Relations (3A)

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3/27/18

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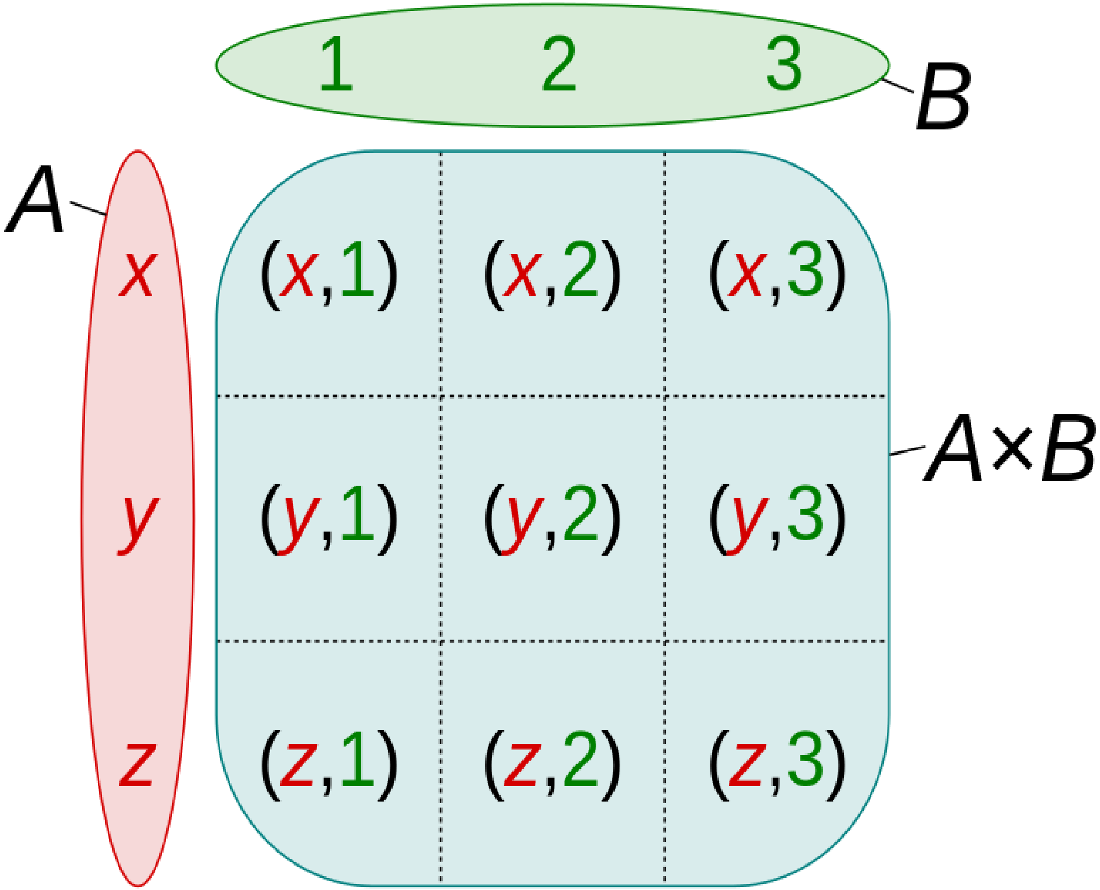
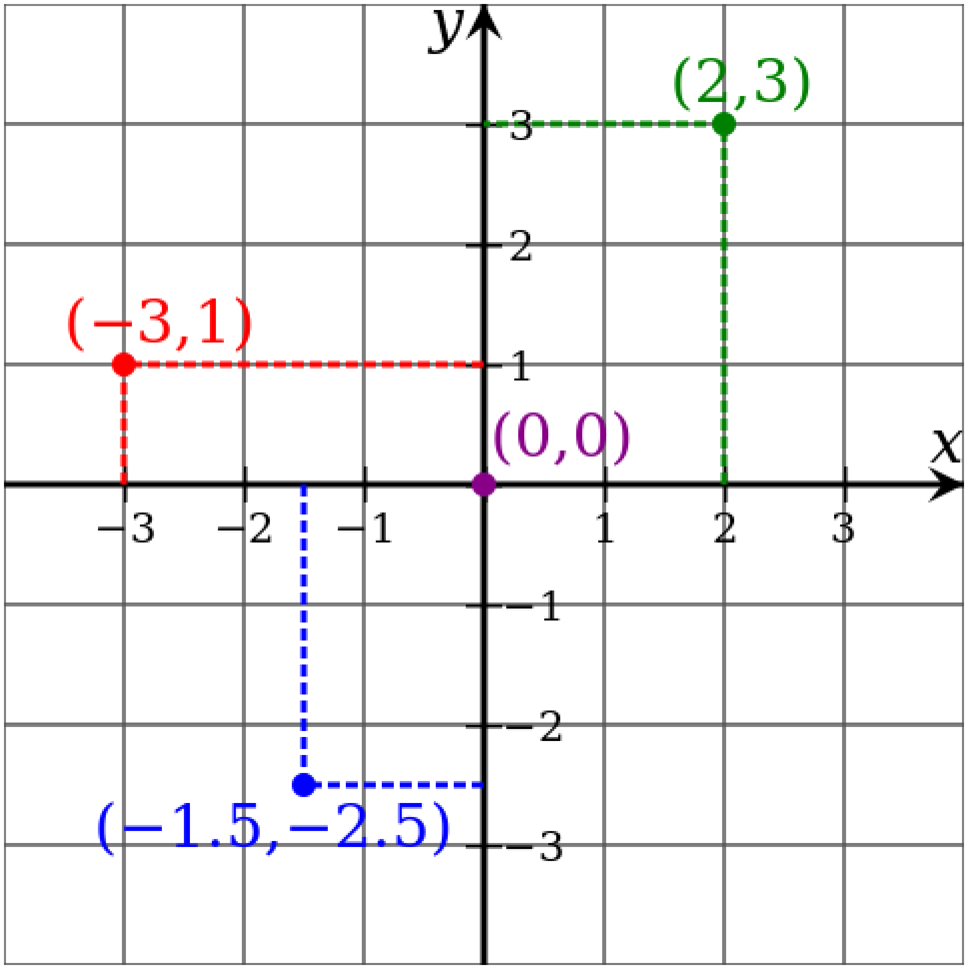
Cartesian Product



Cartesian product A × B
of the sets A = { x , y , z }
and B = { 1 , 2 , 3 }

https://en.wikipedia.org/wiki/Cartesian_product

Cartesian Coordinates



https://en.wikipedia.org/wiki/Cartesian_product

Cartesian coordinates of example points

Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

https://en.wikipedia.org/wiki/Cartesian_product

Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

	1	2	3	4	5
1	1**R**1	1**R**2	1**R**3	1**R**4	1**R**5
2	2**R**1	2**R**2	2**R**3	2**R**4	2**R**5
3	3**R**1	3**R**2	3**R**3	3**R**4	3**R**5
4	1**R**1	4**R**2	4**R**3	4**R**4	4**R**5
5	5**R**1	5**R**2	5**R**3	5**R**4	5**R**5

https://en.wikipedia.org/wiki/Cartesian_product

$$x \mathrel{\mathbf{R}} x$$

$$x \mathrel{\mathbf{R}} y \iff y \mathrel{\mathbf{R}} x$$

$$x \mathrel{\mathbf{R}} y \wedge y \mathrel{\mathbf{R}} z \Rightarrow x \mathrel{\mathbf{R}} z$$

- Reflexive Relation
- Symmetric Relation
- Transitive Relation

https://en.wikipedia.org/wiki/Reflexive_relation

Definitions of Relations

Reflexive: for all x in X it holds that x**Rx.**

Symmetric: for all x and y in X it holds that if x**R**y then y**R**x.

Antisymmetric: for all x and y in X, if x**R**y and y**R**x then x = y.

Transitive: for all x, y and z in X it holds that if x**R**y and y**R**z then x**R**z.

https://en.wikipedia.org/wiki/Reflexive_relation

More Definitions of Relations

A relation **R** on a set ***A*** is called **reflexive**
if (a, a) ∈ **R** for every element a ∈ ***A***.

A relation **R** on a set ***A*** is called **symmetric**
if (b, a) ∈ **R** whenever (a, b) ∈ **R**, for all a, b ∈ ***A***

A relation **R** on a set ***A*** such that for all a, b in **R**,
if (a, b) ∈ **R** and (b, a) ∈ **R**, then a = b is called **anti-symmetric**.

A relation **R** on a set ***A*** is called **transitive**
if whenever (a, b) ∈ **R** and (b, c) ∈ **R**,
then (a, c) ∈ **R**, for all a, b, c ∈ ***A***

https://en.wikipedia.org/wiki/Reflexive_relation

Relation Examples (1)

$x \geq y$

	1	2	3	4	5
1	(1,1)				
2	(2,1)	(2,2)			
3	(3,1)	(3,2)	(3,3)		
4	(4,1)	(4,2)	(4,3)	(4,4)	
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

$x > y$

	1	2	3	4	5
1					
2	(2,1)				
3	(3,1)	(3,2)			
4	(4,1)	(4,2)	(4,3)		
5	(5,1)	(5,2)	(5,3)	(5,4)	

https://en.wikipedia.org/wiki/Cartesian_product

Relation Examples (2)

$$x = y$$

	1	2	3	4	5
1	(1,1)				
2		(2,2)			
3			(3,3)		
4				(4,4)	
5					(5,5)

$$x = y+1$$

	1	2	3	4	5
1					
2	(2,1)				
3		(3,2)			
4			(4,3)		
5				(5,4)	

https://en.wikipedia.org/wiki/Cartesian_product

Relation Examples (3)

$$x + y = 4$$

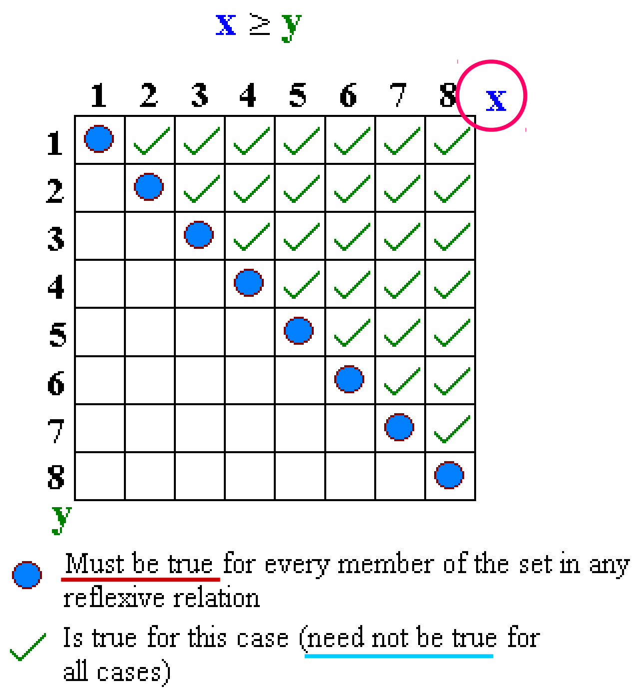
	1	2	3	4	5
1			(1,3)		
2		(2,2)			
3	(3,1)				
4					
5					

$$x + y \leq 4$$

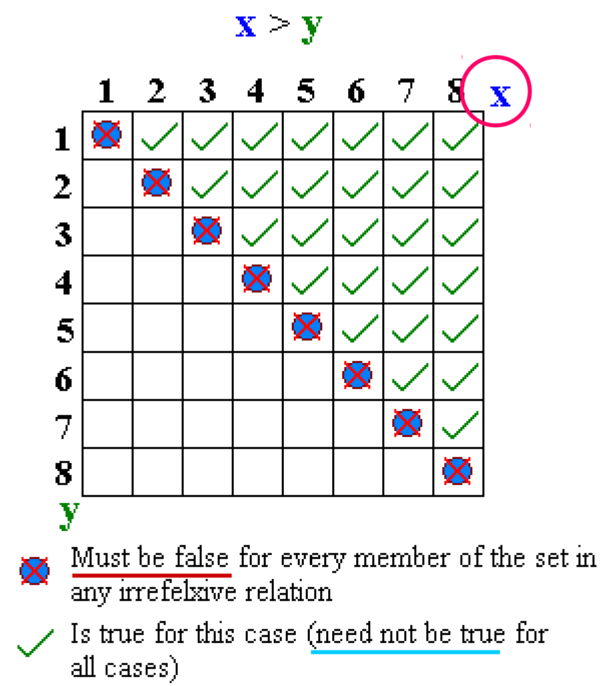
	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)		
2	(2,1)	(2,2)			
3	(3,1)				
4					
5					

https://en.wikipedia.org/wiki/Cartesian_product

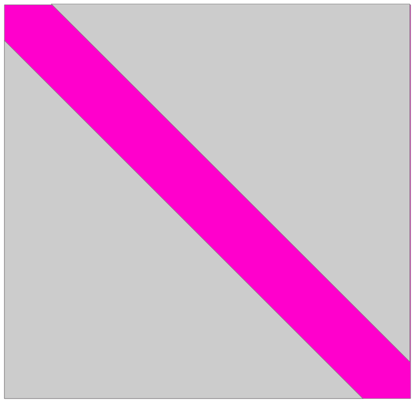
Reflexive Relation Examples



Reflexive Relation

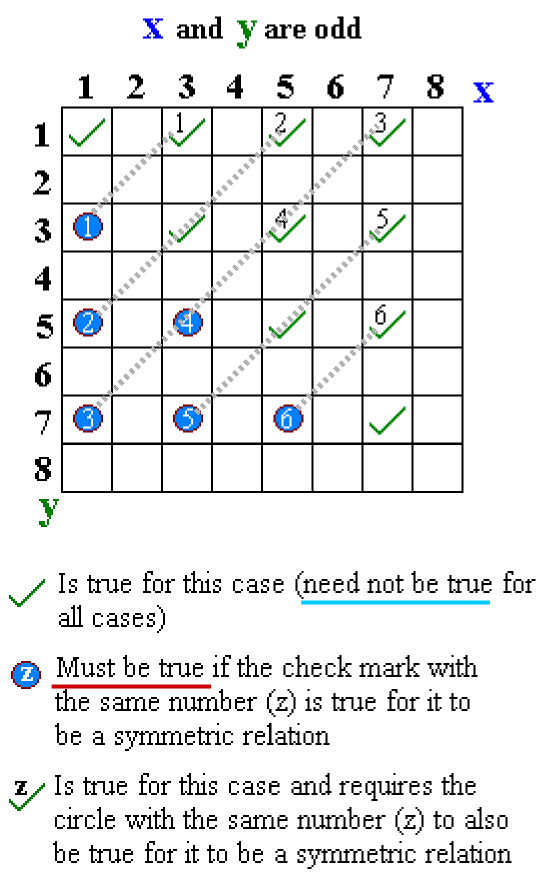


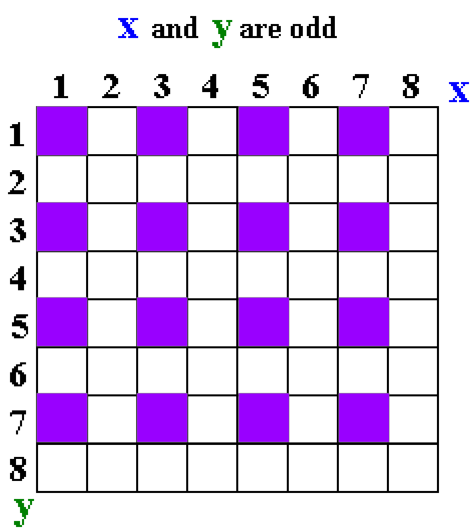
Irreflexive Relation

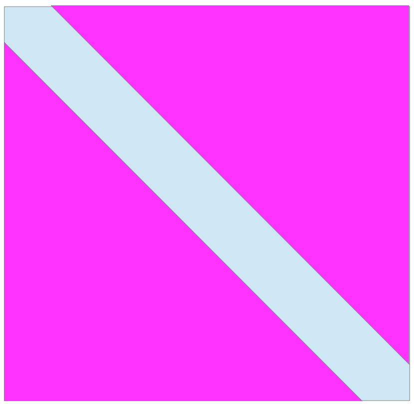


https://en.wikipedia.org/wiki/Reflexive_relation

Symmetric Relation Examples



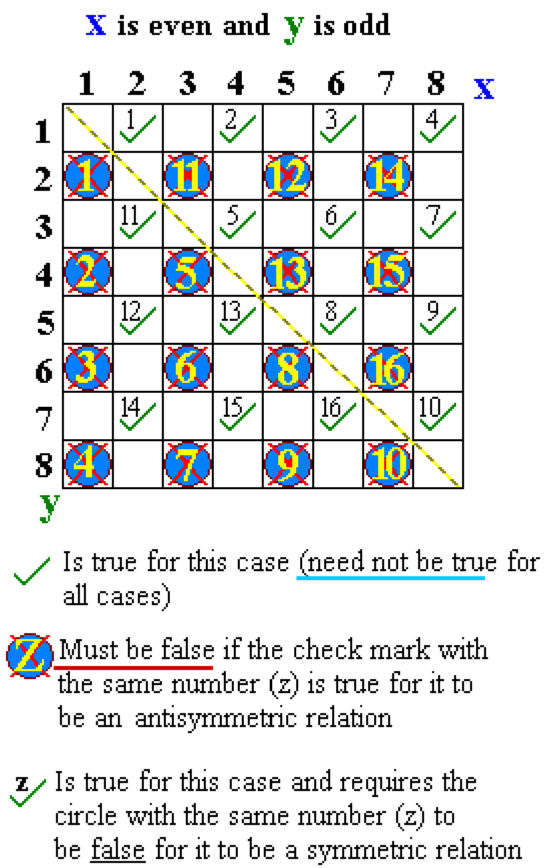


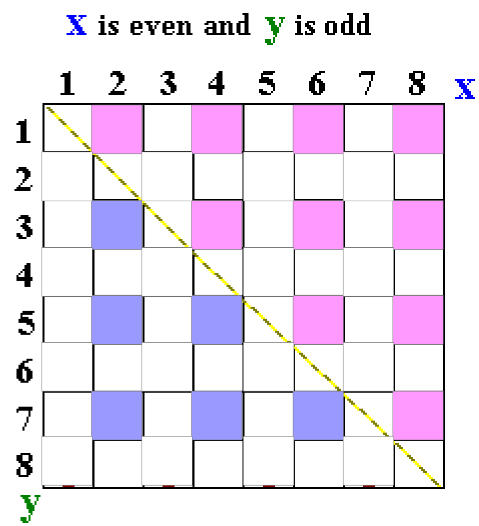


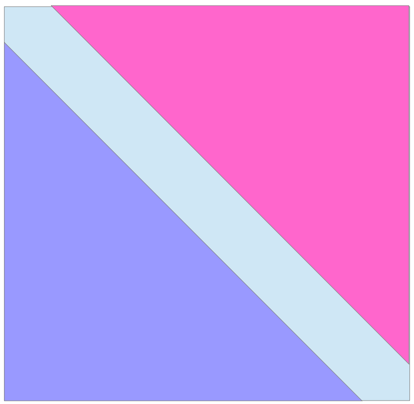
Symmetric Relation

https://en.wikipedia.org/wiki/Cartesian_product

Anti-Symmetric Relation Examples

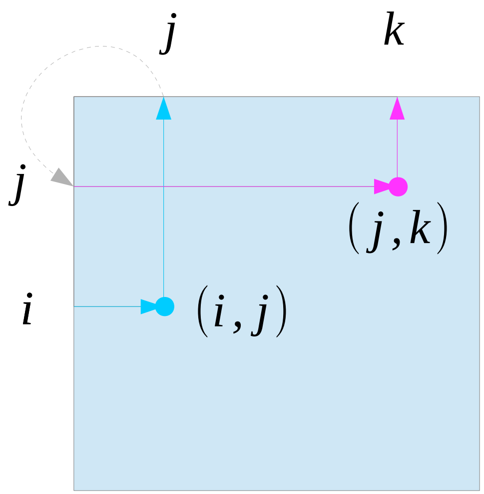


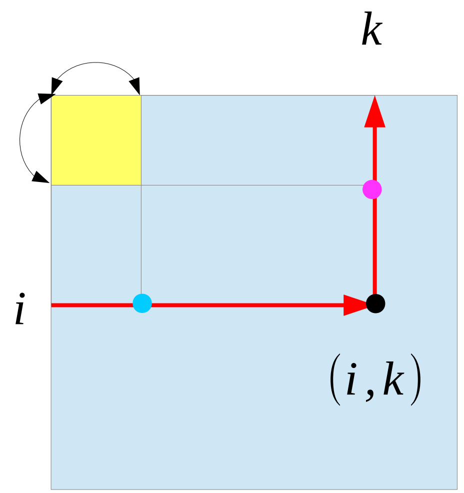




https://en.wikipedia.org/wiki/Cartesian_product

Transitive Relation Examples

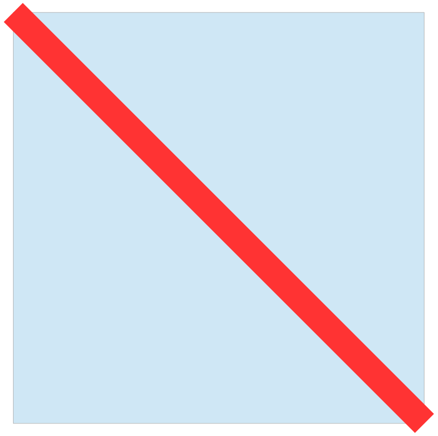




$(i \, R \, j) \wedge (j \, R \, k)$ $(i \, R \, k)$

Reflexive Relation

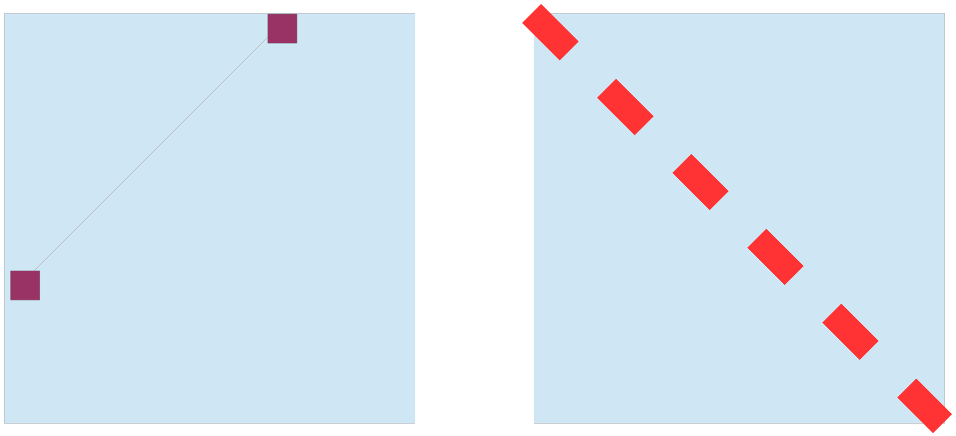
$$\forall x \quad (x, x) \in R$$



All diagonal relations
<u>must</u> exist

Symmetric Relation

$$\forall x, \forall y \; [\; (x, y) \in R \rightarrow (y, x) \in R \;]$$



no relation is mandatory

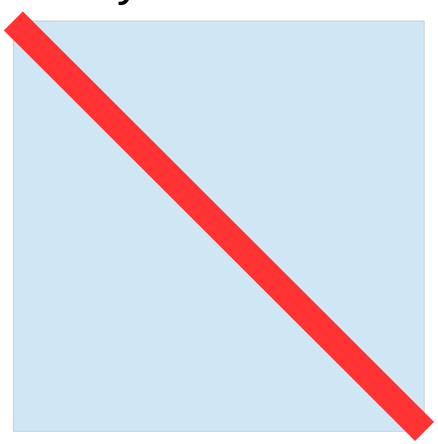
but for any relation,
its symmetric relation must exist
including diagonal relations

symmetric

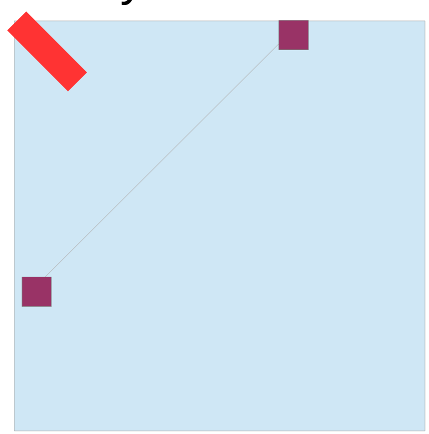
Symmetric Relation Examples

$$\forall x, \forall y \; [\; (x, y) \in R \rightarrow (y, x) \in R \;]$$

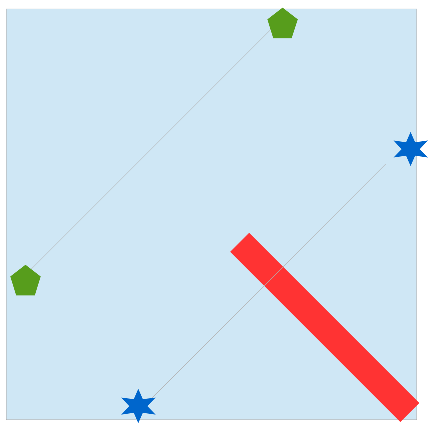
symmetric

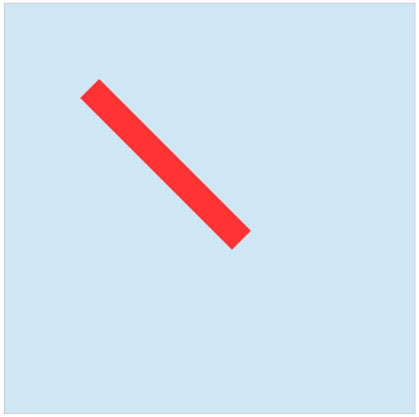


symmetric

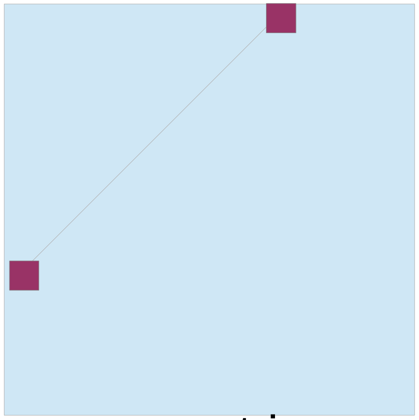


symmetric

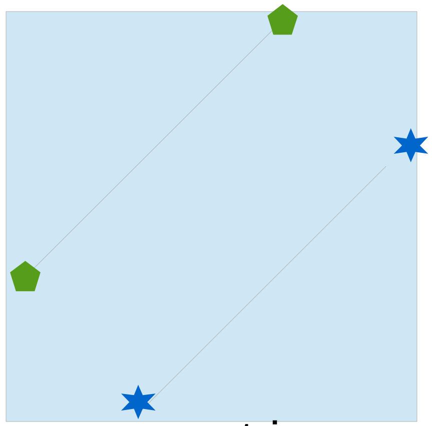




symmetric



symmetric



symmetric

Not Symmetric Relation

$$\neg\{\ \forall x, \forall y\ [\ (x, y) \in R\ ] \rightarrow [\ (y, x) \in R\ ]\ \}$$

$$\exists x, \exists y\ \neg\{\ [\ (x, y) \in R\ ] \rightarrow [\ (y, x) \in R\ ]\ \}$$

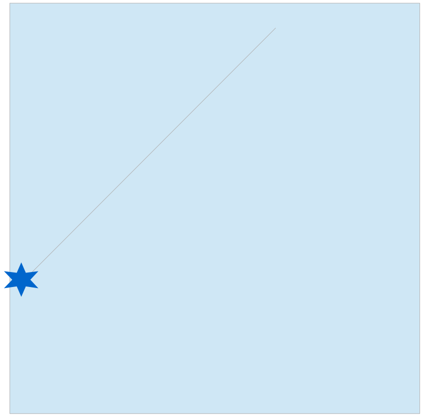
$$\exists x, \exists y\ \neg\{\ \neg[\ (x, y) \in R\ ] \vee [\ (y, x) \in R\ ]\ \}$$

$$\exists x, \exists y\ [\ (x, y) \in R\ ] \wedge \neg[\ (y, x) \in R\ ]$$

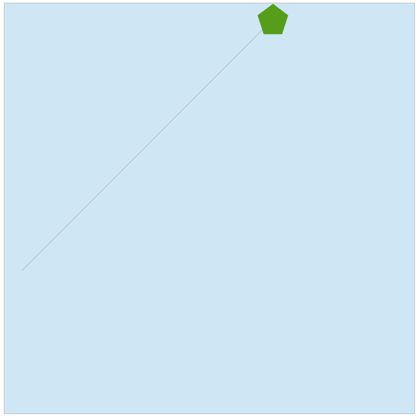
$$\exists x, \exists y\ [\ (x, y) \in R\ ] \wedge [\ (y, x) \notin R\ ]$$

counter example

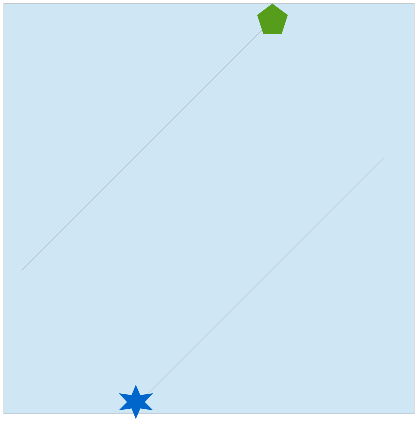
not symmetric



not symmetric

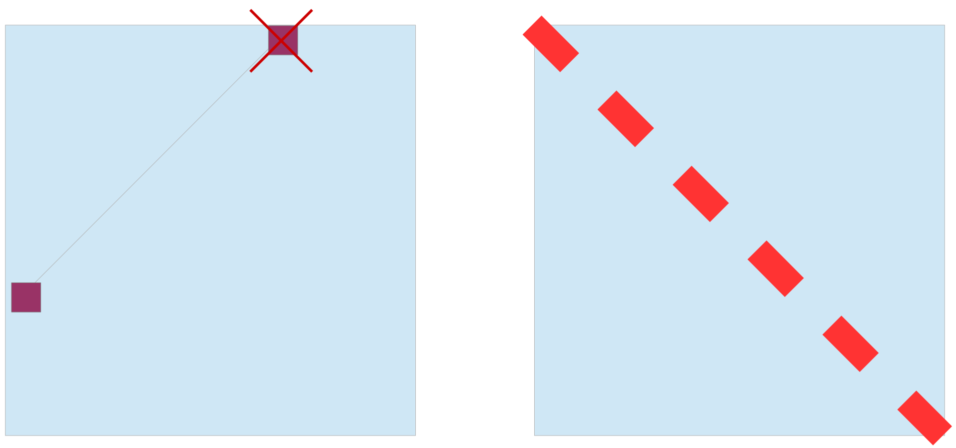


not symmetric



Anti-symmetric Relation

$$\forall x, \forall y \ \left[\left((x, y) \in R \ \wedge \ (y, x) \in R \right) \rightarrow \ x = y \right]$$



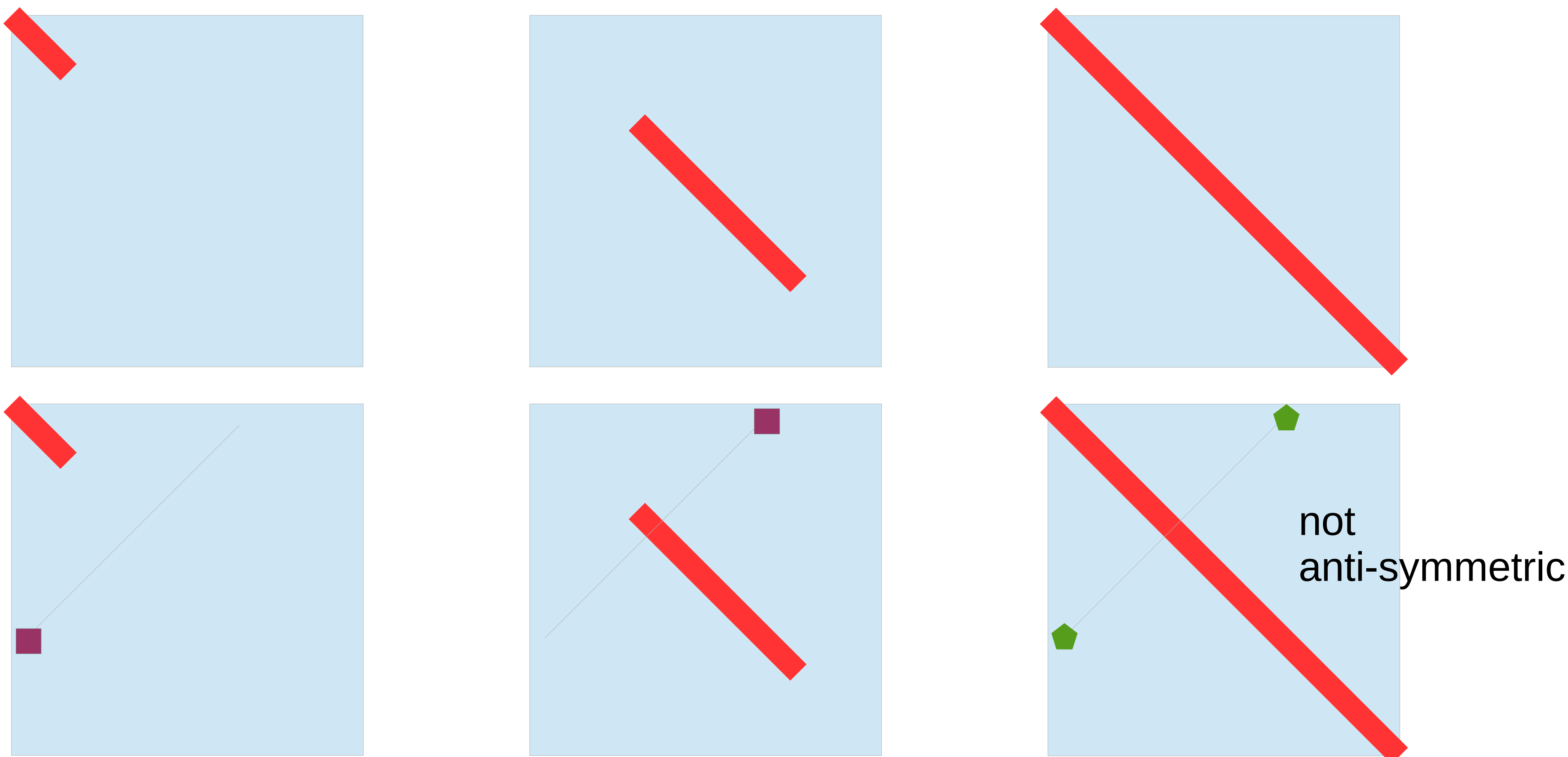
no relation is mandatory

but for any relation,
its symmetric relation must NOT exist
excluding diagonal relations

Anti-symmetric Relation Examples

$$\forall x, \forall y \; \left[\left((x, y) \in R \;\land\; (y, x) \in R \right) \rightarrow \; x = y \right]$$

anti-symmetric　　anti-symmetric　　anti-symmetric



anti-symmetric　　anti-symmetric

Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y \; [\, (x, y) \in R \;\wedge\; (y, x) \in R \,] \rightarrow [\, x = y \,]\}$$

$$\exists x, \exists y \neg\{ [\, (x, y) \in R \;\wedge\; (y, x) \in R \,] \rightarrow [\, x = y \,]\}$$

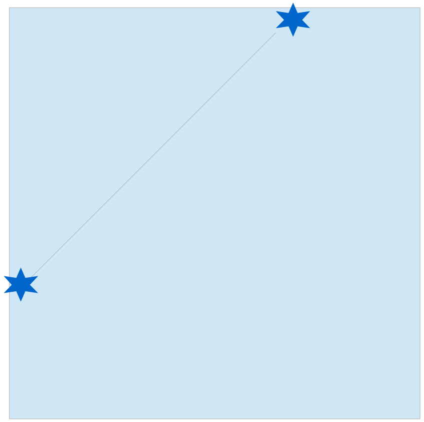
$$\exists x, \exists y \neg\{\neg [\, (x, y) \in R \;\wedge\; (y, x) \in R \,] \vee [\, x = y \,]\}$$

$$\exists x, \exists y \{ [\, (x, y) \in R \;\wedge\; (y, x) \in R \,] \wedge \neg[\, x = y \,]\}$$

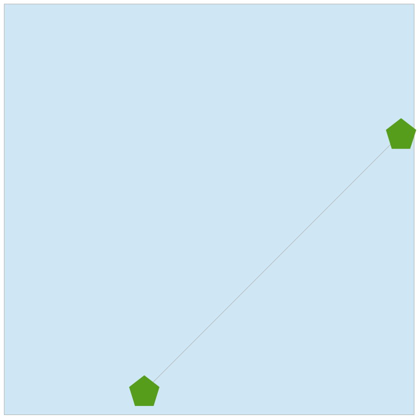
$$\exists x, \exists y \{ [\, (x, y) \in R \;\wedge\; (y, x) \in R \,] \wedge [\, x \neq y \,]\}$$

counter example

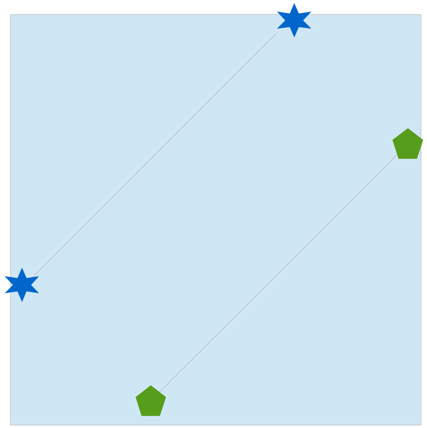
not anti-symmetric



not anti-symmetric



not anti-symmetric



Not Symmetric vs Not Anti-Symmetric Relation

$$\neg\{\ \forall x, \forall y\ [\ (x, y) \in R\ ] \rightarrow [\ (y, x) \in R\ ]\ \}$$ ⟺

$$\exists x, \exists y\ [\ (x, y) \in R\ ] \wedge [\ (y, x) \notin R\ ]$$

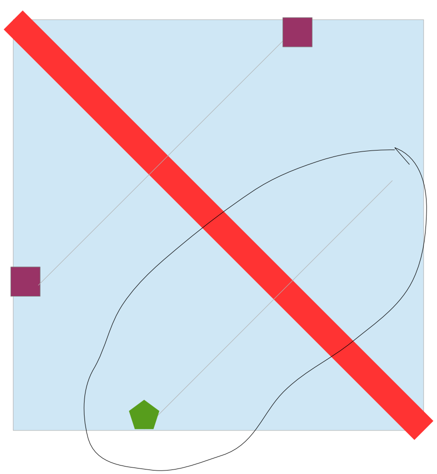
<u>not</u> symmetric

$$\neg\{\forall x, \forall y\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \rightarrow [\ x = y\ ]\}$$ ⟺

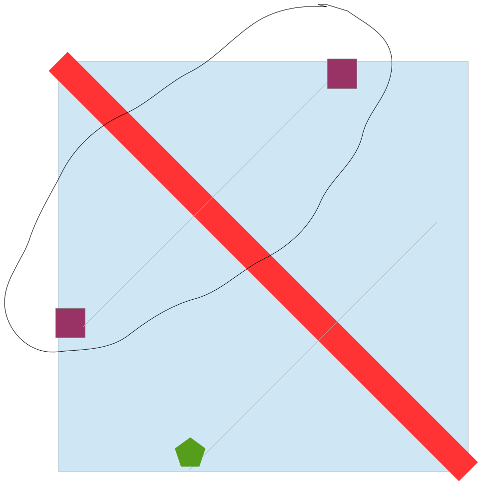
$$\exists x, \exists y\{\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \wedge [\ x \neq y\ ]\}$$

<u>not</u> anti-symmetric

neither symmetric



nor anti-symmetric

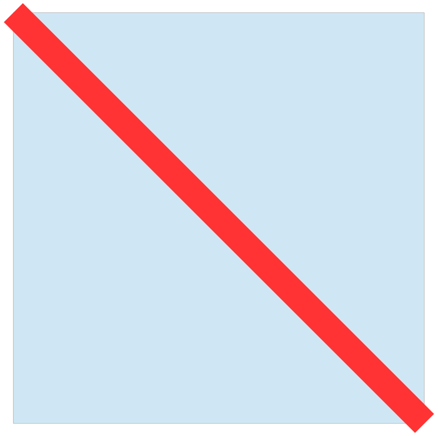


Reflexive, Symmetric, Anti-symmetric

$$\forall x \quad (x, x) \in R$$

$$\forall x, \forall y \; [\; (x, y) \in R \;] \rightarrow [\; (y, x) \in R \;]$$

$$\forall x, \forall y \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \rightarrow [\; x = y \;]$$



Reflexive
Also, symmetric (no relation for (x, y) where x≠y)
Also, anti-symmetric (no relation for (x, y) where x≠y)

Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y \ [\ (x,y) \in R \ \wedge \ (y,x) \in R\ ] \rightarrow [\ x = y\ ]\}$$

$$\exists x, \exists y \neg\{\ [\ (x,y) \in R \ \wedge \ (y,x) \in R\ ] \rightarrow [\ x = y\ ]\}$$

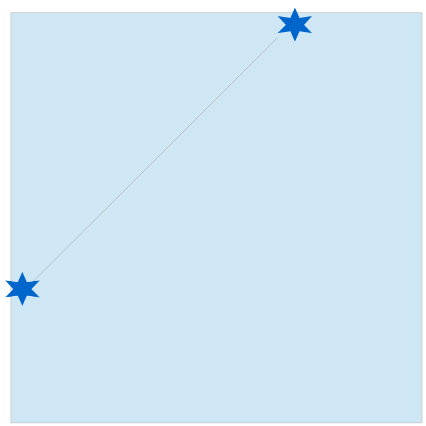
$$\exists x, \exists y \neg\{\neg\ [\ (x,y) \in R \ \wedge \ (y,x) \in R\ ] \vee [\ x = y\ ]\}$$

$$\exists x, \exists y \{\ [\ (x,y) \in R \ \wedge \ (y,x) \in R\ ] \wedge \neg[\ x = y\ ]\}$$

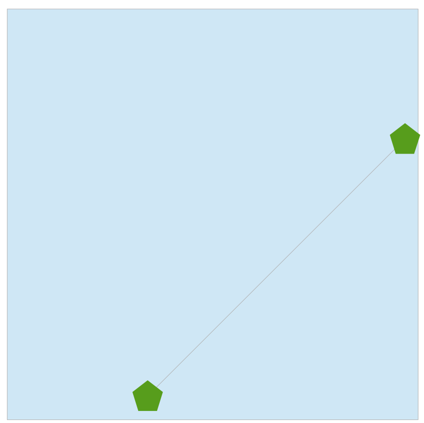
$$\exists x, \exists y \{\ [\ (x,y) \in R \ \wedge \ (y,x) \in R\ ] \wedge [\ x \neq y\ ]\}$$

counter example

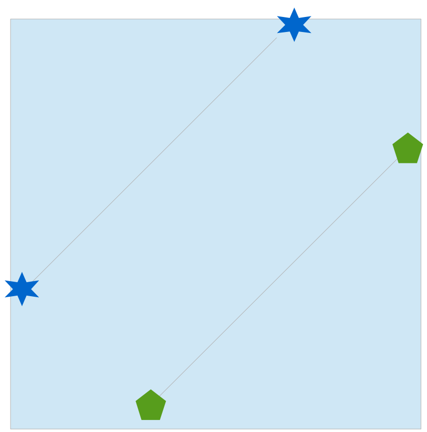
not anti-symmetric



not anti-symmetric



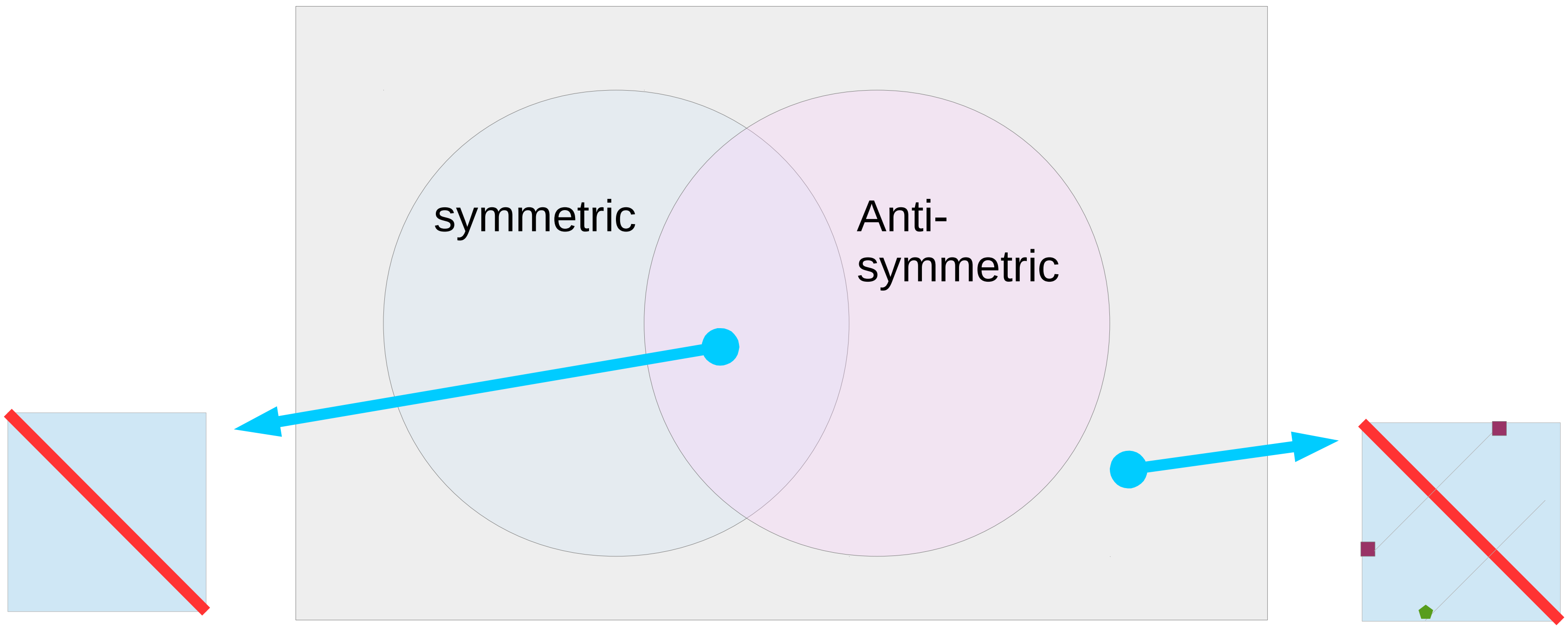
not anti-symmetric



Symmetric vs Anti-Symmetric Relation

not symmetric $\neq$ anti-symmetric

not anti-symmetric $\neq$ symmetric



References

[1] http://en.wikipedia.org/
[2]