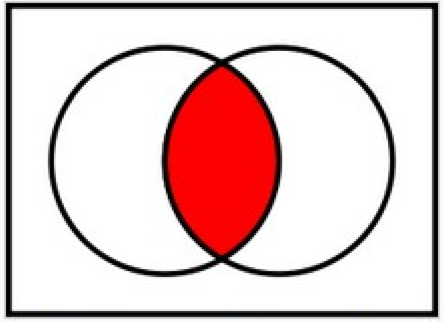
Properties of Unions

- $A \cup B = B \cup A$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \subseteq (A \cup B)$.
- $A \cup A = A$.
- $A \cup U = U$.
- $A \cup \varnothing = A$.
- $A \subseteq B$ if and only if $A \cup B = B$.

https://en.wikipedia.org/wiki/Set_(mathematics)

Intersections

A new set can also be constructed by determining which members two sets have "in common". The *intersection* of A and B, denoted by $A \cap B$, is the set of all things that are members of both A and B. If $A \cap B = \varnothing$, then A and B are said to be *disjoint*.



The **intersection** of A and B, denoted $A \cap B$.

https://en.wikipedia.org/wiki/Set_(mathematics)

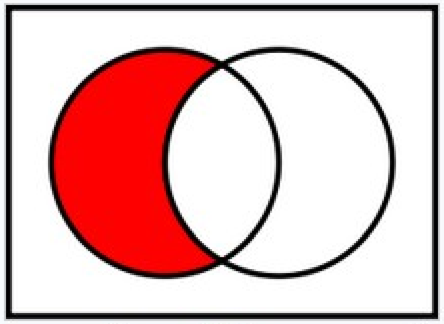
Properties of Intersections

- $A \cap B = B \cap A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$.
- $A \cap B \subseteq A$.
- $A \cap A = A$.
- $A \cap U = A$.
- $A \cap \varnothing = \varnothing$.
- $A \subseteq B$ if and only if $A \cap B = A$.

https://en.wikipedia.org/wiki/Set_(mathematics)

Complements

Two sets can also be "subtracted". The *relative complement* of B in A (also called the *set-theoretic difference* of A and B), denoted by $A \setminus B$ (or $A - B$), is the set of all elements that are members of A but not members of B. Note that it is valid to "subtract" members of a set that are not in the set, such as removing the element *green* from the set {1, 2, 3}; doing so has no effect.



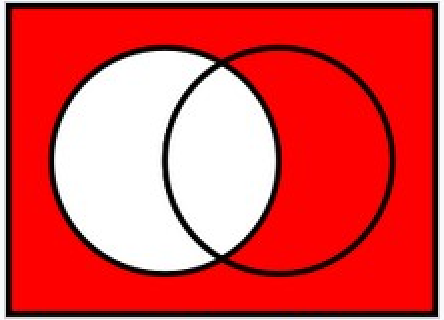
The **relative complement** of B in A

https://en.wikipedia.org/wiki/Set_(mathematics)

Complements

In certain settings all sets under discussion are considered to be subsets of a given universal set U. In such cases, $U \setminus A$ is called the *absolute complement* or simply *complement* of A, and is denoted by A'.

- $A' = U \setminus A$



The **complement** of A in U

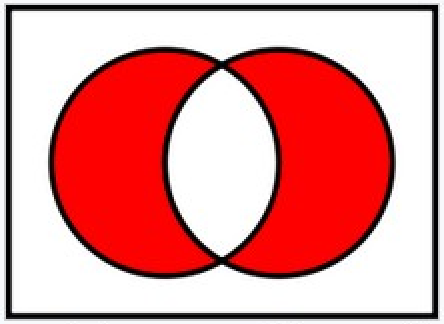
https://en.wikipedia.org/wiki/Set_(mathematics)

Complements

An extension of the complement is the symmetric difference, defined for sets *A*, *B* as

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

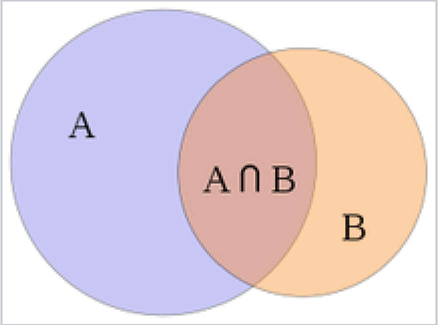
For example, the symmetric difference of {7,8,9,10} and {9,10,11,12} is the set {7,8,11,12}. The power set of any set becomes a Boolean ring with symmetric difference as the addition of the ring (with the empty set as neutral element) and intersection as the multiplication of the ring.



The **symmetric difference** of *A* and *B*

https://en.wikipedia.org/wiki/Set_(mathematics)

Inclusion and Exclusion



The inclusion-exclusion principle can be used to calculate the size of the union of sets: the size of the union is the size of the two sets, minus the size of their intersection.

The inclusion–exclusion principle is a counting technique that can be used to count the number of elements in a union of two sets, if the size of each set and the size of their intersection are known. It can be expressed symbolically as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A more general form of the principle can be used to find the cardinality of any finite union of sets:

https://en.wikipedia.org/wiki/Set_(mathematics)

De Morgan's Law

If A and B are any two sets then,

- $(A \cup B)' = A' \cap B'$

The complement of A union B equals the complement of A intersected with the complement of B.

- $(A \cap B)' = A' \cup B'$

The complement of A intersected with B is equal to the complement of A union to the complement of B.

https://en.wikipedia.org/wiki/Set_(mathematics)

The Same Cardinality

https://en.wikipedia.org/wiki/Set_(mathematics)

References

[1] http://en.wikipedia.org/
[2]

Matrix (2A)

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Row Vector, Column Vector, Square Matrix

Name	Size	Example	Description
Row vector	$1 \times n$	$\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$	A matrix with one row, sometimes used to represent a vector
Column vector	$n \times 1$	$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$	A matrix with one column, sometimes used to represent a vector
Square matrix	$n \times n$	$\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$	A matrix with the same number of rows and columns, sometimes used to represent a linear transformation from a vector space to itself, such as reflection, rotation, or shearing.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \in \mathbb{R}^{m \times n}.$$

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$a_{i,j} = f(i, j).$$

$$a_{ij} = i - j.$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Matrix Addition

Addition	The *sum* **A**+**B** of two *m*-by-*n* matrices **A** and **B** is calculated entrywise: $(\mathbf{A} + \mathbf{B})_{i,j} = \mathbf{A}_{i,j} + \mathbf{B}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$.

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Scalar Multiplication

Scalar multiplication	The product $c\mathbf{A}$ of a number c (also called a scalar in the parlance of abstract algebra) and a matrix $\mathbf{A}$ is computed by multiplying every entry of $\mathbf{A}$ by c: $(c\mathbf{A})_{i,j} = c \cdot \mathbf{A}_{i,j}$. This operation is called *scalar multiplication*, but its result is not named "scalar product" to avoid confusion, since "scalar product" is sometimes used as a synonym for "inner product".

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Transposition

Transposition	The *transpose* of an *m*-by-*n* matrix **A** is the *n*-by-*m* matrix $\mathbf{A}^T$ (also denoted $\mathbf{A}^{tr}$ or ${}^t\mathbf{A}$) formed by turning rows into columns and vice versa: $(\mathbf{A}^T)_{i,j} = \mathbf{A}_{j,i}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

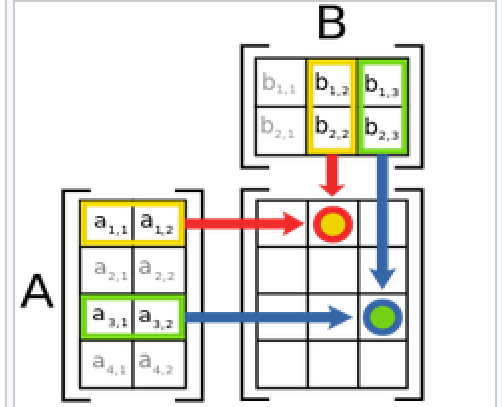
Matrix Multiplication

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their *matrix product* **AB** is the *m*-by-*p* matrix whose entries are given by dot product of the corresponding row of **A** and the corresponding column of **B**:

$$[\mathbf{AB}]_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j} = \sum_{r=1}^{n} A_{i,r}B_{r,j},$$

where $1 \le i \le m$ and $1 \le j \le p$.[13] For example, the underlined entry 2340 in the product is calculated as $(2 \times 1000) + (3 \times 100) + (4 \times 10) = 2340$:

$$\begin{bmatrix} \underline{2} & \underline{3} & \underline{4} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \underline{1000} \\ 1 & \underline{100} \\ 0 & \underline{10} \end{bmatrix} = \begin{bmatrix} 3 & \underline{2340} \\ 0 & 1000 \end{bmatrix}.$$



Schematic depiction of the matrix product **AB** of two matrices **A** and **B**.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Properties

Matrix multiplication satisfies the rules (**AB**)**C** = **A**(**BC**) (associativity), and (**A**+**B**)**C** = **AC**+**BC** as well as **C**(**A**+**B**) = **CA**+**CB** (left and right distributivity), whenever the size of the matrices is such that the various products are defined.[14] The product **AB** may be defined without **BA** being defined, namely if **A** and **B** are m-by-n and n-by-k matrices, respectively, and $m \neq k$. Even if both products are defined, they need not be equal, that is, generally

$$\mathbf{AB} \neq \mathbf{BA},$$

that is, matrix multiplication is not commutative, in marked contrast to (rational, real, or complex) numbers whose product is independent of the order of the factors. An example of two matrices not commuting with each other is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}.$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Matrix Types

Name	Example with n = 3
Diagonal matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
Lower triangular matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
Upper triangular matrix	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Identity Matrix

Identity matrix [edit]

Main article: Identity matrix

The *identity matrix* $\mathbf{I}_n$ of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0, for example,

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

It is a square matrix of order n, and also a special kind of diagonal matrix. It is called an identity matrix because multiplication with it leaves a matrix unchanged:

$\mathbf{A}\mathbf{I}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$ for any m-by-n matrix $\mathbf{A}$.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Inverse Matrix

In relation to its adjugate [edit]

The adjugate of a matrix A can be used to find the inverse of A as follows:

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

In relation to the identity matrix [edit]

It follows from the theory of matrices that if

$$\mathbf{AB} = \mathbf{I}$$

for *finite square* matrices **A** and **B**, then also

$$\mathbf{BA} = \mathbf{I}$$ [1]

https://en.wikipedia.org/wiki/Invertible_matrix

Inverse Matrix Examples

Consider the following 2-by-2 matrix:

$$\mathbf{A} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix}.$$

The matrix $\mathbf{A}$ is invertible. To check this, one can compute that $\det \mathbf{A} = -1/2$, which is non-zero.

As an example of a non-invertible, or singular, matrix, consider the matrix

$$\mathbf{B} = \begin{pmatrix} -1 & \frac{3}{2} \\ \frac{2}{3} & -1 \end{pmatrix}.$$

The determinant of $\mathbf{B}$ is 0, which is a necessary and sufficient condition for a matrix to be non-invertible.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Solving Linear Equations

A set of linear equations

$$a\,x + b\,y = e$$
$$c\,x + d\,y = f$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

If the inverse matrix exists

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} de - bf \\ -ce + af \end{bmatrix}$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Determinant of order 3 only

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Copy and concatenate

Rule of Sarrus

$$\begin{matrix} + & + & + \end{matrix} \qquad\qquad\qquad \begin{matrix} - & - & - \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \qquad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Linear Equations

(Eq 1) $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$

(Eq 2) $a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

(Eq 3) $a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} = \\ = \\ \\ = \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Cramer's Rule (1) – solutions

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\boldsymbol{x}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\boldsymbol{b}}$$

$$\boldsymbol{A}_1 = \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix} \quad \boldsymbol{A}_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}$$

$$x_1 = \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} \qquad x_2 = \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} \qquad x_n = \frac{det(\boldsymbol{A}_n)}{det(\boldsymbol{A})}$$

References

[1] http://en.wikipedia.org/
[2]