

# Relations (3A)

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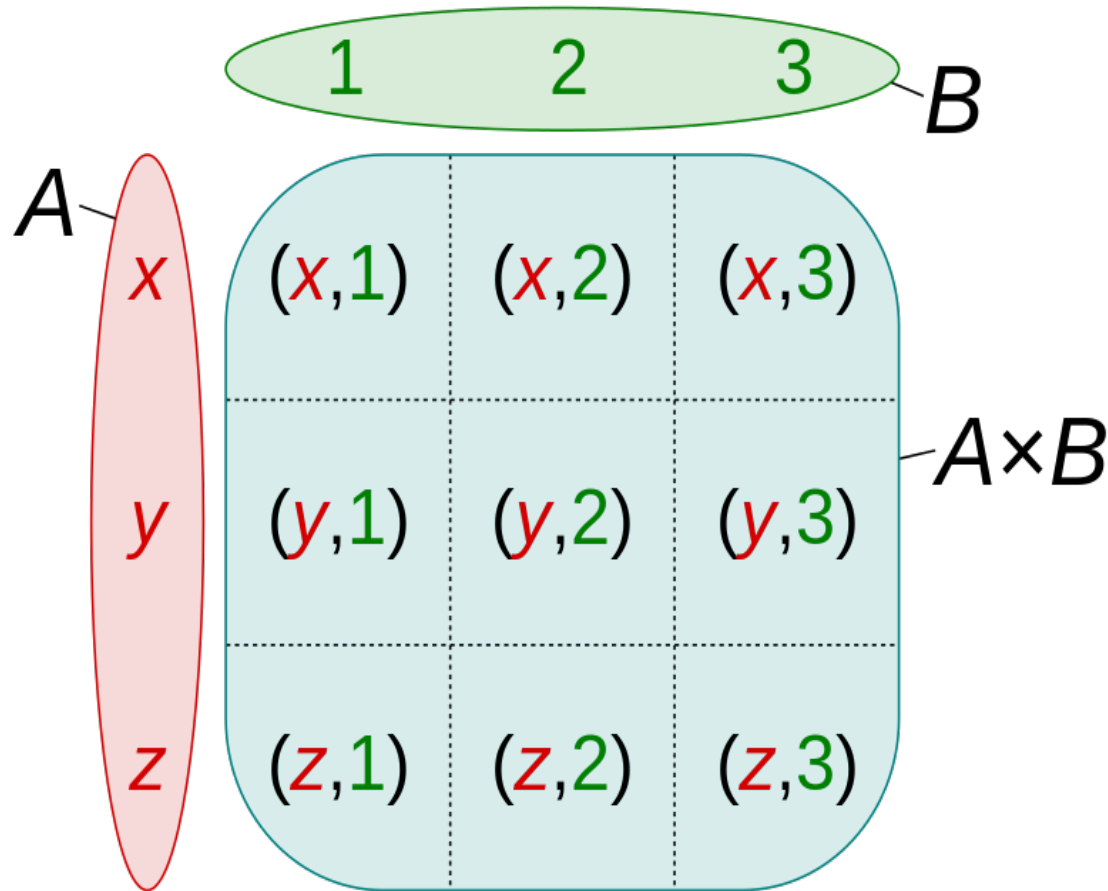
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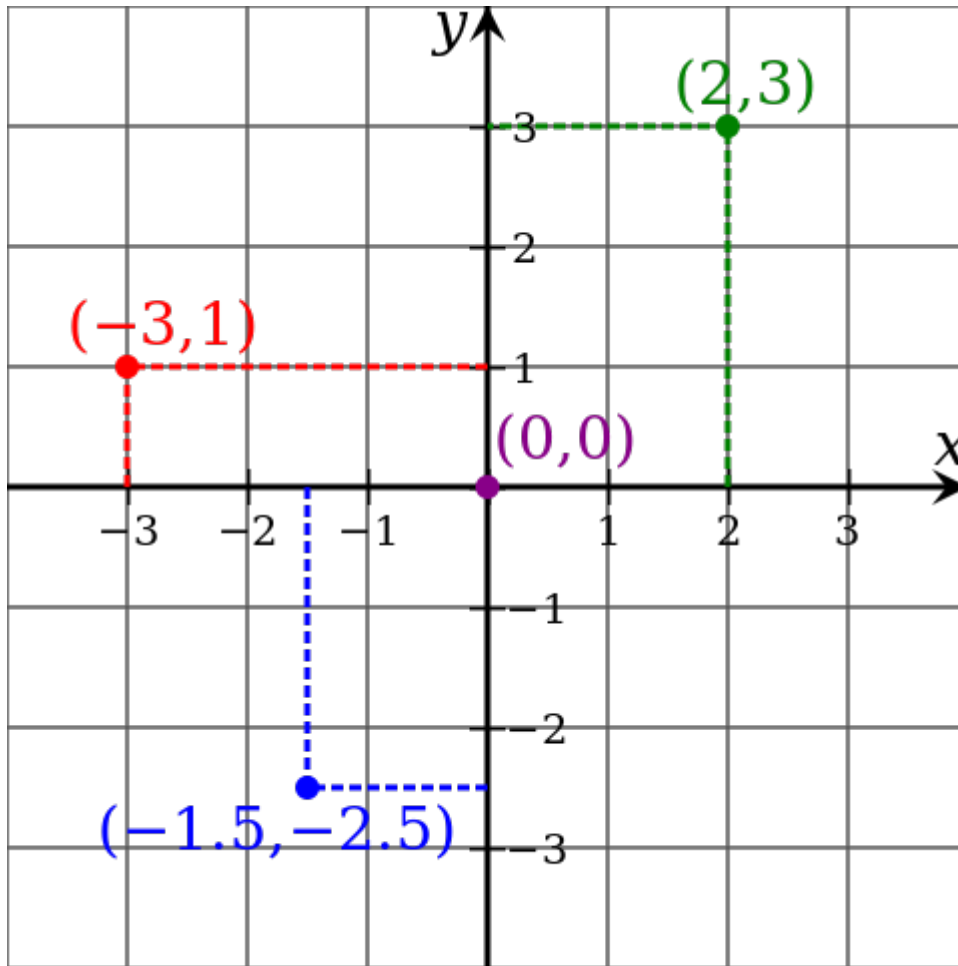
# Cartesian Product



Cartesian product  $A \times B$   
of the sets  $A = \{x, y, z\}$   
and  $B = \{1, 2, 3\}$

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Cartesian Coordinates



[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

Cartesian coordinates of  
example points

# Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

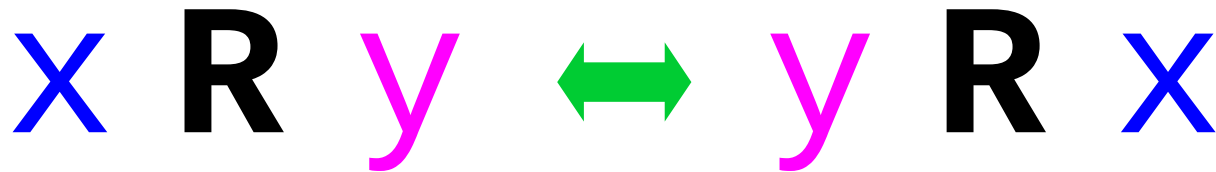
	1	2	3	4	5
1	1R1	1R2	1R3	1R4	1R5
2	2R1	2R2	2R3	2R4	2R5
3	3R1	3R2	3R3	3R4	3R5
4	1R1	4R2	4R3	4R4	4R5
5	5R1	5R2	5R3	5R4	5R5

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Types of Relations (1)



- Reflexive Relation
- Symmetric Relation
- Transitive Relation



[https://en.wikipedia.org/wiki/Reflexive\\_relation](https://en.wikipedia.org/wiki/Reflexive_relation)

# Definitions of Relations

**Reflexive:** for all  $x$  in  $X$  it holds that  $xRx$ .

**Symmetric:** for all  $x$  and  $y$  in  $X$  it holds that if  $xRy$  then  $yRx$ .

**Antisymmetric:** for all  $x$  and  $y$  in  $X$ , if  $xRy$  and  $yRx$  then  $x = y$ .

**Transitive:** for all  $x$ ,  $y$  and  $z$  in  $X$  it holds that if  $xRy$  and  $yRz$  then  $xRz$ .

[https://en.wikipedia.org/wiki/Reflexive\\_relation](https://en.wikipedia.org/wiki/Reflexive_relation)



# More Definitions of Relations

A relation  $R$  on a set  $A$  is called **reflexive** if  $(a, a) \in R$  for every element  $a \in A$ .

A relation  $R$  on a set  $A$  is called **symmetric** if  $(b, a) \in R$  whenever  $(a, b) \in R$ , for all  $a, b \in A$

A relation  $R$  on a set  $A$  such that for all  $a, b$  in  $A$ , if  $(a, b) \in R$  and  $(b, a) \in R$ , then  $a = b$  is called **anti-symmetric**.

A relation  $R$  on a set  $A$  is called **transitive** if whenever  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R$ , for all  $a, b, c \in A$

[https://en.wikipedia.org/wiki/Reflexive\\_relation](https://en.wikipedia.org/wiki/Reflexive_relation)

# Relation Examples (1)

$$x \geq y$$

	1	2	3	4	5
1	(1,1)				
2	(2,1)	(2,2)			
3	(3,1)	(3,2)	(3,3)		
4	(4,1)	(4,2)	(4,3)	(4,4)	
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

$$x > y$$

	1	2	3	4	5
1					
2	(2,1)				
3	(3,1)	(3,2)			
4	(4,1)	(4,2)	(4,3)		
5	(5,1)	(5,2)	(5,3)	(5,4)	

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Relation Examples (2)

$$x = y$$

	1	2	3	4	5
1	(1,1)				
2		(2,2)			
3			(3,3)		
4				(4,4)	
5					(5,5)

$$x = y + 1$$

	1	2	3	4	5
1					
2	(2,1)				
3		(3,2)			
4			(4,3)		
5				(5,4)	

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Relation Examples (3)

$$x + y = 4$$

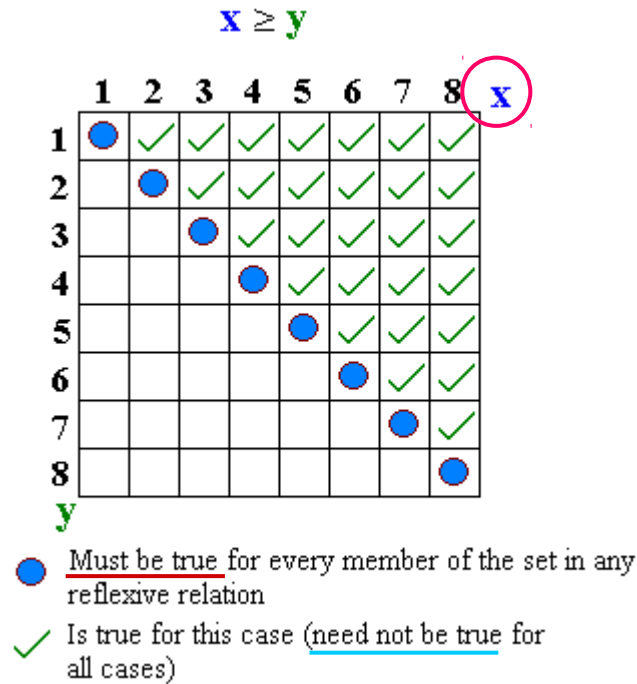
	1	2	3	4	5
1			(1,3)		
2		(2,2)			
3	(3,1)				
4					
5					

$$x + y \leq 4$$

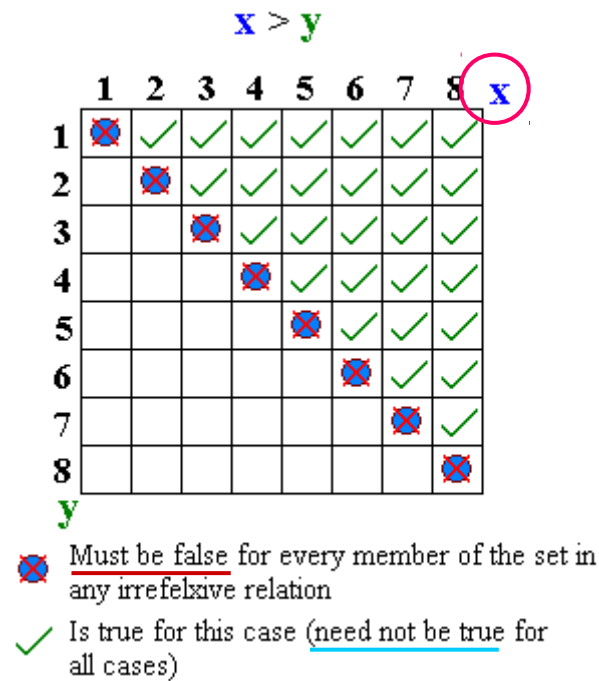
	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)		
2	(2,1)	(2,2)			
3	(3,1)				
4					
5					

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

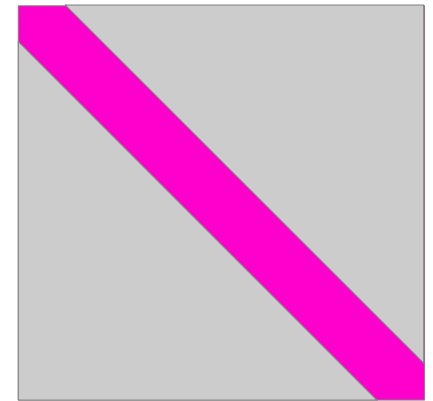
# Reflexive Relation Examples



Reflexive Relation



Irreflexive Relation



[https://en.wikipedia.org/wiki/Reflexive\\_relation](https://en.wikipedia.org/wiki/Reflexive_relation)

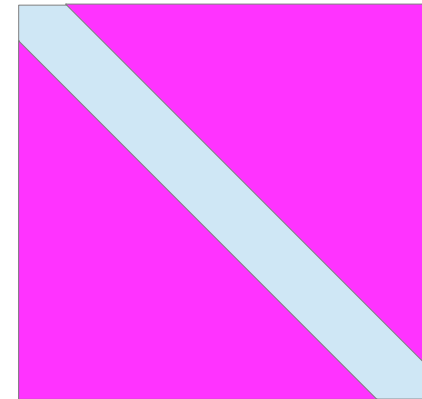
# Symmetric Relation Examples

**X and y are odd**

	1	2	3	4	5	6	7	8	x
1	✓		1		2		3		
2									
3	①		✓		4		5		
4									
5	②		④		✓		6		
6									
7	③		⑤		⑦		✓		
8									
y									

**X and y are odd**

	1	2	3	4	5	6	7	8	x
1	■		■		■		■		
2									
3	■		■		■		■		
4									
5	■		■		■		■		
6									
7	■		■		■		■		
8									
y									



- ✓ Is true for this case (need not be true for all cases)
- ① Must be true if the check mark with the same number (z) is true for it to be a symmetric relation
- ② Is true for this case and requires the circle with the same number (z) to also be true for it to be a symmetric relation

## Symmetric Relation

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Anti-Symmetric Relation Examples

**X is even and y is odd**

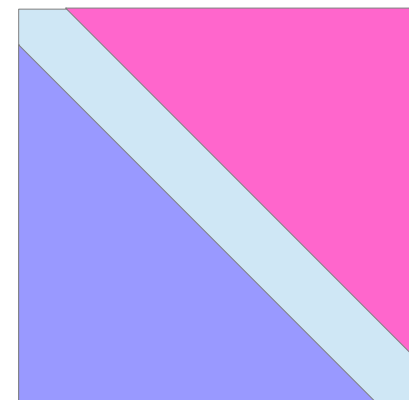
	1	2	3	4	5	6	7	8	x
1	✓			✓		✓		✓	
2	1	11		12		13		14	
3			✓		✓		✓		✓
4	2		5	13		15			
5		12		✓	✓	8		9	
6	3		6		8	16			
7		✓		✓		✓		✓	
8	4		7		9	10			

y

**X is even and y is odd**

	1	2	3	4	5	6	7	8	x
1									
2									
3									
4									
5									
6									
7									
8									

y



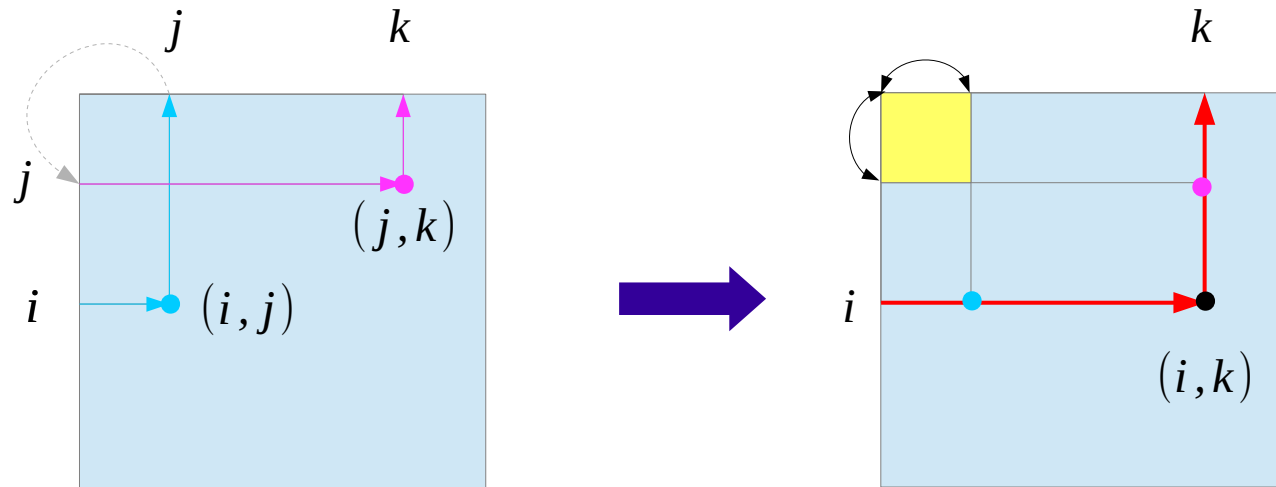
✓ Is true for this case (need not be true for all cases)

✗ Must be false if the check mark with the same number (z) is true for it to be an antisymmetric relation

z ✓ Is true for this case and requires the circle with the same number (z) to be false for it to be a symmetric relation

[https://en.wikipedia.org/wiki/Cartesian\\_product](https://en.wikipedia.org/wiki/Cartesian_product)

# Transitive Relation Examples



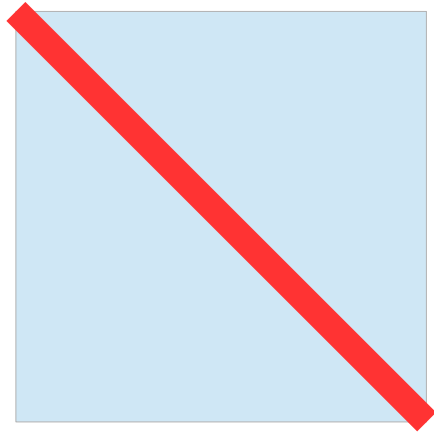
$$(\mathbf{i R j}) \wedge (\mathbf{j R k})$$

$$(\mathbf{i R k})$$



# Reflexive Relation

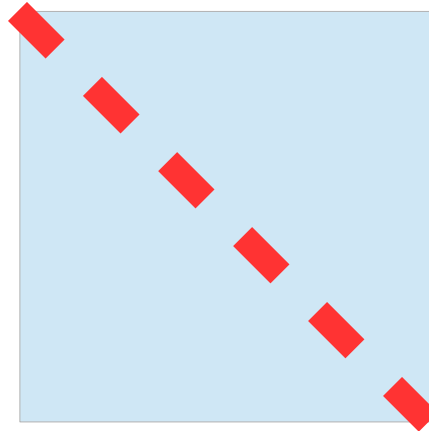
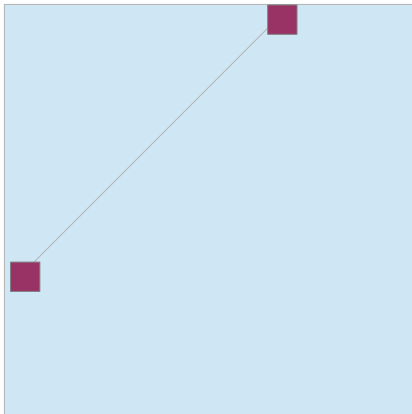
$$\forall x \quad (x, x) \in R$$



All diagonal relations  
must exist

# Symmetric Relation

$$\forall x, \forall y \left[ (x, y) \in R \rightarrow (y, x) \in R \right]$$



no relation is mandatory

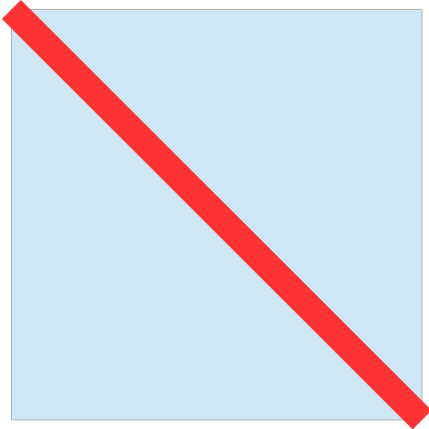
but for any relation,  
its symmetric relation must exist  
including diagonal relations

symmetric

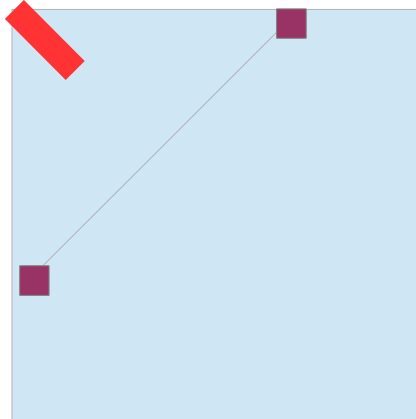
# Symmetric Relation Examples

$$\forall x, \forall y \left[ (x, y) \in R \rightarrow (y, x) \in R \right]$$

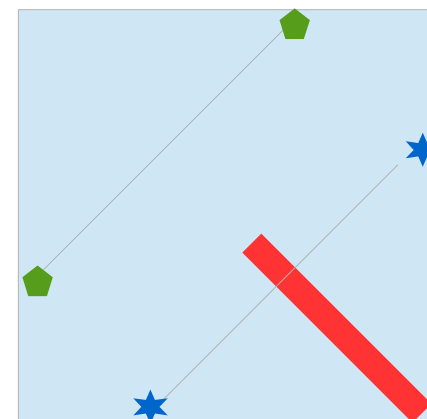
symmetric



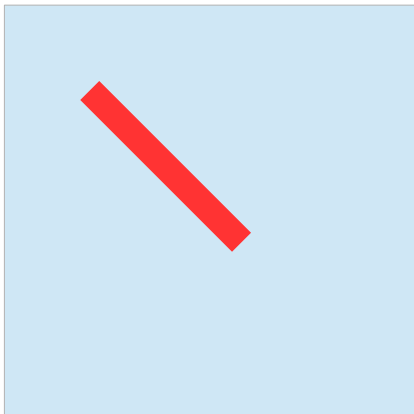
symmetric



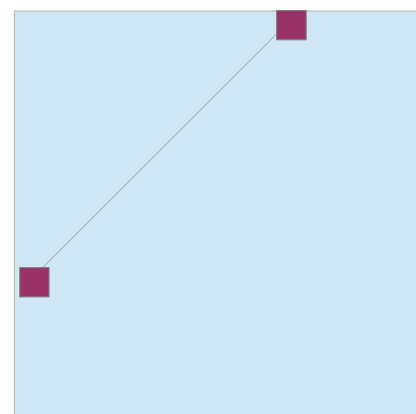
symmetric



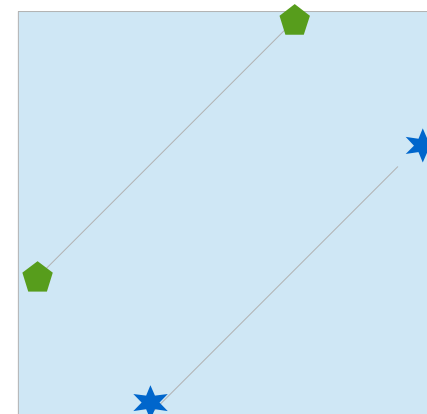
symmetric



symmetric



symmetric



# Not Symmetric Relation

$$\neg \{ \forall x, \forall y [ (x, y) \in R ] \rightarrow [ (y, x) \in R ] \}$$

$$\exists x, \exists y \neg \{ [ (x, y) \in R ] \rightarrow [ (y, x) \in R ] \}$$

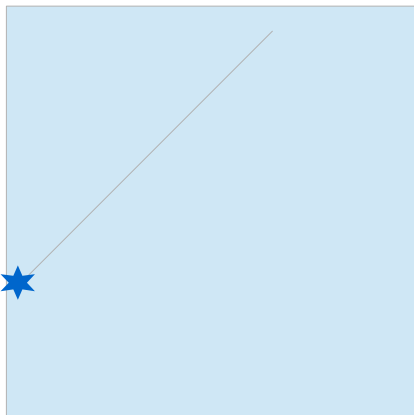
$$\exists x, \exists y \neg \{ \neg [ (x, y) \in R ] \vee [ (y, x) \in R ] \}$$

$$\exists x, \exists y [ (x, y) \in R ] \wedge \neg [ (y, x) \in R ]$$

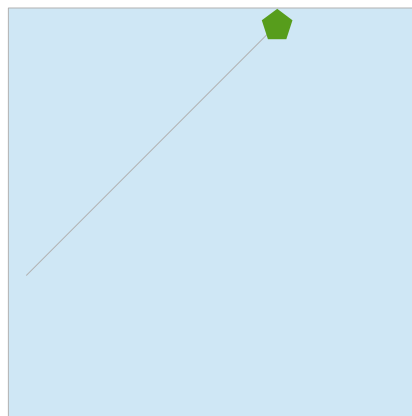
$$\exists x, \exists y [ (x, y) \in R ] \wedge [ (y, x) \notin R ]$$

counter example

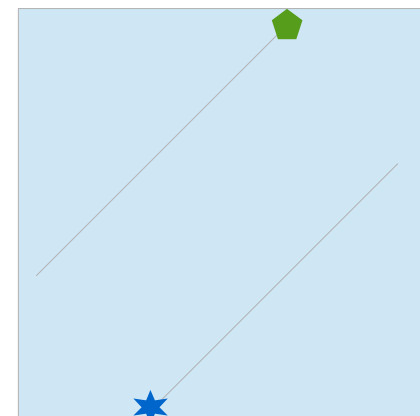
not symmetric



not symmetric

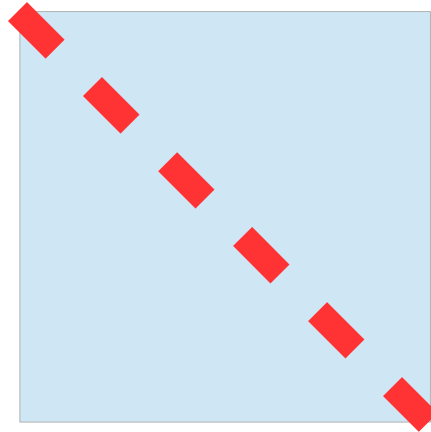
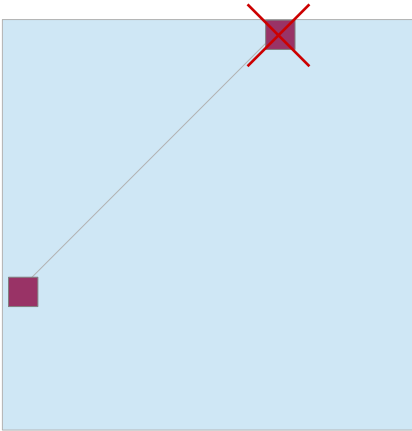


not symmetric



# Anti-symmetric Relation

$$\forall x, \forall y \left[ \left( (x, y) \in R \wedge (y, x) \in R \right) \rightarrow x = y \right]$$



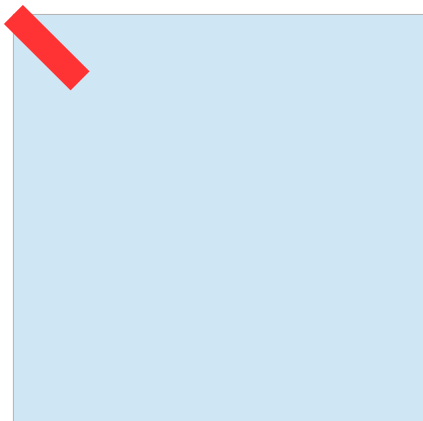
no relation is mandatory

but for any relation,  
its symmetric relation must NOT exist  
excluding diagonal relations

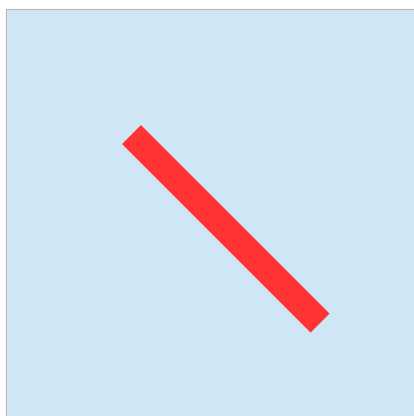
# Anti-symmetric Relation Examples

$$\forall x, \forall y \left[ \left( (x, y) \in R \wedge (y, x) \in R \right) \rightarrow x = y \right]$$

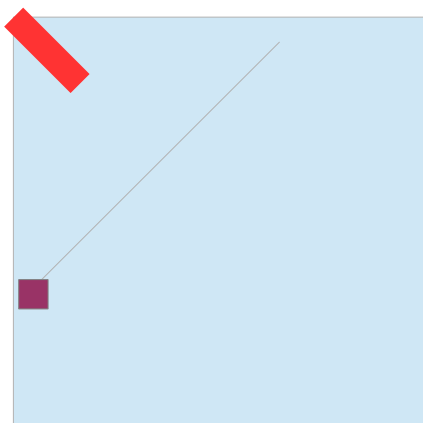
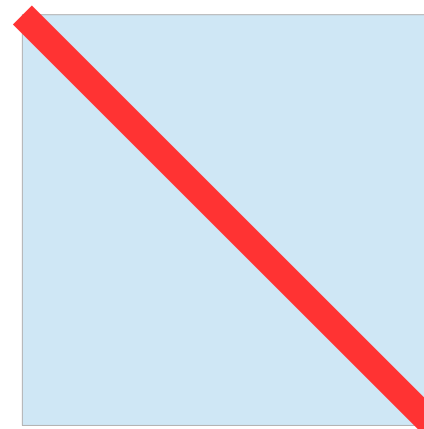
anti-symmetric



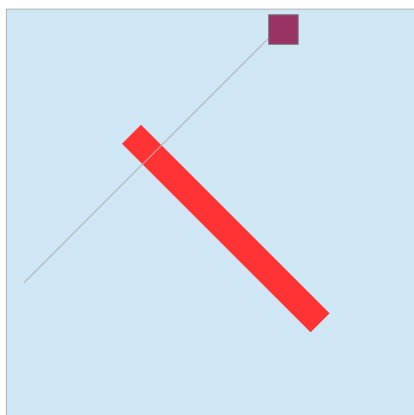
anti-symmetric



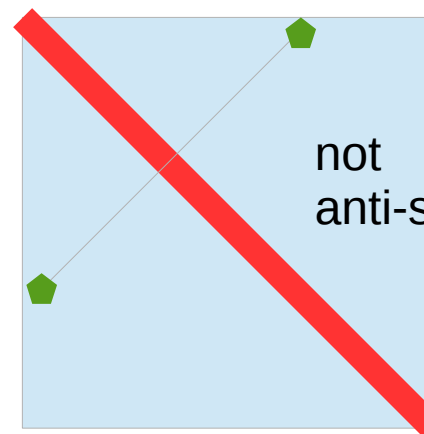
anti-symmetric



anti-symmetric



anti-symmetric



not  
anti-symmetric

# Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ]\}$$

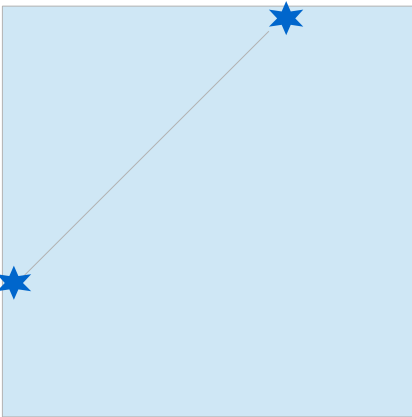
$$\exists x, \exists y \neg\{ [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ]\}$$

$$\exists x, \exists y \neg\{\neg [ (x, y) \in R \wedge (y, x) \in R ] \vee [ x = y ]\}$$

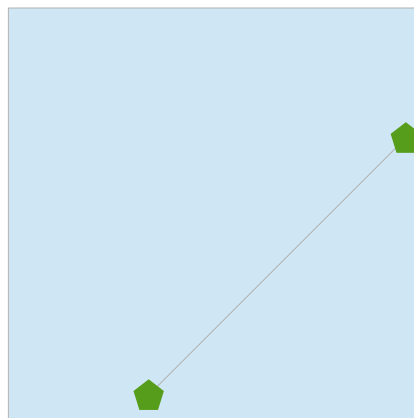
$$\exists x, \exists y\{ [ (x, y) \in R \wedge (y, x) \in R ] \wedge \neg [ x = y ]\}$$

$$\exists x, \exists y\{ [ (x, y) \in R \wedge (y, x) \in R ] \wedge [ x \neq y ]\} \quad \text{counter example}$$

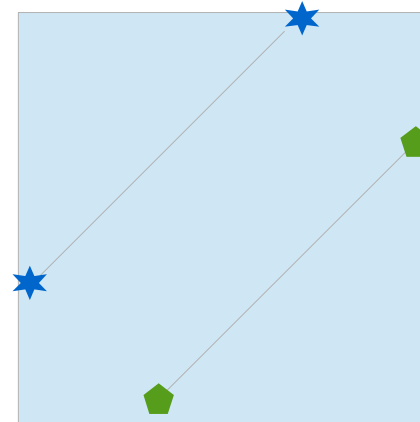
not anti-symmetric



not anti-symmetric



not anti-symmetric



# Not Symmetric vs Not Anti-Symmetric Relation

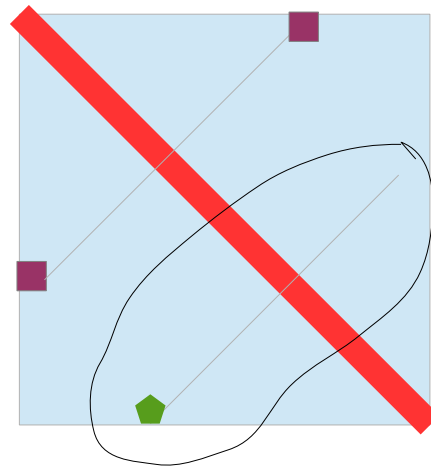
$$\neg \{ \forall x, \forall y [ (x, y) \in R ] \rightarrow [ (y, x) \in R ] \} \iff$$
$$\exists x, \exists y [ (x, y) \in R ] \wedge [ (y, x) \notin R ]$$

not symmetric

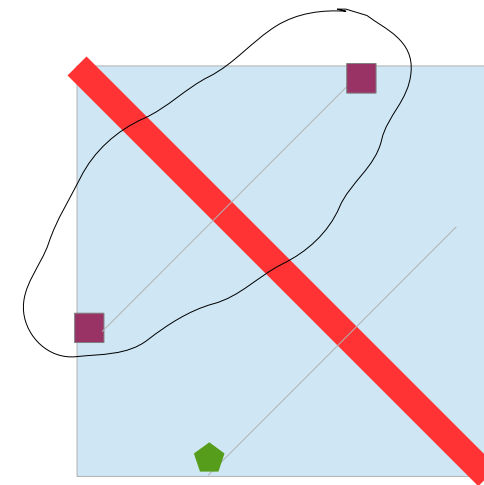
$$\neg \{ \forall x, \forall y [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ] \} \iff$$
$$\exists x, \exists y \{ [ (x, y) \in R \wedge (y, x) \in R ] \wedge [ x \neq y ] \}$$

not anti-symmetric

neither  
symmetric



nor  
anti-symmetric



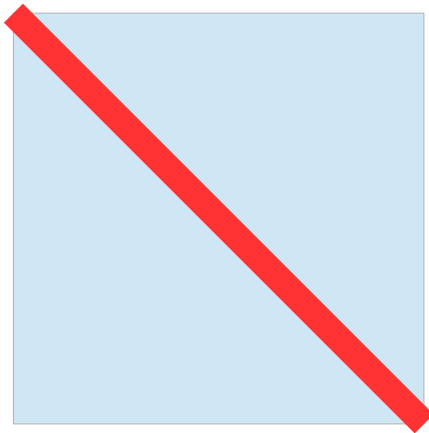


# Reflexive, Symmetric, Anti-symmetric

$$\forall x \ (x, x) \in R$$

$$\forall x, \forall y \ [ (x, y) \in R ] \rightarrow [ (y, x) \in R ]$$

$$\forall x, \forall y \ [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ]$$



Reflexive

Also, symmetric

Also, anti-symmetric

(no relation for  $(x, y)$  where  $x \neq y$ )

(no relation for  $(x, y)$  where  $x \neq y$ )

# Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ]\}$$

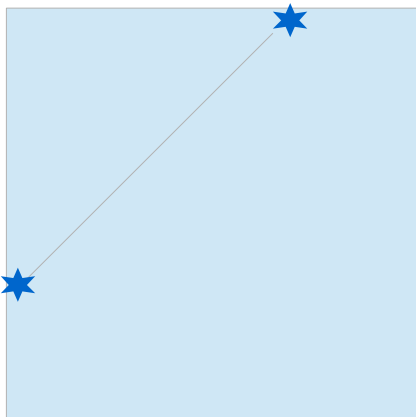
$$\exists x, \exists y \neg\{ [ (x, y) \in R \wedge (y, x) \in R ] \rightarrow [ x = y ]\}$$

$$\exists x, \exists y \neg\{\neg [ (x, y) \in R \wedge (y, x) \in R ] \vee [ x = y ]\}$$

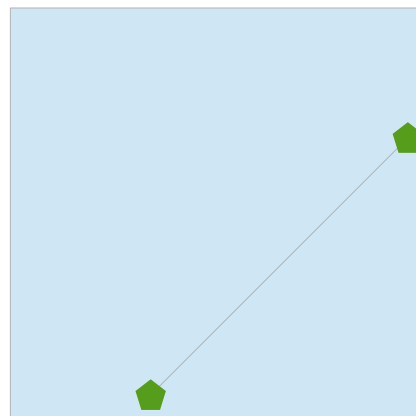
$$\exists x, \exists y\{ [ (x, y) \in R \wedge (y, x) \in R ] \wedge \neg [ x = y ]\}$$

$$\exists x, \exists y\{ [ (x, y) \in R \wedge (y, x) \in R ] \wedge [ x \neq y ]\} \quad \text{counter example}$$

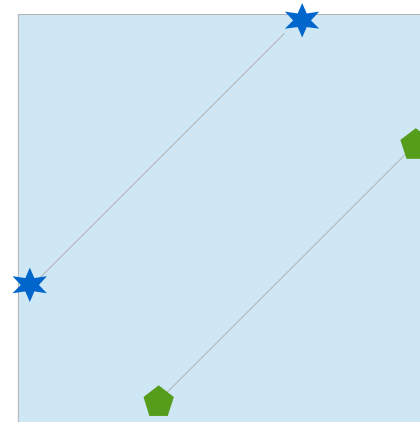
not anti-symmetric



not anti-symmetric

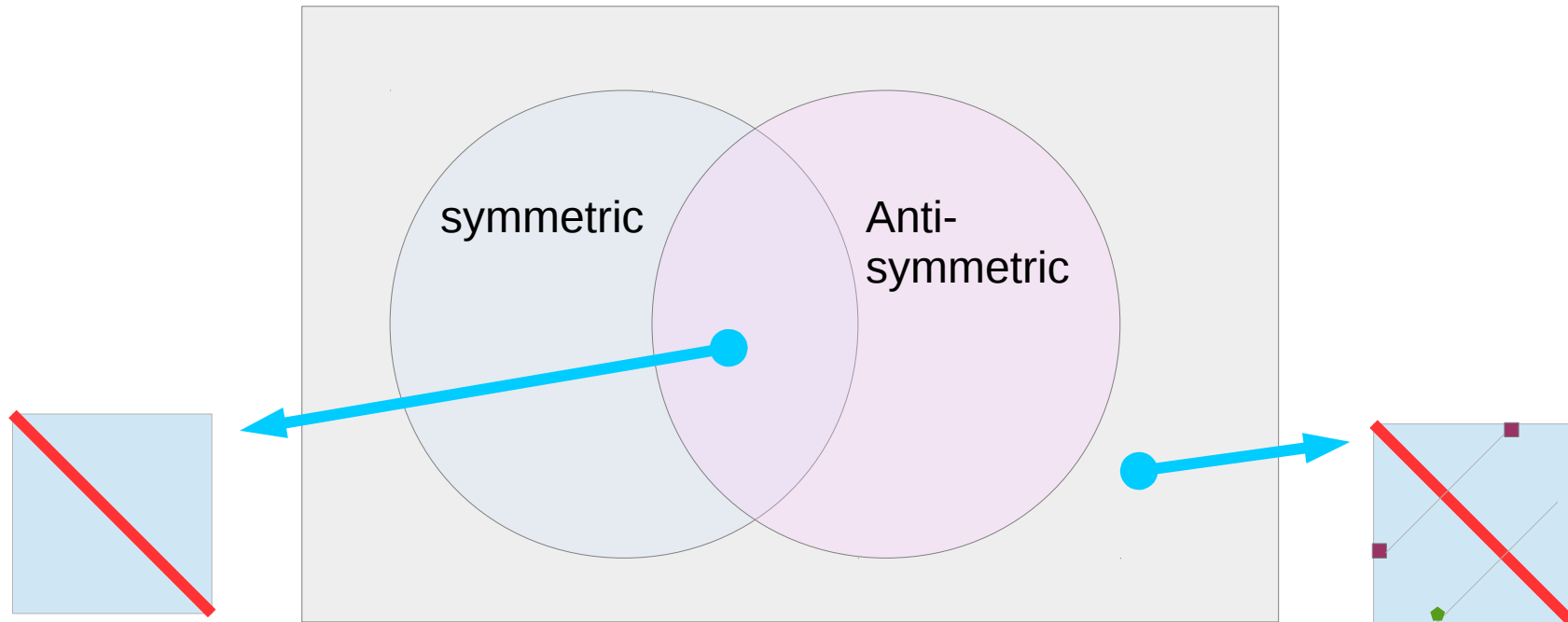


not anti-symmetric



# Symmetric vs Anti-Symmetric Relation

not symmetric  $\neq$  anti-symmetric  
not anti-symmetric  $\neq$  symmetric



## References

- [1] <http://en.wikipedia.org/>
- [2]