

# Laurent Series and z-Transform

## - Geometric Series

### Applications

(A)

20210130 Sat

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# Unshifted Geometric Sequences

Causal

$$\frac{1}{1 - az} \rightarrow u(n)$$

Anti-causal

$$\frac{1}{1 - az^{-1}} \rightarrow u(-n)$$

Positive Exponent

$$az, az^{-1} \rightarrow a^n$$

Negative Exponent

$$a^{-1}z, az^{-1} \rightarrow a^{-n}$$

Positive Exponent

unshifted (1)

$$\frac{1}{1 - az} \quad |z| < a^{-1} \quad a^n u(n)$$

complementary (7)

$$\frac{az}{1 - az} \quad |z| < a^{-1} \quad a^n u(n-1)$$

unshifted (5)

$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1)$$

complementary (3)

$$-\frac{1}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^n u(-n)$$

Negative Exponent

unshifted (2)

$$\frac{1}{1 - a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n)$$

complementary (8)

$$\frac{a^{-1}z}{1 - a^{-1}z} \quad |z| < a^{-1} \quad a^{-n} u(n-1)$$

unshifted (6)

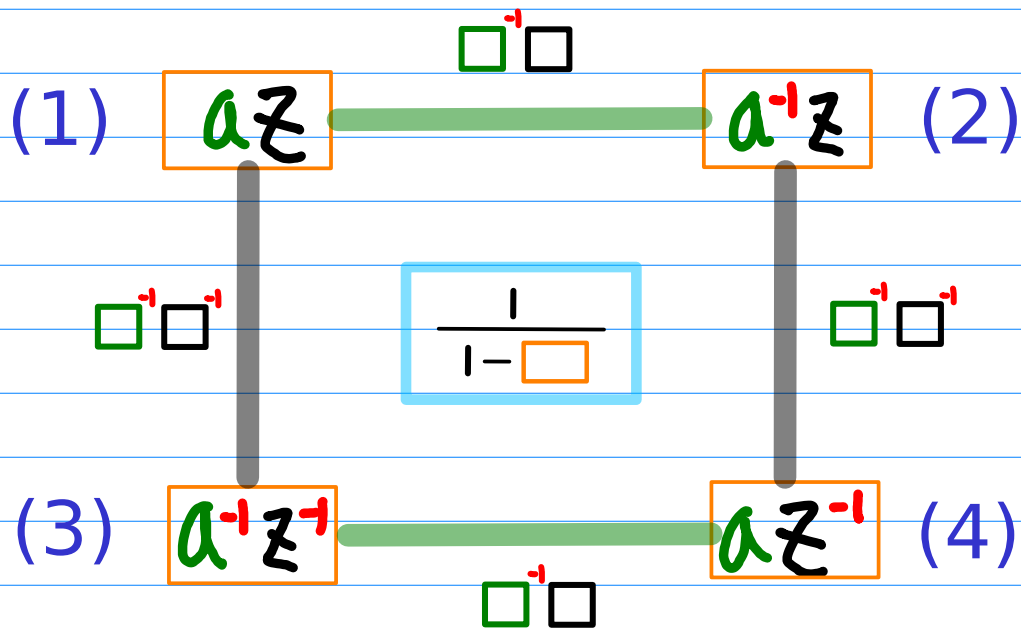
$$-\frac{az^{-1}}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n-1)$$

complementary (4)

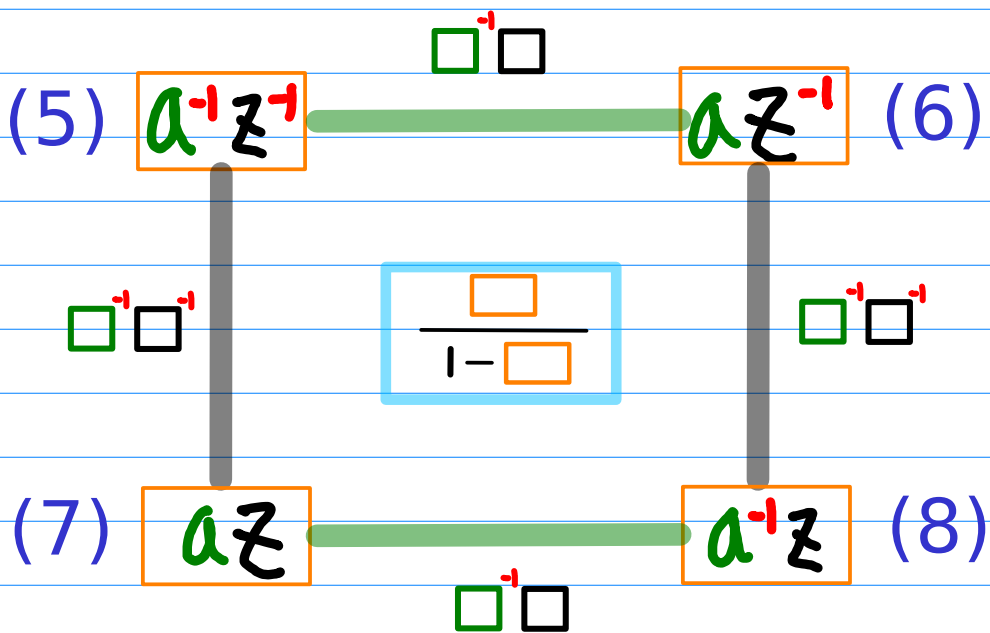
$$-\frac{1}{1 - az^{-1}} \quad |z| > a^{-1} \quad a^{-n} u(-n)$$

# Numbering the basic elements - (1) CR

## unshifted geometric sequences

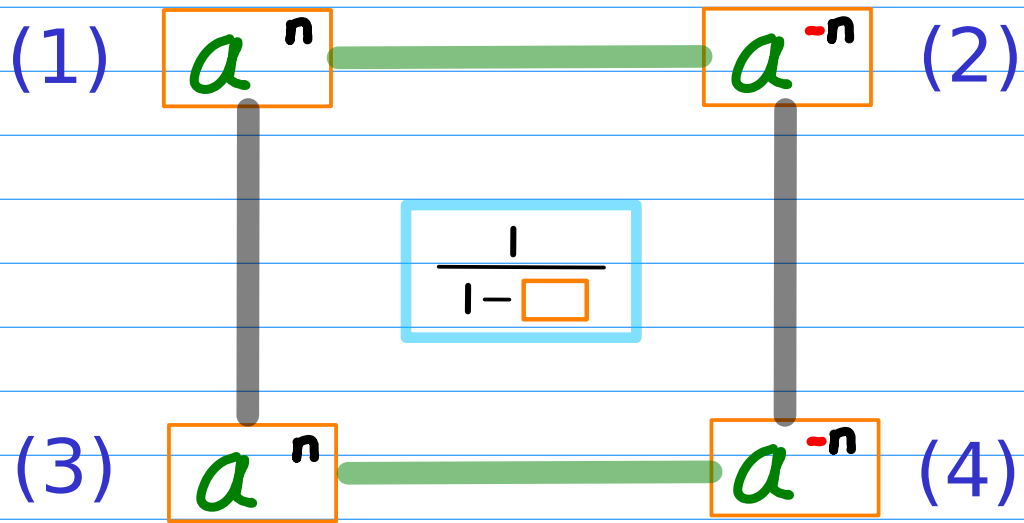


## complementary geometric sequences

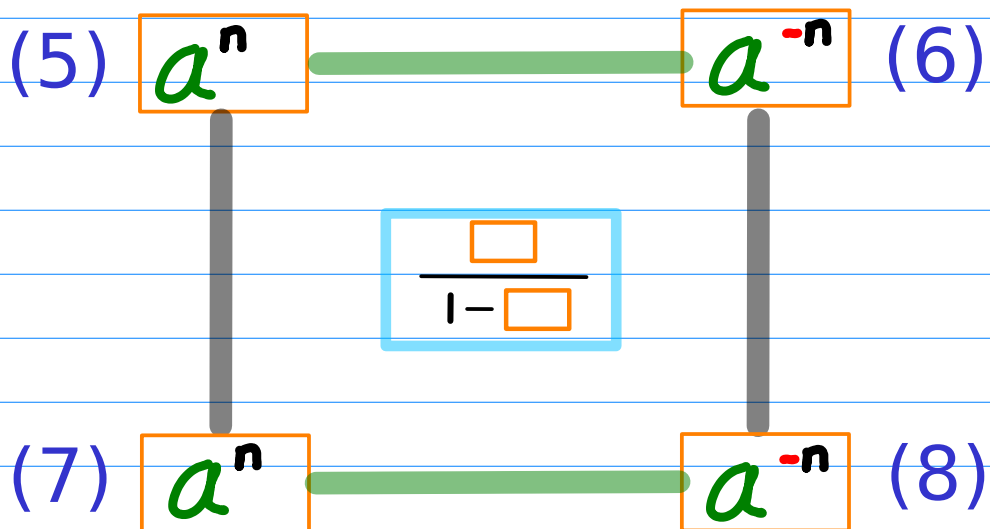


# Numbering the basic elements - (2) Power

## unshifted geometric sequences

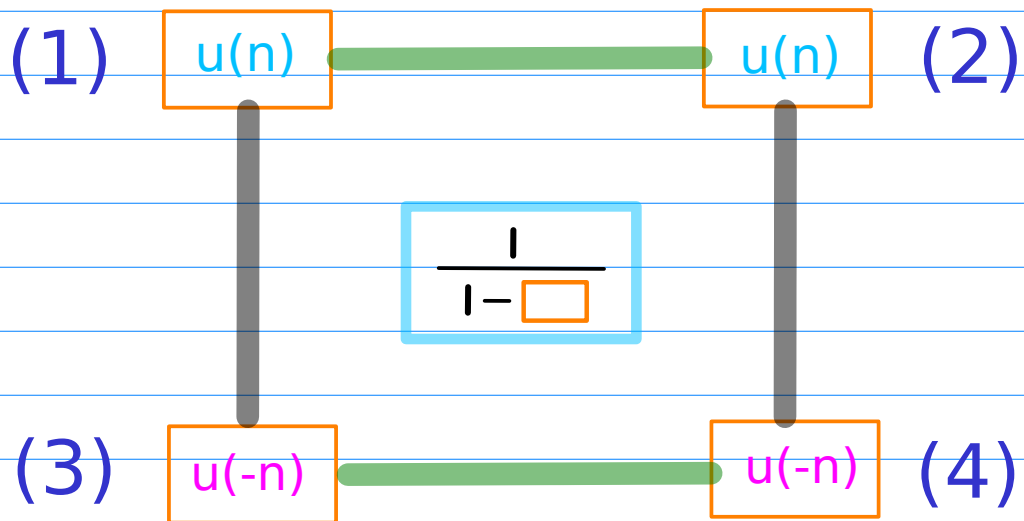


## complementary geometric sequences

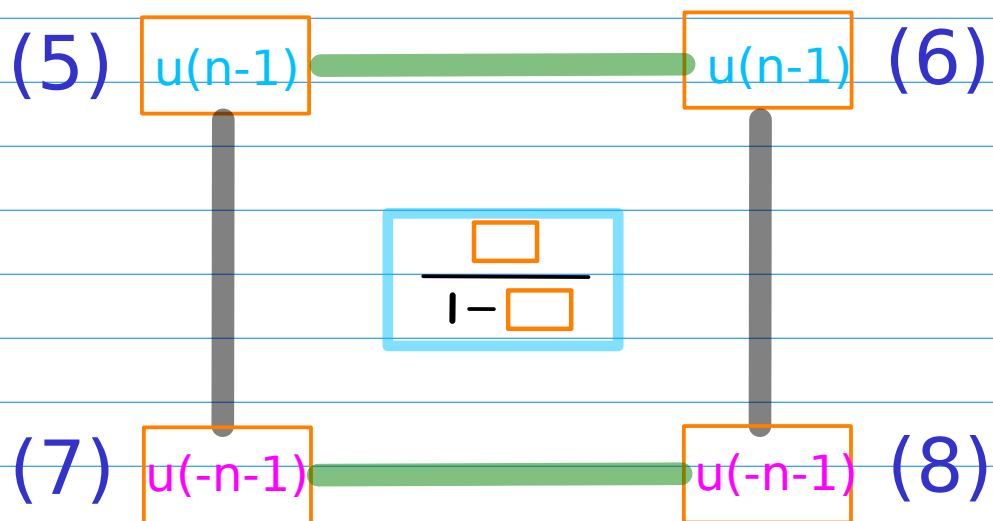


# Numbering the basic elements - (3) Range

## unshifted geometric sequences

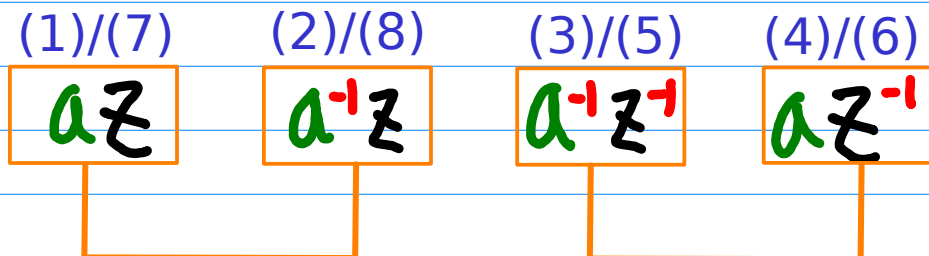


## complementary geometric sequences

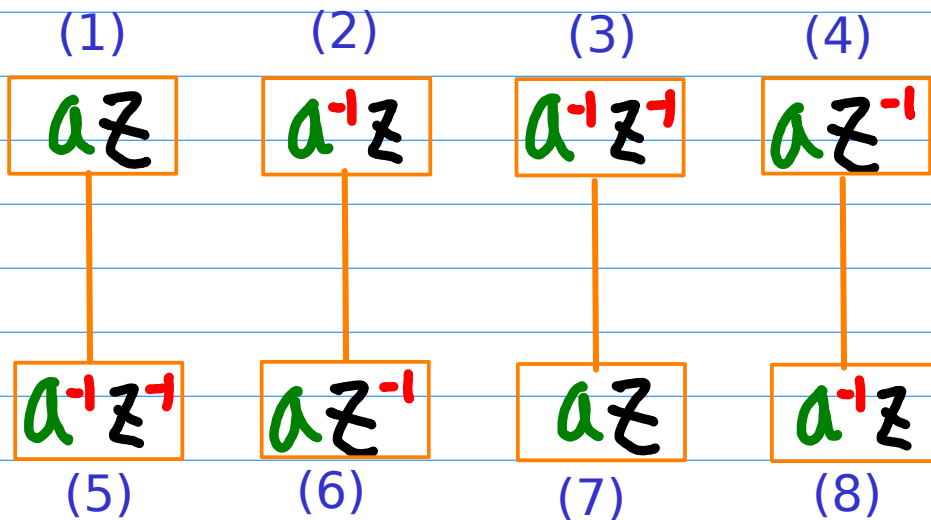


# Inverse Relations

inverse power  $\square^{\cdot} \square$



complementary  $\square^{\cdot} \square^{\cdot}$



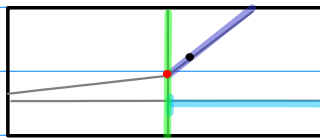
# Geometric Series Form Combinations with a unit start term unshifted

(1)  $\frac{1}{1 - az}$   $a^n u(n)$

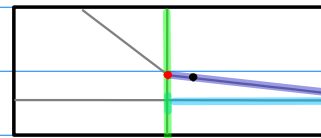
(2)  $\frac{1}{1 - a^{-1}z}$   $a^{-n} u(n)$

(3)  $-\frac{1}{1 - a^{-1}z^{-1}}$   $-a^n u(-n)$

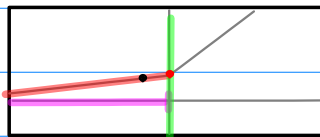
(4)  $-\frac{1}{1 - az^{-1}}$   $-a^{-n} u(-n)$



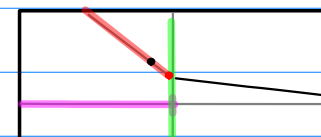
$a^n u(n)$



$a^{-n} u(n)$



$a^n u(-n)$



$a^{-n} u(-n)$



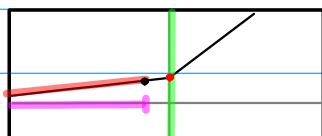
# Geometric Series Form Combinations with a common-ratio start term **unshifted, complementary**

$$(5) \quad \frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \quad -a^n u(-n-1)$$

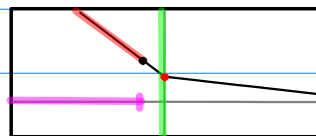
$$(6) \quad \frac{az^{-1}}{1 - az^{-1}} \quad -a^{-n} u(-n-1)$$

$$(7) \quad \frac{az}{1 - az} \quad a^n u(n-1)$$

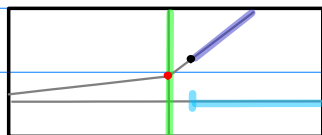
$$(8) \quad \frac{a^{-1}z}{1 - a^{-1}z} \quad a^{-n} u(n-1)$$



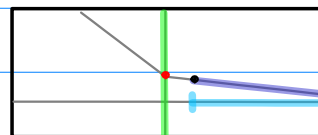
$$a^n u(-n-1)$$



$$a^{-n} u(-n-1)$$

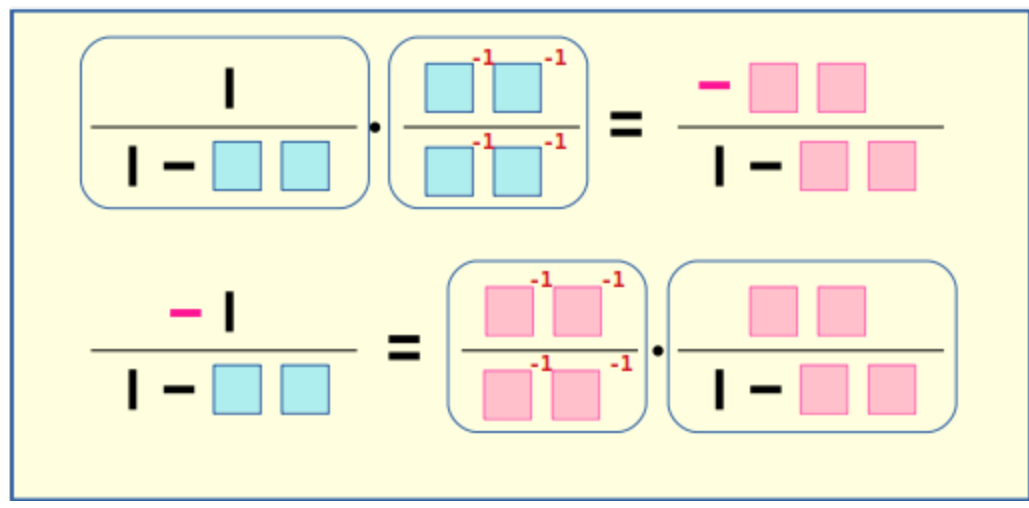
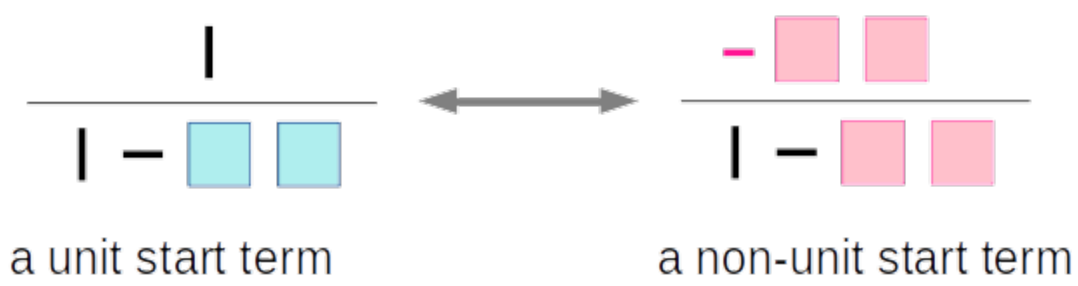
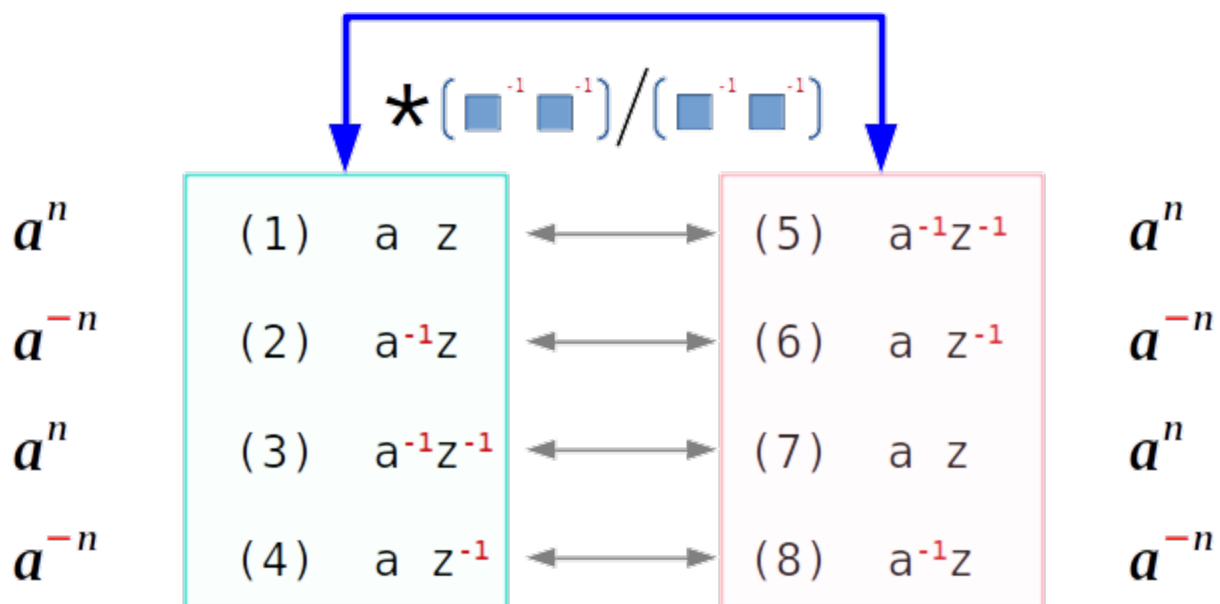


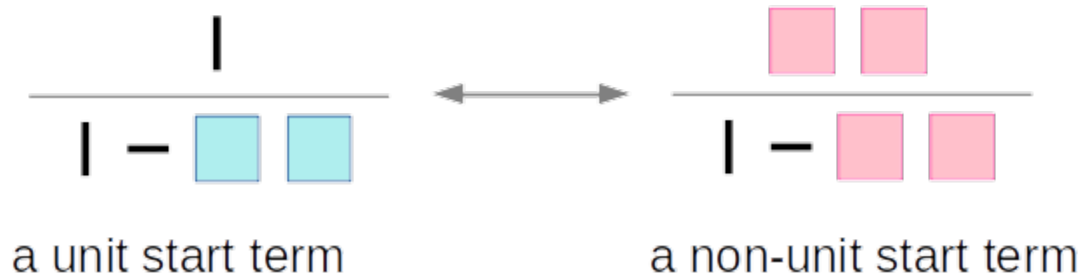
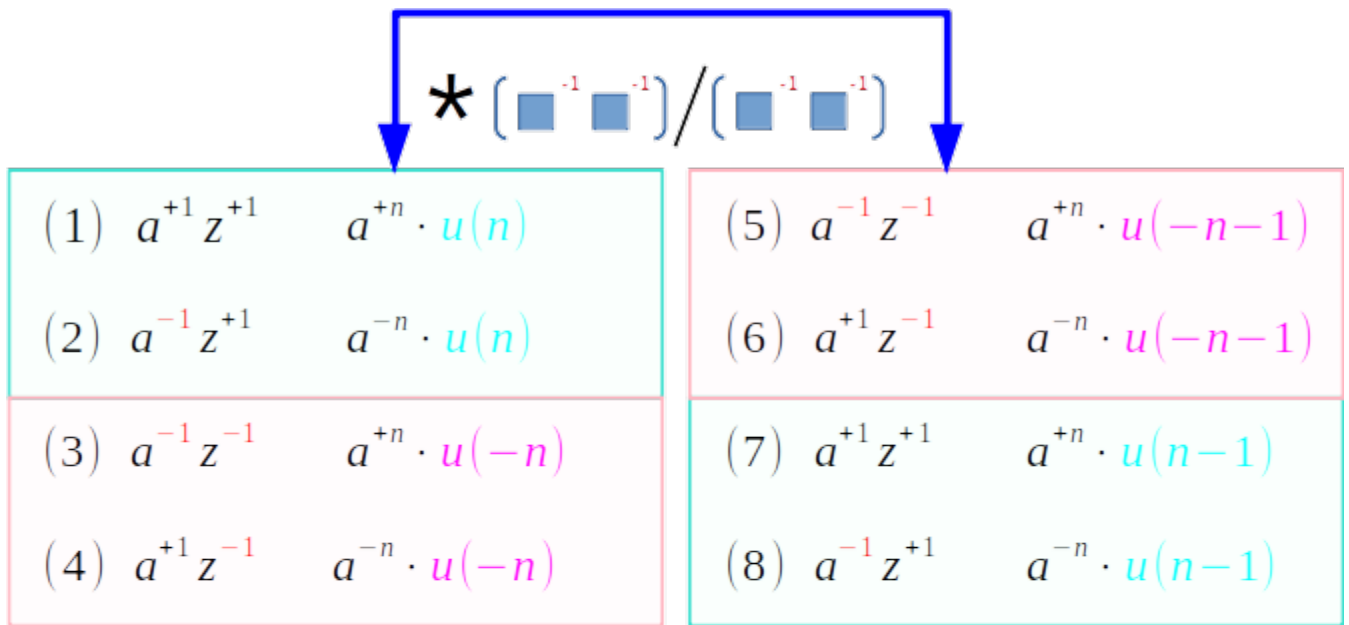
$$a^n u(n-1)$$



$$a^{-n} u(n-1)$$

# Complementary Relations





Positive Exponent	Negative Exponent
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

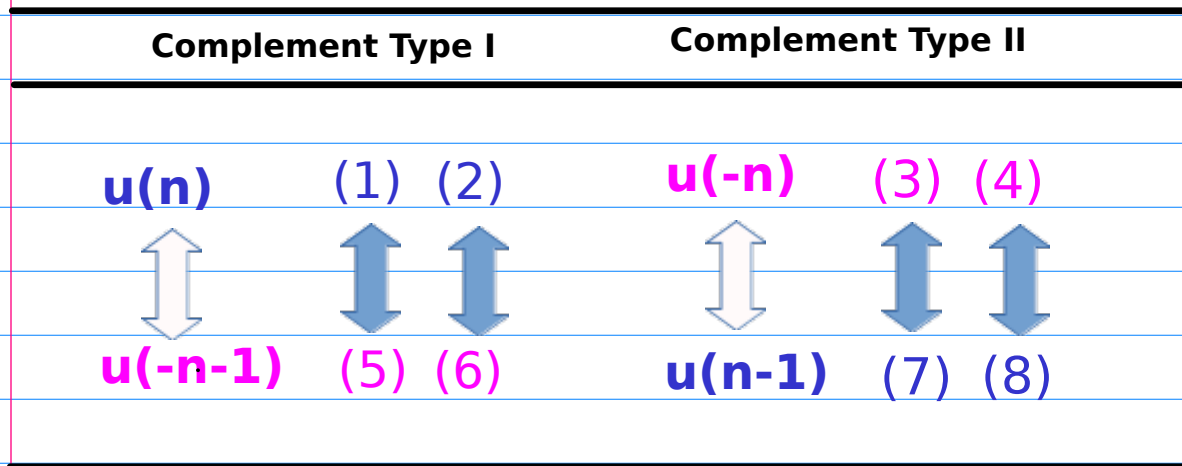
$u(n)$	(1)	(2)	Causal
$u(-n)$	(3)	(4)	Anti-Causal
$u(-n-1)$	(5)	(6)	Anti-Causal
$u(n-1)$	(7)	(8)	Causal

ranges include the origin	(1)	(2)	$u(n)$
	(3)	(4)	$u(-n)$
ranges exclude the origin	(5)	(6)	$u(-n-1)$
	(7)	(8)	$u(n-1)$

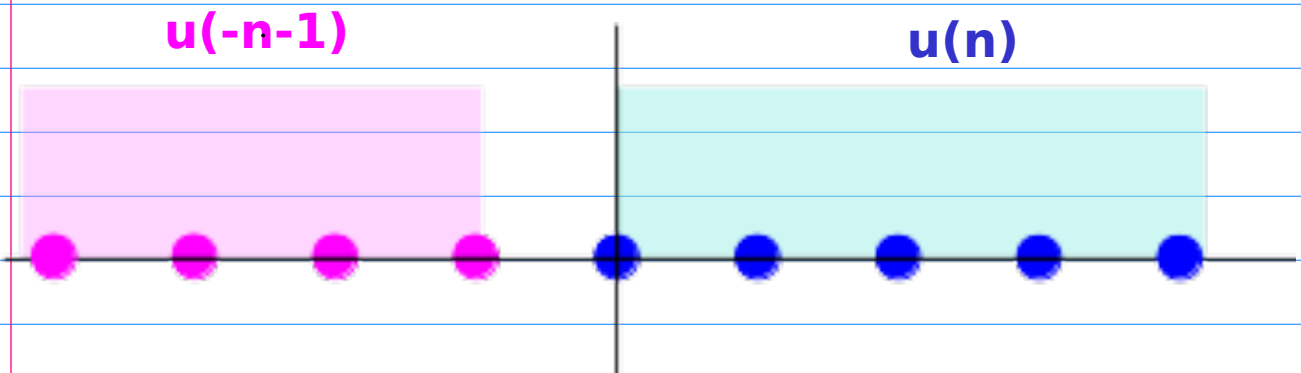
Complement Type I

$u(n)$	(1)	(2)	Causal
$u(-n)$	(3)	(4)	Anti-Causal
$u(-n-1)$	(5)	(6)	Anti-Causal
$u(n-1)$	(7)	(8)	Causal

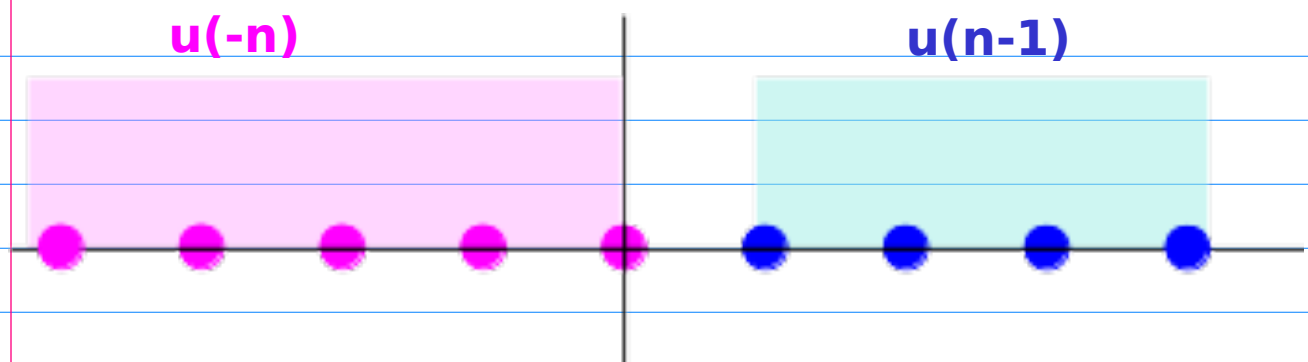
Complement Type II







Complement Type I







Complement Type II











<b>Causal</b>	$u(n)$	(1)		(2)
	$u(n-1)$	(7)		(8)
<b>Anti-Causal</b>	$u(-n-1)$	(5)		(6)
	$u(-n)$	(3)		(4)

butterfly pair ordering

<b>Complement Type I</b>	$u(n)$		(1)	(2)
	$u(-n-1)$		(5)	(6)
<b>Complement Type II</b>	$u(-n)$		(3)	(4)
	$u(n-1)$		(7)	(8)

complementary pair ordering

		<b>Positive Exponent</b>	<b>Negative Exponent</b>
<b>Complement Type I</b>	$u(n)$	(1) 	(2) 
	$u(-n-1)$	(5) 	(6) 
<b>Complement Type II</b>	$u(-n)$	(3) 	(4) 
	$u(n-1)$	(7) 	(8) 

complementary pair ordering

# Shifted Geometric Sequences

## Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

## Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift

## Positive Exponent

(1) \*a, (7) /z

$$\frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$$

(1) \*z, (7) /a

$$\frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$$

(5) \*a, (3) /z

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$$

(5) \*z, (3) /a

$$-\frac{a^{-1}}{1-az^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$$

## Negative Exponent

(2) /a, (8) /z

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a \quad a^{-n-1} u(n)$$

(2) \*z, (8) \*a

$$\frac{z}{1-a^{-1}z} \quad |z| < a \quad a^{-n+1} u(n-1)$$

(6) /a, (4) /z

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n-1} u(-n-1)$$

(6) \*z, (4) \*a

$$-\frac{a}{1-az^{-1}} \quad |z| > a \quad a^{-n+1} u(-n)$$

Left Shifted

Right Shifted

Left Shifted

Right Shifted

multiplying  $a$  or  $a^{-1}$

multiplying  $z$  or  $z^{-1}$

## Exponent Shifting

$$* a$$

$$a^{n+1} \leftarrow a^n$$

Left Shift

$$a^{-n+1} \leftarrow a^{-n}$$

Right Shift

$$* a^{-1}$$

$$a^{n-1} \leftarrow a^n$$

Right Shift

$$a^{-n-1} \leftarrow a^{-n}$$

Left Shift

## Exponent & Range Shifting

$$* z$$

$$n \leftarrow n-1$$

Right Shift

$$* z^{-1}$$

$$n \leftarrow n+1$$

Left Shift



# Combinations of Shifted Geometric Series (1)

Positive Exponent

/z  $n \leftarrow n+1$

\*z  $n \leftarrow n-1$

(1)  $\frac{1}{1-az} \quad |z| < a^{-1}$   $\xrightarrow{*a}$   $\frac{a}{1-az} \quad |z| < a^{-1}$  **Left Shifted**  
 $a^n u(n)$   $\xrightarrow{/z}$   $a^{n+1} u(n)$

(7)  $\frac{az}{1-az} \quad |z| < a^{-1}$   $\xrightarrow{/a}$   $\frac{z}{1-az} \quad |z| < a^{-1}$  **Right Shifted**  
 $a^n u(n-1)$   $\xrightarrow{*z}$   $a^{n-1} u(n-1)$

(5)  $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$   $\xrightarrow{*a}$   $-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$  **Left Shifted**  
 $a^n u(-n-1)$   $\xrightarrow{/z}$   $a^{n+1} u(-n-1)$

(3)  $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$   $\xrightarrow{/a}$   $-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$  **Right Shifted**  
 $a^n u(-n)$   $\xrightarrow{*z}$   $a^{n-1} u(-n)$

Causal	$u(n)$	(1)	(2)	butterfly pair ordering
	$u(n-1)$	(7)	(8)	
Anti-Causal	$u(-n-1)$	(5)	(6)	
	$u(-n)$	(3)	(4)	

# Combinations of Shifted Geometric Series (2)

Negative Exponent

/z  $n \leftarrow n+1$

\*z  $n \leftarrow n-1$

(2)  $\frac{1}{1-a^1z} \quad |z| < a$   $\xrightarrow{/a}$   $\frac{a^{-1}}{1-a^1z} \quad |z| < a$  **Left Shifted**  
 $a^{-n} u(n)$   $\xrightarrow{/z}$   $a^{-n-1} u(n)$

(8)  $\frac{a^1z}{1-a^1z} \quad |z| < a$   $\xrightarrow{*a}$   $\frac{z}{1-a^1z} \quad |z| < a$  **Right Shifted**  
 $a^{-n} u(n-1)$   $\xrightarrow{*z}$   $a^{-n+1} u(n-1)$

(6)  $-\frac{a^1z^{-1}}{1-a^1z^{-1}} \quad |z| > a$   $\xrightarrow{/a}$   $-\frac{z^{-1}}{1-a^1z^{-1}} \quad |z| > a$  **Left Shifted**  
 $a^{-n} u(-n-1)$   $\xrightarrow{/z}$   $a^{-n-1} u(-n-1)$

(4)  $-\frac{1}{1-a^1z^{-1}} \quad |z| > a$   $\xrightarrow{*a}$   $-\frac{a}{1-a^1z^{-1}} \quad |z| > a$  **Right Shifted**  
 $a^{-n} u(-n)$   $\xrightarrow{*z}$   $a^{-n+1} u(-n)$

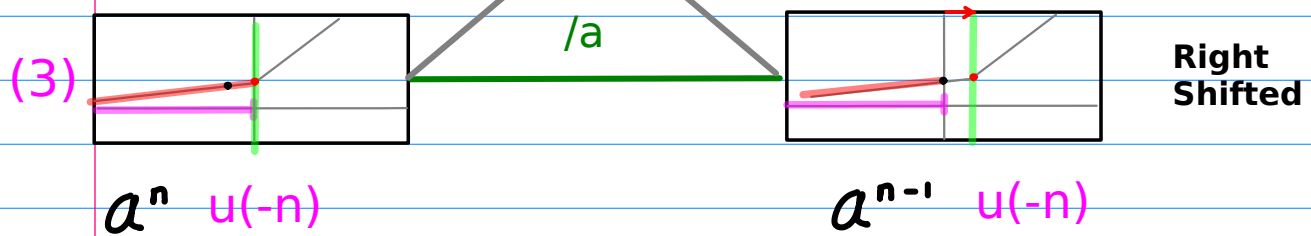
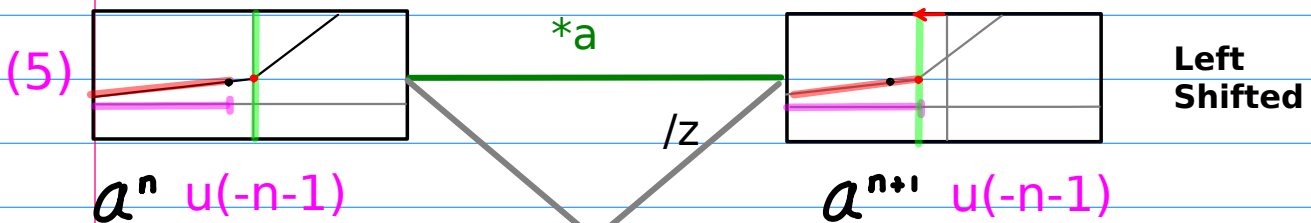
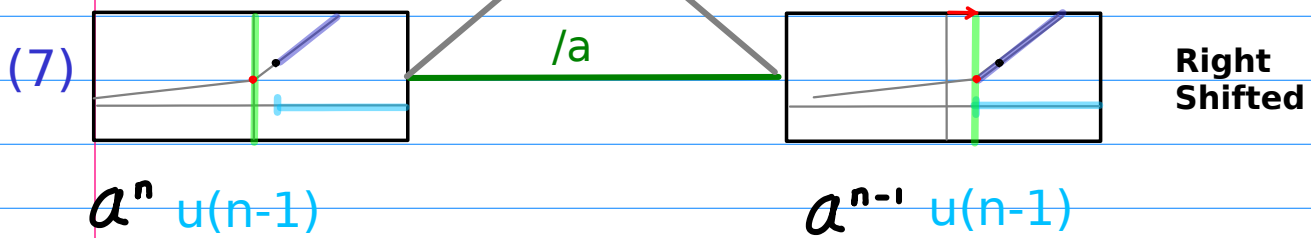
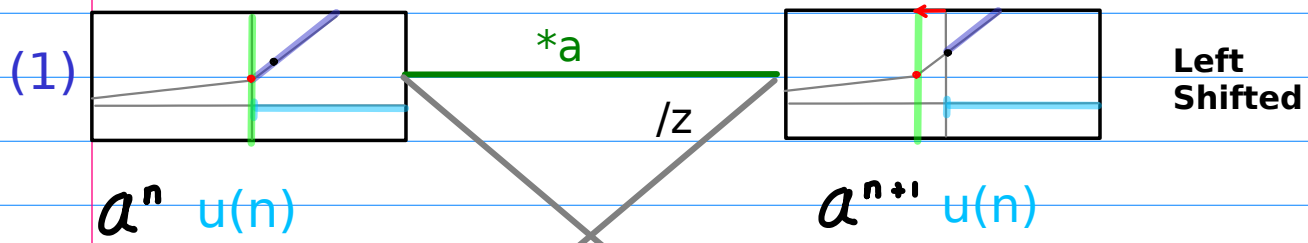
Causal	$u(n)$	(1)	(2)	butterfly pair ordering
	$u(n-1)$	(7)	(8)	
Anti-Causal	$u(-n-1)$	(5)	(6)	
	$u(-n)$	(3)	(4)	

# Graphs of Shifted Geometric Series (1)

Positive Exponent

$/z \quad n \leftarrow n+1$

$*z \quad n \leftarrow n-1$



Causal	$u(n)$	(1)	(2)
	$u(n-1)$	(7)	(8)
Anti-Causal	$u(-n-1)$	(5)	(6)
	$u(-n)$	(3)	(4)

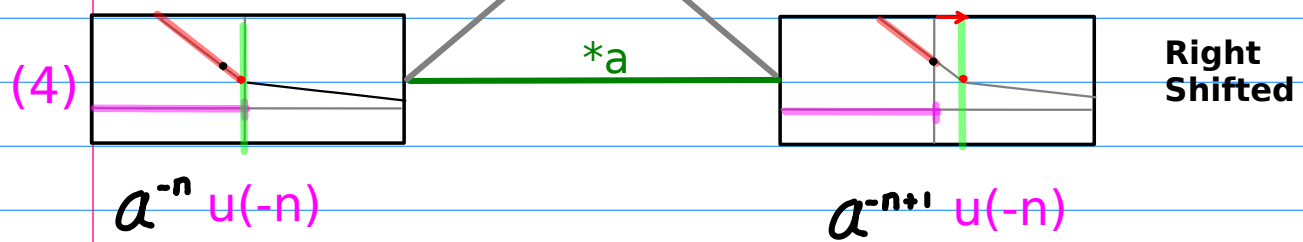
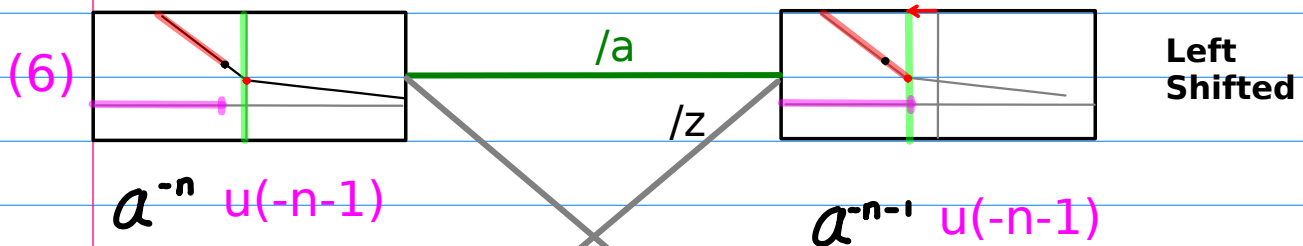
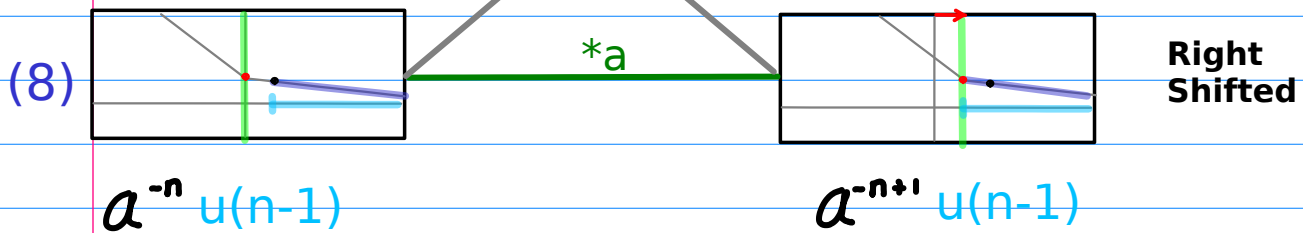
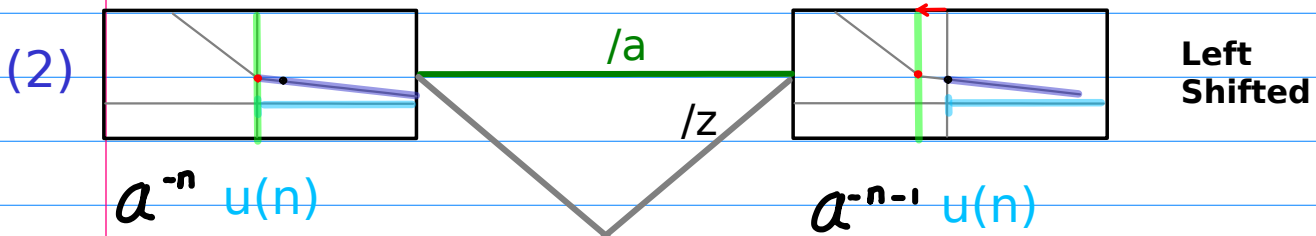
butterfly pair ordering

# Graphs of Shifted Geometric Series (2)

Negative Exponent

$/z \quad n \leftarrow n+1$

$*z \quad n \leftarrow n-1$



Causal	$u(n)$	(1)	(2)	butterfly pair ordering
	$u(n-1)$	(7)	(8)	
Anti-Causal	$u(-n-1)$	(5)	(6)	
	$u(-n)$	(3)	(4)	

# Shifting Geometric Series by $*a$ or $/a$

$$\star \left( \begin{array}{c} \blacksquare^{-1} \\ \blacksquare^{-1} \end{array} \right) / \left( \begin{array}{c} \blacksquare^{-1} \\ \blacksquare^{-1} \end{array} \right)$$

(1)  $a^{+1} z^{+1}$   $a^{+n} \cdot u(n)$   
 $\leftarrow$   $\star a$   $a^{+n+1} \cdot u(n)$

(2)  $a^{-1} z^{+1}$   $a^{-n} \cdot u(n)$   
 $\leftarrow$   $/a$   $a^{-n-1} \cdot u(n)$

(5)  $a^{-1} z^{-1}$   $a^{+n} \cdot u(-n-1)$   
 $\leftarrow$   $\star a$   $a^{+n+1} \cdot u(-n-1)$

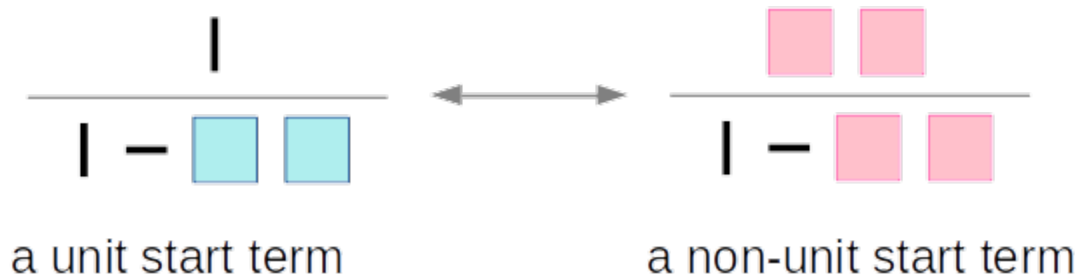
(6)  $a^{+1} z^{-1}$   $a^{-n} \cdot u(-n-1)$   
 $\leftarrow$   $/a$   $a^{-n-1} \cdot u(-n-1)$

(3)  $a^{-1} z^{-1}$   $a^{+n} \cdot u(-n)$   
 $\rightarrow$   $/a$   $a^{+n-1} \cdot u(-n)$

(4)  $a^{+1} z^{-1}$   $a^{-n} \cdot u(-n)$   
 $\rightarrow$   $\star a$   $a^{-n+1} \cdot u(-n)$

(7)  $a^{+1} z^{+1}$   $a^{+n} \cdot u(n-1)$   
 $\rightarrow$   $/a$   $a^{+n-1} \cdot u(n-1)$

(8)  $a^{-1} z^{+1}$   $a^{-n} \cdot u(n-1)$   
 $\rightarrow$   $\star a$   $a^{-n+1} \cdot u(n-1)$



# Shifting Geometric Series by $*z$ or $/z$

$$* \left( \begin{array}{|c|} \hline \square^{-1} \square^{-1} \\ \hline \end{array} \right) / \left( \begin{array}{|c|} \hline \square^{-1} \square^{-1} \\ \hline \end{array} \right)$$

(1)  $a^{+1} z^{+1}$   $a^{+n} \cdot u(n)$   
 $\Rightarrow$   $* z$   $a^{+n-1} \cdot u(n-1)$

(2)  $a^{-1} z^{+1}$   $a^{-n} \cdot u(n)$   
 $\Rightarrow$   $* z$   $a^{-n+1} \cdot u(n-1)$

(5)  $a^{-1} z^{-1}$   $a^{+n} \cdot u(-n-1)$   
 $\Rightarrow$   $* z$   $a^{+n-1} \cdot u(-n)$

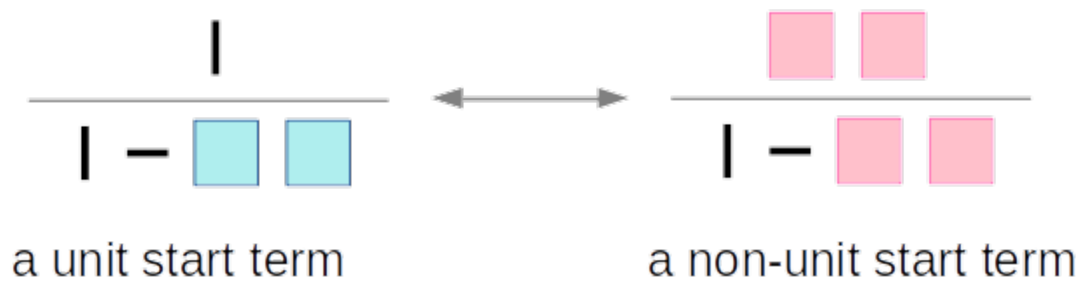
(6)  $a^{+1} z^{-1}$   $a^{-n} \cdot u(-n-1)$   
 $\Rightarrow$   $* z$   $a^{-n+1} \cdot u(-n)$

(3)  $a^{-1} z^{-1}$   $a^{+n} \cdot u(-n)$   
 $\Leftarrow$   $/ z$   $a^{+n+1} \cdot u(-n-1)$

(4)  $a^{+1} z^{-1}$   $a^{-n} \cdot u(-n)$   
 $\Leftarrow$   $/ z$   $a^{-n-1} \cdot u(-n-1)$

(7)  $a^{+1} z^{+1}$   $a^{+n} \cdot u(n-1)$   
 $\Leftarrow$   $/ z$   $a^{+n+1} \cdot u(n)$

(8)  $a^{-1} z^{+1}$   $a^{-n} \cdot u(n-1)$   
 $\Leftarrow$   $/ z$   $a^{-n-1} \cdot u(n)$



$u(n)$	(1)	(2)
$u(-n)$	(3)	(4)
$u(-n-1)$	(5)	(6)
$u(n-1)$	(7)	(8)

row major ordering

(1) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n)$ $\leftarrow$ $* a$ $a^{+n+1} \cdot u(n)$	(2) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n)$ $\leftarrow$ $/ a$ $a^{-n-1} \cdot u(n)$
(3) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n)$ $\rightarrow$ $/ a$ $a^{+n-1} \cdot u(-n)$	(4) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n)$ $\rightarrow$ $* a$ $a^{-n+1} \cdot u(-n)$
(5) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n-1)$ $\leftarrow$ $* a$ $a^{+n+1} \cdot u(-n-1)$	(6) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n-1)$ $\leftarrow$ $/ a$ $a^{-n-1} \cdot u(-n-1)$
(7) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n-1)$ $\rightarrow$ $/ a$ $a^{+n-1} \cdot u(n-1)$	(8) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n-1)$ $\rightarrow$ $* a$ $a^{-n+1} \cdot u(n-1)$

(1) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n)$ $\Rightarrow$ $* z$ $a^{+n-1} \cdot u(n-1)$	(2) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n)$ $\Rightarrow$ $* z$ $a^{-n+1} \cdot u(n-1)$
(3) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n)$ $\Leftarrow$ $/ z$ $a^{+n+1} \cdot u(-n-1)$	(4) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n)$ $\Leftarrow$ $/ z$ $a^{-n-1} \cdot u(-n-1)$
(5) $a^{-1}z^{-1}$ $a^{+n} \cdot u(-n-1)$ $\Rightarrow$ $* z$ $a^{+n-1} \cdot u(-n)$	(6) $a^{+1}z^{-1}$ $a^{-n} \cdot u(-n-1)$ $\Rightarrow$ $* z$ $a^{-n+1} \cdot u(-n)$
(7) $a^{+1}z^{+1}$ $a^{+n} \cdot u(n-1)$ $\Leftarrow$ $/ z$ $a^{+n+1} \cdot u(n)$	(8) $a^{-1}z^{+1}$ $a^{-n} \cdot u(n-1)$ $\Leftarrow$ $/ z$ $a^{-n-1} \cdot u(n)$

Causal	$u(n)$	(1)	(2)
	$u(n-1)$	(7)	(8)
Anti-Causal	$u(-n-1)$	(5)	(6)
	$u(-n)$	(3)	(4)

butterfly pair ordering

(1) $a^{+1}z^{+1}$	$a^{+n} \cdot u(n)$	(2) $a^{-1}z^{+1}$	$a^{-n} \cdot u(n)$
← $* a$	$a^{+n+1} \cdot u(n)$	← $/ a$	$a^{-n-1} \cdot u(n)$
(7) $a^{+1}z^{+1}$	$a^{+n} \cdot u(n-1)$	(8) $a^{-1}z^{+1}$	$a^{-n} \cdot u(n-1)$
→ $/ a$	$a^{+n-1} \cdot u(n-1)$	→ $* a$	$a^{-n+1} \cdot u(n-1)$
(5) $a^{-1}z^{-1}$	$a^{+n} \cdot u(-n-1)$	(6) $a^{+1}z^{-1}$	$a^{-n} \cdot u(-n-1)$
← $* a$	$a^{+n+1} \cdot u(-n-1)$	← $/ a$	$a^{-n-1} \cdot u(-n-1)$
(3) $a^{-1}z^{-1}$	$a^{+n} \cdot u(-n)$	(4) $a^{+1}z^{-1}$	$a^{-n} \cdot u(-n)$
→ $/ a$	$a^{+n-1} \cdot u(-n)$	→ $* a$	$a^{-n+1} \cdot u(-n)$

(1) $a^{+1}z^{+1}$	$a^{+n} \cdot u(n)$	(2) $a^{-1}z^{+1}$	$a^{-n} \cdot u(n)$
⇒ $* z$	$a^{+n-1} \cdot u(n-1)$	⇒ $* z$	$a^{-n+1} \cdot u(n-1)$
(7) $a^{+1}z^{+1}$	$a^{+n} \cdot u(n-1)$	(8) $a^{-1}z^{+1}$	$a^{-n} \cdot u(n-1)$
⇐ $/ z$	$a^{+n+1} \cdot u(n)$	⇐ $/ z$	$a^{-n-1} \cdot u(n)$
(5) $a^{-1}z^{-1}$	$a^{+n} \cdot u(-n-1)$	(6) $a^{+1}z^{-1}$	$a^{-n} \cdot u(-n-1)$
⇒ $* z$	$a^{+n-1} \cdot u(-n)$	⇒ $* z$	$a^{-n+1} \cdot u(-n)$
(3) $a^{-1}z^{-1}$	$a^{+n} \cdot u(-n)$	(4) $a^{+1}z^{-1}$	$a^{-n} \cdot u(-n)$
⇐ $/ z$	$a^{+n+1} \cdot u(-n-1)$	⇐ $/ z$	$a^{-n-1} \cdot u(-n-1)$



<b>Complement Type I</b>	<b>u(n)</b>	(1)	(2)
	<b>u(-n-1)</b>	(5)	(6)
<b>Complement Type II</b>	<b>u(-n)</b>	(3)	(4)
	<b>u(n-1)</b>	(7)	(8)

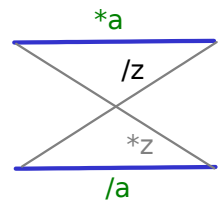
**complementary pair ordering**

(1) $a^{+1} z^{+1}$	$a^{+n} \cdot u(n)$	(2) $a^{-1} z^{+1}$	$a^{-n} \cdot u(n)$
← $* a$	$a^{+n+1} \cdot u(n)$	← $/ a$	$a^{-n-1} \cdot u(n)$
(5) $a^{-1} z^{-1}$	$a^{+n} \cdot u(-n-1)$	(6) $a^{+1} z^{-1}$	$a^{-n} \cdot u(-n-1)$
← $* a$	$a^{+n+1} \cdot u(-n-1)$	← $/ a$	$a^{-n-1} \cdot u(-n-1)$
(3) $a^{-1} z^{-1}$	$a^{+n} \cdot u(-n)$	(4) $a^{+1} z^{-1}$	$a^{-n} \cdot u(-n)$
→ $/ a$	$a^{+n-1} \cdot u(-n)$	→ $* a$	$a^{-n+1} \cdot u(-n)$
(7) $a^{+1} z^{+1}$	$a^{+n} \cdot u(n-1)$	(8) $a^{-1} z^{+1}$	$a^{-n} \cdot u(n-1)$
→ $/ a$	$a^{+n-1} \cdot u(n-1)$	→ $* a$	$a^{-n+1} \cdot u(n-1)$

(1) $a^{+1} z^{+1}$	$a^{+n} \cdot u(n)$	(2) $a^{-1} z^{+1}$	$a^{-n} \cdot u(n)$
⇒ $* z$	$a^{+n-1} \cdot u(n-1)$	⇒ $* z$	$a^{-n+1} \cdot u(n-1)$
(5) $a^{-1} z^{-1}$	$a^{+n} \cdot u(-n-1)$	(6) $a^{+1} z^{-1}$	$a^{-n} \cdot u(-n-1)$
⇒ $* z$	$a^{+n-1} \cdot u(-n)$	⇒ $* z$	$a^{-n+1} \cdot u(-n)$
(3) $a^{-1} z^{-1}$	$a^{+n} \cdot u(-n)$	(4) $a^{+1} z^{-1}$	$a^{-n} \cdot u(-n)$
⇐ $/ z$	$a^{+n+1} \cdot u(-n-1)$	⇐ $/ z$	$a^{-n-1} \cdot u(-n-1)$
(7) $a^{+1} z^{+1}$	$a^{+n} \cdot u(n-1)$	(8) $a^{-1} z^{+1}$	$a^{-n} \cdot u(n-1)$
⇐ $/ z$	$a^{+n+1} \cdot u(n)$	⇐ $/ z$	$a^{-n-1} \cdot u(n)$

# Shifted Geometric Series (1)

by multiplying  $a$  or  $a^{-1}$



## Positive Exponent

(1) ←  $\frac{1}{1-az} \quad |z| < a^{-1} \quad a^n u(n) \times a \quad \frac{a}{1-az} \quad |z| < a^{-1} \quad a^{n+1} u(n)$

(7) →  $\frac{az}{1-az} \quad |z| < a^{-1} \quad a^n u(n-1) \times a^{-1} \quad \frac{z}{1-az} \quad |z| < a^{-1} \quad a^{n-1} u(n-1)$

(5) ←  $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n-1) \times a \quad -\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n+1} u(-n-1)$

(3) →  $-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^n u(-n) \times a^{-1} \quad -\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad a^{n-1} u(-n)$

## Negative Exponent

(2) ←  $\frac{1}{1-a^{-1}z} \quad |z| < a \quad a^{-n} u(n) \times a^{-1} \quad \frac{a^{-1}}{1-a^{-1}z} \quad |z| < a \quad a^{-n-1} u(n)$

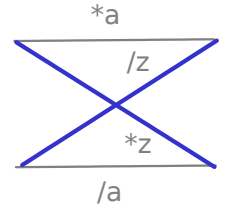
(8) →  $\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a \quad a^{-n} u(n-1) \times a \quad \frac{z}{1-a^{-1}z} \quad |z| < a \quad a^{-n+1} u(n-1)$

(6) ←  $-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n} u(-n-1) \times a^{-1} \quad -\frac{z^{-1}}{1-az^{-1}} \quad |z| > a \quad a^{-n-1} u(-n-1)$

(4) →  $-\frac{1}{1-az^{-1}} \quad |z| > a \quad a^{-n} u(-n) \times a \quad -\frac{a}{1-az^{-1}} \quad |z| > a \quad a^{-n+1} u(-n)$

# Shifted Geometric Series (2)

by multiplying  $z$  or  $z^{-1}$



## Positive Exponent

$$\begin{array}{l} (1) \\ \leftarrow \end{array} \quad \frac{az}{1-az} \quad |z| < a^{-1} \quad \begin{array}{l} n \leftarrow n+1 \\ a^n u(n-1) \end{array} \times z^{-1} \quad \boxed{\frac{a}{1-az} \quad |z| < a^{-1}} \quad a^{n+1} u(n)$$

$$\begin{array}{l} (7) \\ \Rightarrow \end{array} \quad \frac{1}{1-az} \quad |z| < a^{-1} \quad \begin{array}{l} n \leftarrow n-1 \\ a^n u(n) \end{array} \times z \quad \boxed{\frac{z}{1-az} \quad |z| < a^{-1}} \quad a^{n-1} u(n-1)$$

$$\begin{array}{l} (5) \\ \leftarrow \end{array} \quad -\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad \begin{array}{l} n \leftarrow n+1 \\ a^n u(-n) \end{array} \times z^{-1} \quad \boxed{-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}} \quad a^{n+1} u(-n-1)$$

$$\begin{array}{l} (3) \\ \Rightarrow \end{array} \quad -\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1} \quad \begin{array}{l} n \leftarrow n-1 \\ a^n u(-n-1) \end{array} \times z \quad \boxed{-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}} \quad a^{n-1} u(-n)$$

## Negative Exponent

$$\begin{array}{l} (2) \\ \leftarrow \end{array} \quad \frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a \quad \begin{array}{l} n \leftarrow n+1 \\ a^{-n} u(n-1) \end{array} \times z^{-1} \quad \boxed{\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a} \quad a^{-n-1} u(n)$$

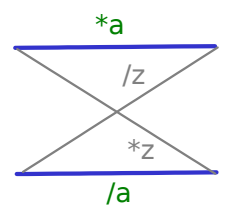
$$\begin{array}{l} (8) \\ \Rightarrow \end{array} \quad \frac{1}{1-a^{-1}z} \quad |z| < a \quad \begin{array}{l} n \leftarrow n-1 \\ a^{-n} u(n) \end{array} \times z \quad \boxed{\frac{z}{1-a^{-1}z} \quad |z| < a} \quad a^{-n+1} u(n-1)$$

$$\begin{array}{l} (6) \\ \leftarrow \end{array} \quad -\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a \quad \begin{array}{l} n \leftarrow n+1 \\ a^{-n} u(-n) \end{array} \times z^{-1} \quad \boxed{-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a} \quad a^{-n-1} u(-n-1)$$

$$\begin{array}{l} (4) \\ \Rightarrow \end{array} \quad -\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a \quad \begin{array}{l} n \leftarrow n-1 \\ a^{-n} u(-n-1) \end{array} \times z \quad \boxed{-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a} \quad a^{-n+1} u(-n)$$

# Shifted Geometric Series (3)

by multiplying  $a$  or  $a^{-1}$  Assume  $a > 1$



- (1)  $a^n u(n)$      $*a \leftarrow$      $a^{n+1} u(n)$
- (2)  $a^{-n} u(n)$      $/a \leftarrow$      $a^{-n-1} u(n)$
- (3)  $a^n u(-n)$      $/a \rightarrow$      $a^{n-1} u(-n)$
- (4)  $a^{-n} u(-n)$      $*a \rightarrow$      $a^{-n+1} u(-n)$
- (5)  $a^n u(-n-1)$      $*a \leftarrow$      $a^{n+1} u(-n-1)$
- (6)  $a^{-n} u(-n-1)$      $/a \leftarrow$      $a^{-n-1} u(-n-1)$
- (7)  $a^n u(n-1)$      $/a \rightarrow$      $a^{n-1} u(n-1)$
- (8)  $a^{-n} u(n-1)$      $*a \rightarrow$      $a^{-n+1} u(n-1)$

## row major ordering

(1)	(2)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(3)	(4)	$/a$	$*a$	$\rightarrow$	$\rightarrow$
(5)	(6)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(7)	(8)	$/a$	$*a$	$\rightarrow$	$\rightarrow$

## complementary pair ordering

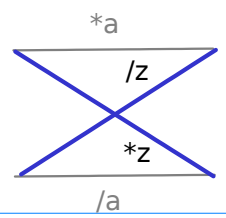
(1)	(2)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(5)	(6)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(3)	(4)	$/a$	$*a$	$\rightarrow$	$\rightarrow$
(7)	(8)	$/a$	$*a$	$\rightarrow$	$\rightarrow$

## butterfly pair ordering

(1)	(2)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(7)	(8)	$/a$	$*a$	$\rightarrow$	$\rightarrow$
(5)	(6)	$*a$	$/a$	$\leftarrow$	$\leftarrow$
(3)	(4)	$/a$	$*a$	$\rightarrow$	$\rightarrow$

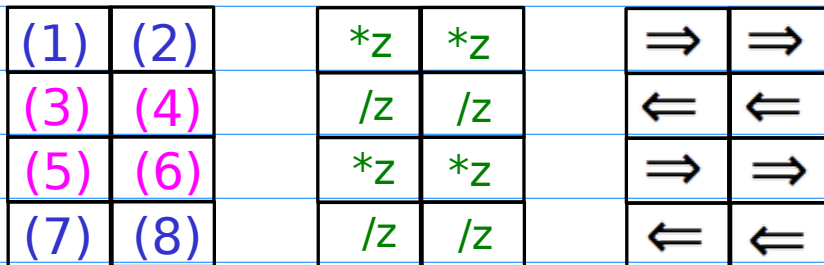
# Shifted Geometric Series (4)

by multiplying  $z$  or  $z^{-1}$  Assume  $a > 1$

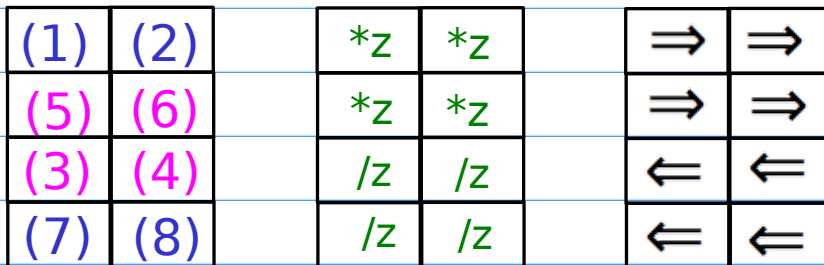


- (1)  $a^n u(n)$        $*z \Rightarrow a^{n-1} u(n-1)$
- (2)  $a^{-n} u(n)$        $*z \Rightarrow a^{-n+1} u(n-1)$
- (3)  $a^n u(-n)$        $/z \Leftarrow a^{n+1} u(-n-1)$
- (4)  $a^{-n} u(-n)$        $/z \Leftarrow a^{-n-1} u(-n-1)$
- (5)  $a^n u(-n-1)$        $*z \Rightarrow a^{n-1} u(-n)$
- (6)  $a^{-n} u(-n-1)$        $*z \Rightarrow a^{-n+1} u(-n)$
- (7)  $a^n u(n-1)$        $/z \Leftarrow a^{n+1} u(n)$
- (8)  $a^{-n} u(n-1)$        $/z \Leftarrow a^{-n-1} u(n)$

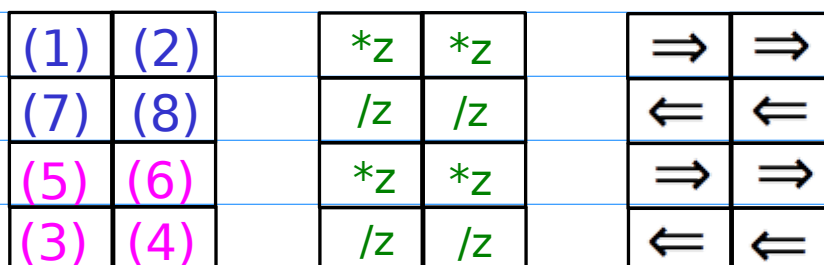
## row major ordering

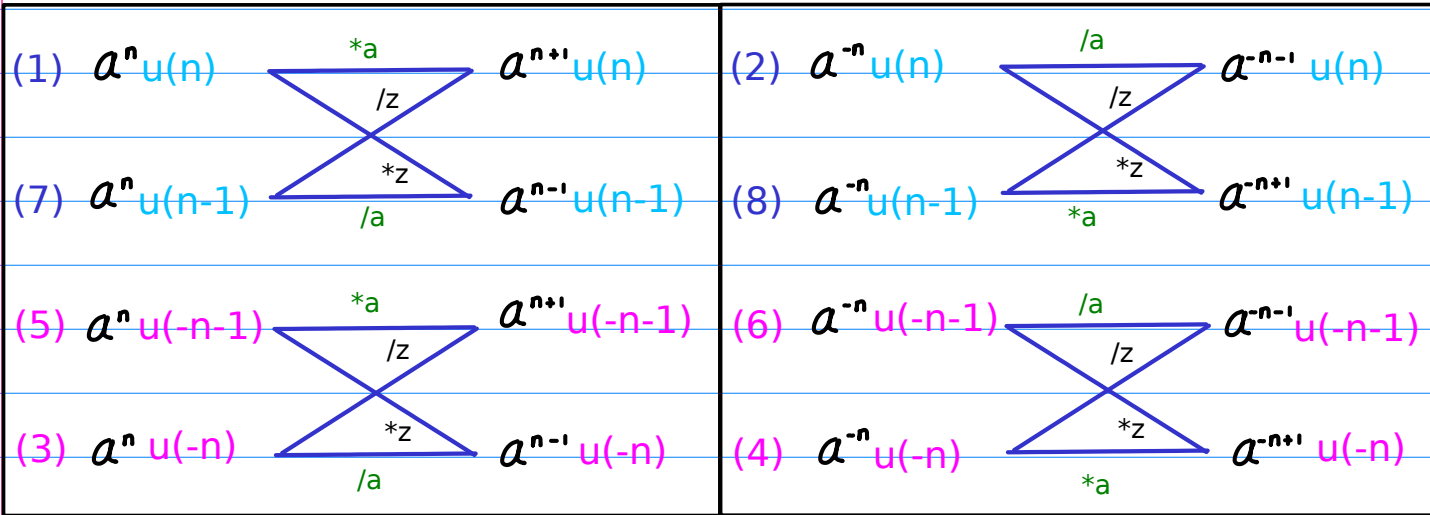


## complementary pair ordering



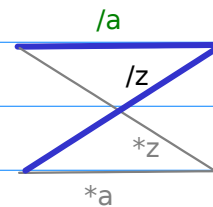
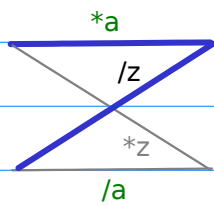
## butterfly pair ordering





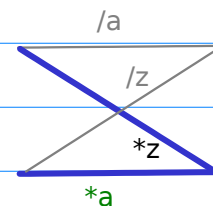
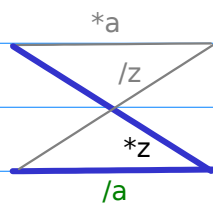
← (1)  $*a = (7) /z$  ←

← (2)  $/a = (8) /z$  ←



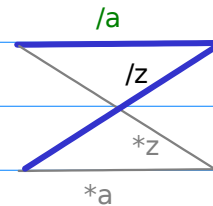
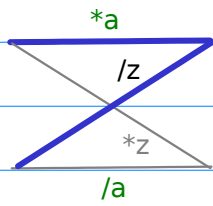
⇒ (1)  $*z = (7) /a$  →

⇒ (2)  $*z = (8) *a$  →



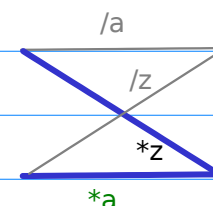
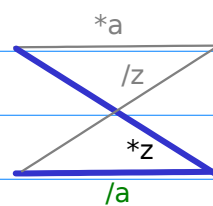
← (5)  $*a = (3) /z$  ←

← (6)  $/a = (4) /z$  ←



⇒ (5)  $*z = (3) /a$  →

⇒ (6)  $*z = (4) *a$  →



$\leftarrow (1) *a = (7) /z \rightleftarrows$   
 $\Rightarrow (1) *z = (7) /a \rightarrow$   
 $\rightleftarrows (3) /z = (5) *a \leftarrow$   
 $\rightarrow (3) /a = (5) *z \Rightarrow$

$\leftarrow (2) /a = (8) /z \rightleftarrows$   
 $\Rightarrow (2) *z = (8) *a \rightarrow$   
 $\rightleftarrows (4) /z = (6) /a \leftarrow$   
 $\rightarrow (4) *a = (6) *z \Rightarrow$

**row major ordering**

$\leftarrow (1) *a \quad *z \Rightarrow$   
 $\rightarrow (3) /a \quad /z \rightleftarrows$   
 $\leftarrow (5) *a \quad *z \Rightarrow$   
 $\rightarrow (7) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$   
 $\rightarrow (4) *a \quad /z \rightleftarrows$   
 $\leftarrow (6) /a \quad *z \Rightarrow$   
 $\rightarrow (8) *a \quad /z \rightleftarrows$

**complementary pair ordering**

$\leftarrow (1) *a \quad *z \Rightarrow$   
 $\leftarrow (5) *a \quad *z \Rightarrow$   
 $\rightarrow (3) /a \quad /z \rightleftarrows$   
 $\rightarrow (7) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$   
 $\leftarrow (6) /a \quad *z \Rightarrow$   
 $\rightarrow (4) *a \quad /z \rightleftarrows$   
 $\rightarrow (8) *a \quad /z \rightleftarrows$

**butterfly pair ordering**

$\leftarrow (1) *a \quad *z \Rightarrow$   
 $\rightarrow (7) /a \quad /z \rightleftarrows$   
 $\leftarrow (5) *a \quad *z \Rightarrow$   
 $\rightarrow (3) /a \quad /z \rightleftarrows$

$\leftarrow (2) /a \quad *z \Rightarrow$   
 $\rightarrow (8) *a \quad /z \rightleftarrows$   
 $\leftarrow (6) /a \quad *z \Rightarrow$   
 $\rightarrow (4) *a \quad /z \rightleftarrows$

Assume  $a > 1$

- (1)  $a^n u(n)$        $*a \leftarrow a^{n+1} u(n)$   
 (7)  $a^n u(n-1)$      $/a \rightarrow a^{n-1} u(n-1)$   
 (5)  $a^n u(-n-1)$      $*a \leftarrow a^{n+1} u(-n-1)$   
 (3)  $a^n u(-n)$          $/a \rightarrow a^{n-1} u(-n)$   
 (2)  $a^{-n} u(n)$          $/a \leftarrow a^{-n-1} u(n)$   
 (8)  $a^{-n} u(n-1)$      $*a \rightarrow a^{-n+1} u(n-1)$   
 (6)  $a^{-n} u(-n-1)$     $/a \leftarrow a^{-n-1} u(-n-1)$   
 (4)  $a^{-n} u(-n)$        $*a \rightarrow a^{-n+1} u(-n)$

$2^n u(n)$	$2^{n+1} u(n)$
$2^n u(n-1)$	$2^{n-1} u(n-1)$
$2^n u(-n-1)$	$2^{n+1} u(-n-1)$
$2^n u(-n)$	$2^{n-1} u(-n)$

$(\frac{1}{2})^{-n} u(n)$	$(\frac{1}{2})^{-n-1} u(n)$
$(\frac{1}{2})^{-n} u(n-1)$	$(\frac{1}{2})^{-n+1} u(n-1)$
$(\frac{1}{2})^{-n} u(-n-1)$	$(\frac{1}{2})^{-n-1} u(-n-1)$
$(\frac{1}{2})^{-n} u(-n)$	$(\frac{1}{2})^{-n+1} u(-n)$

$(\frac{1}{2})^n u(n)$	$(\frac{1}{2})^{n+1} u(n)$
$(\frac{1}{2})^n u(n-1)$	$(\frac{1}{2})^{n-1} u(n-1)$
$(\frac{1}{2})^n u(-n-1)$	$(\frac{1}{2})^{n+1} u(-n-1)$
$(\frac{1}{2})^n u(-n)$	$(\frac{1}{2})^{n-1} u(-n)$

$2^{-n} u(n)$	$2^{-n-1} u(n)$
$2^{-n} u(n-1)$	$2^{-n+1} u(n-1)$
$2^{-n} u(-n-1)$	$2^{-n-1} u(-n-1)$
$2^{-n} u(-n)$	$2^{-n+1} u(-n)$



Complement Type I	$u(n)$	(1)	(2)
	$u(-n-1)$	(5)	(6)
Complement Type II	$u(-n)$	(3)	(4)
	$u(n-1)$	(7)	(8)

complementary pair ordering

$$\begin{array}{l} (1) \quad a \quad z \quad \mathbf{a}^n \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{a}^{-n} \\ (6) \quad a \quad z^{-1} \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{a}^n \\ (7) \quad a \quad z \quad \mathbf{a}^n \end{array}$$

$$\begin{array}{l} (4) \quad a \quad z^{-1} \quad \mathbf{a}^{-n} \\ (8) \quad a^{-1}z \quad \mathbf{a}^{-n} \end{array}$$

$$\begin{array}{l} (1) \quad a \quad z \quad \mathbf{*a} \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{/a} \\ (6) \quad a \quad z^{-1} \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{/a} \\ (7) \quad a \quad z \quad \mathbf{/a} \end{array}$$

$$\begin{array}{l} (4) \quad a \quad z^{-1} \quad \mathbf{*a} \\ (8) \quad a^{-1}z \quad \mathbf{*a} \end{array}$$

$$\begin{array}{l} (1) \quad a \quad z \quad \mathbf{*z} \\ (5) \quad a^{-1}z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (2) \quad a^{-1}z \quad \mathbf{*z} \\ (6) \quad a \quad z^{-1} \quad \mathbf{*z} \end{array}$$

$$\begin{array}{l} (3) \quad a^{-1}z^{-1} \quad \mathbf{/z} \\ (7) \quad a \quad z \quad \mathbf{/z} \end{array}$$

$$\begin{array}{l} (4) \quad a \quad z^{-1} \quad \mathbf{/z} \\ (8) \quad a^{-1}z \quad \mathbf{/z} \end{array}$$

# Geometric Series Combinations

complementary pair ordering

(1)

unit	$\frac{1}{1-az}$ $ z  < a^{-1}$	$a^n u(n)$
non-unit	$-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z  > a^{-1}$	$-a^n u(-n-1)$

(2)

unit	$\frac{1}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^n u(n)$
non-unit	$-\frac{a^nz^{-1}}{1-a^nz^{-1}}$ $ z  > a$	$-(\frac{1}{a})^n u(-n-1)$

(5)

(3)

unit	$-\frac{1}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^n u(-n)$
non-unit	$\frac{az}{1-az}$ $ z  < a^{-1}$	$a^n u(n-1)$

(6)

(4)

unit	$-\frac{1}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^n u(-n)$
non-unit	$\frac{a^nz}{1-a^nz}$ $ z  < a$	$(\frac{1}{a})^n u(n-1)$

(7)

(8)

		Positive Exponent	Negative Exponent
Complement Type I	$u(n)$	(1)	(2)
	$u(-n-1)$	(5)	(6)
Complement Type II	$u(-n)$	(3)	(4)
	$u(n-1)$	(7)	(8)

# Shifted Combinations (I) by scaling $*a$ $/a$

complementary  
pair  
ordering

(1)  $*a$

$\frac{a}{1-az}$ $ z  < a^{-1}$	$a^{n+1} u(n)$
$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^{n+1} u(-n-1)$

(2)  $/a$

$\frac{a^{-1}}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^{n+1} u(n)$
$-\frac{z^{-1}}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^{n+1} u(-n-1)$

(5)  $*a$

(3)  $/a$

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^{n-1} u(-n)$
$\frac{z}{1-az}$ $ z  < a^{-1}$	$a^{n-1} u(n-1)$

(6)  $/a$

(4)  $*a$

$-\frac{a}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^{n-1} u(-n)$
$\frac{z}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^{n-1} u(n-1)$

(7)  $/a$

(8)  $*a$

	Positive Exponent	Negative Exponent		
Complement Type I	(1) $u(n)$	(2) $u(n)$	$*a$ $u(n)$	$/a$ $u(n)$
	(5) $u(-n-1)$	(6) $u(-n-1)$	$*a$ $u(-n-1)$	$/a$ $u(-n-1)$
Complement Type II	(3) $u(-n)$	(4) $u(-n)$	$/a$ $u(-n)$	$*a$ $u(-n)$
	(7) $u(n-1)$	(8) $u(n-1)$	$/a$ $u(n-1)$	$*a$ $u(n-1)$

# Shifted Combinations (II) by scaling $*z /z$

complementary  
pair  
ordering

(1)  $*z$

$\frac{z}{1-az}$ $ z  < a^{-1}$	$a^{n-1} u(n-1)$
$-\frac{a}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^{n-1} u(-n)$

(2)  $*z$

$\frac{z}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^{n-1} u(n-1)$
$-\frac{a}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^{n-1} u(-n)$

(5)  $*z$

(3)  $/z$

$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z  > a^{-1}$	$-a^{n+1} u(-n-1)$
$\frac{a}{1-az}$ $ z  < a^{-1}$	$a^{n+1} u(n)$

(7)  $/z$

(6)  $*z$

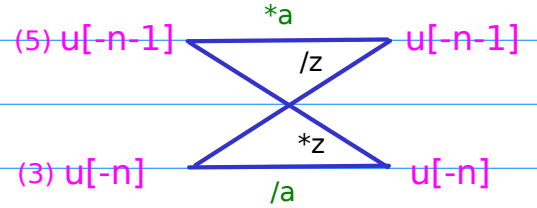
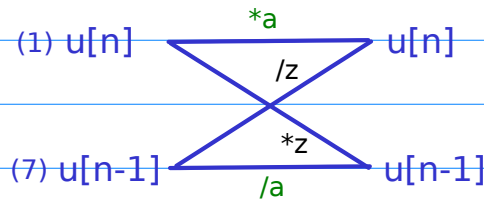
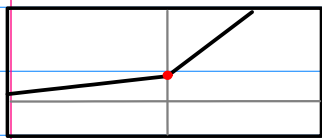
(4)  $/z$

$-\frac{z^{-1}}{1-az^{-1}}$ $ z  > a$	$-(\frac{1}{a})^{n+1} u(-n-1)$
$\frac{a^{-1}}{1-a^{-1}z}$ $ z  < a$	$(\frac{1}{a})^{n+1} u(n)$

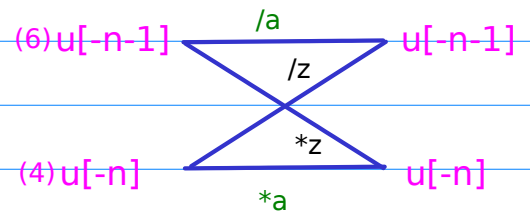
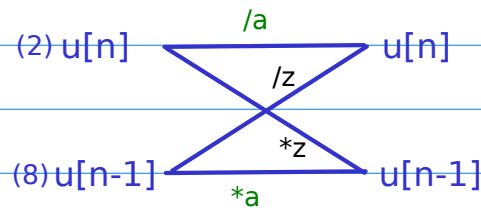
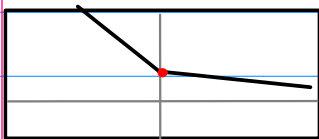
(8)  $/z$

	Positive Exponent	Negative Exponent		
Complement Type I	(1) $u(n)$	(2) $u(n)$	$*z$ $u(n-1)$	$*z$ $u(n-1)$
	(5) $u(-n-1)$	(6) $u(-n-1)$	$*z$ $u(-n)$	$*z$ $u(-n)$
Complement Type II	(3) $u(-n)$	(4) $u(-n)$	$/z$ $u(-n-1)$	$/z$ $u(-n-1)$
	(7) $u(n-1)$	(8) $u(n-1)$	$/z$ $u(n)$	$/z$ $u(n)$

$a^n$



$a^{-n}$



<b>Causal</b>	$u(n)$	(1)	(2)	<b>butterfly pair ordering</b>
	$u(n-1)$	(7)	(8)	
<b>Anti-Causal</b>	$u(-n-1)$	(5)	(6)	
	$u(-n)$	(3)	(4)	

<b>Positive Exponent</b>	<b>Negative Exponent</b>
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

**row major ordering**

$$(1) a^n u(n) \begin{array}{c} *a \\ /z \\ *z \\ /a \end{array} a^{n+1} u(n)$$

$$(7) a^n u(n-1) \begin{array}{c} *z \\ /a \end{array} a^{n-1} u(n-1)$$

$$(5) a^n u(-n-1) \begin{array}{c} *a \\ /z \\ *z \\ /a \end{array} a^{n+1} u(-n-1)$$

$$(3) a^n u(-n) \begin{array}{c} *z \\ /a \end{array} a^{n-1} u(-n)$$

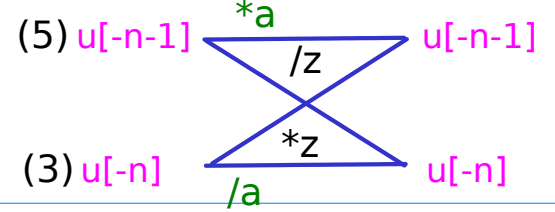
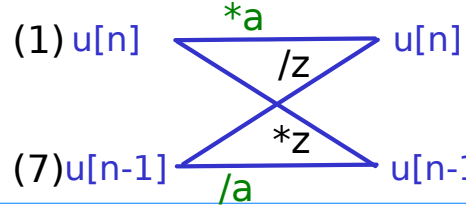
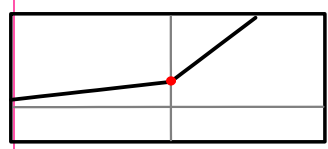
$$(2) a^{-n} u(n) \begin{array}{c} /a \\ /z \\ *z \\ *a \end{array} a^{-n-1} u(n)$$

$$(8) a^{-n} u(n-1) \begin{array}{c} *z \\ /a \end{array} a^{-n+1} u(n-1)$$

$$(6) a^{-n} u(-n-1) \begin{array}{c} /a \\ /z \\ *z \\ *a \end{array} a^{-n-1} u(-n-1)$$

$$(4) a^{-n} u(-n) \begin{array}{c} *z \\ /a \end{array} a^{-n+1} u(-n)$$

$a^n$



(1)  $*a$

$\frac{1}{1-az} \quad  z  < a^{-1}$	$\frac{a}{1-az} \quad  z  < a^{-1}$
$\frac{az}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$

(7)  $/a$

(1)  $*a$

$a^n u(n)$ $(a^0, a^1, a^2, \dots)$	$a^{n+1} u(n)$ $(a^1, a^2, a^3, \dots)$
$a^n u(n-1)$ $(a^1, a^2, a^3, \dots)$	$a^{n-1} u(n-1)$ $(a^0, a^1, a^2, \dots)$

(7)  $/a$

(1)  $*z$

$\frac{1}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$
$\frac{az}{1-az} \quad  z  < a^{-1}$	$\frac{a}{1-az} \quad  z  < a^{-1}$

(7)  $/z$

(1)  $*z$

$a^n u(n)$ $(a^0, a^1, a^2, \dots)$	$a^{n-1} u(n-1)$ $(a^0, a^1, a^2, \dots)$
$a^n u(n-1)$ $(a^1, a^2, a^3, \dots)$	$a^{n+1} u(n)$ $(a^1, a^2, a^3, \dots)$

(7)  $/z$

(5)  $*a$

$-\frac{a'z^{-1}}{1-a'z^{-1}} \quad  z  > a^{-1}$	$-\frac{z^{-1}}{1-a'z^{-1}} \quad  z  > a^{-1}$
$-\frac{1}{1-a'z^{-1}} \quad  z  > a^{-1}$	$-\frac{a'}{1-a'z^{-1}} \quad  z  > a^{-1}$

(3)  $/a$

(5)  $*a$

$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^1}, \frac{1}{a^0})$
$-a^n u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$

(3)  $/a$

(5)  $*z$

$-\frac{a'z^{-1}}{1-a'z^{-1}} \quad  z  > a^{-1}$	$-\frac{a'}{1-a'z^{-1}} \quad  z  > a^{-1}$
$-\frac{1}{1-a'z^{-1}} \quad  z  > a^{-1}$	$-\frac{z^{-1}}{1-a'z^{-1}} \quad  z  > a^{-1}$

(3)  $/z$

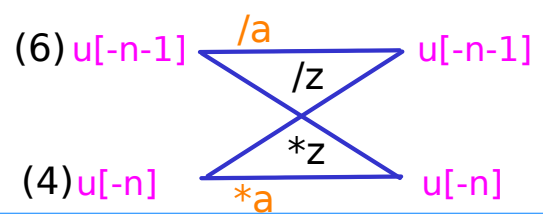
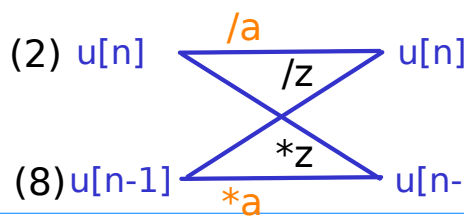
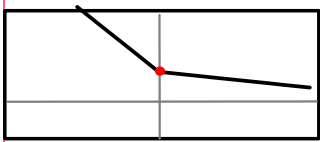
(5)  $*z$

$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$
$-a^n u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^1}, \frac{1}{a^0})$

(3)  $/z$

butterfly pair ordering

$a^{-n}$



(2)  $/a$

$\frac{1}{1-a^{-1}z} \quad  z  < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$
$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$

(2)  $/a$

$(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
$(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$

(8)  $*a$

(8)  $*a$

(2)  $*z$

$\frac{1}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$
$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$

(2)  $*z$

$(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^3}, \dots)$
$(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$

(8)  $/z$

(8)  $/z$

(6)  $/a$

$-\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$
$-\frac{1}{1-az^{-1}} \quad  z  > a$	$-\frac{a}{1-az^{-1}} \quad  z  > a$

(6)  $/a$

$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$
$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^1, a^0, a^{-1})$

(4)  $*a$

(4)  $*a$

(6)  $*z$

$-\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$-\frac{a}{1-az^{-1}} \quad  z  > a$
$-\frac{1}{1-az^{-1}} \quad  z  > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$

(6)  $*z$

$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$
$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$

(4)  $/z$

(4)  $/z$

butterfly pair ordering



# Scale by **a**

## 1. Geometric Series

complementary  
pair  
ordering

(1)

**\*a**

(2)

**/a**

	$\frac{1}{1-az} \quad  z  < a^{-1}$	$\frac{a}{1-az} \quad  z  < a^{-1}$	$\frac{1}{1-a^{-1}z} \quad  z  < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad  z  < a$
Comp.ROC	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{az^{-1}}{1-az^{-1}} \quad  z  > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad  z  > a$

(5)

**\*a**

(6)

**/a**

(3)

**/a**

(4)

**\*a**

	$-\frac{1}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad  z  > a^{-1}$	$-\frac{1}{1-az^{-1}} \quad  z  > a$	$-\frac{a}{1-az^{-1}} \quad  z  > a$
Comp.ROC	$\frac{az}{1-az} \quad  z  < a^{-1}$	$\frac{z}{1-az} \quad  z  < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z} \quad  z  < a$	$\frac{z}{1-a^{-1}z} \quad  z  < a$

(7)

**/a**

(8)

**\*a**

(1)	$a^n u(n)$	$\begin{array}{c} \xrightarrow{*a} \\ /z \\ \xrightarrow{*z} \end{array}$	$a^{n+1} u(n)$	(2)	$a^{-n} u(n)$	$\begin{array}{c} \xrightarrow{/a} \\ /z \\ \xrightarrow{*z} \end{array}$	$a^{-n-1} u(n)$
(7)	$a^n u(n-1)$	$\begin{array}{c} \xrightarrow{*z} \\ /a \end{array}$	$a^{n-1} u(n-1)$	(8)	$a^{-n} u(n-1)$	$\begin{array}{c} \xrightarrow{*a} \\ /z \end{array}$	$a^{-n+1} u(n-1)$
(5)	$a^n u(-n-1)$	$\begin{array}{c} \xrightarrow{*a} \\ /z \\ \xrightarrow{*z} \end{array}$	$a^{n+1} u(-n-1)$	(6)	$a^{-n} u(-n-1)$	$\begin{array}{c} \xrightarrow{/a} \\ /z \\ \xrightarrow{*z} \end{array}$	$a^{-n-1} u(-n-1)$
(3)	$a^n u(-n)$	$\begin{array}{c} \xrightarrow{*z} \\ /a \end{array}$	$a^{n-1} u(-n)$	(4)	$a^{-n} u(-n)$	$\begin{array}{c} \xrightarrow{*a} \\ /z \end{array}$	$a^{-n+1} u(-n)$

# Scale by **a**

## 2. Sequences

complementary  
pair  
ordering

(1)

**\*a**

$a^n u(n)$
$- a^n u(-n-1)$

$a^{n+1} u(n)$
$- a^{n+1} u(-n-1)$

(2)

**/a**

$(\frac{1}{a})^n u(n)$
$-(\frac{1}{a})^n u(-n-1)$

$(\frac{1}{a})^{n+1} u(n)$
$-(\frac{1}{a})^{n+1} u(-n-1)$

Comp.ROC

(5)

**\*a**

$- a^n u(-n)$
$a^n u(n-1)$

$- a^{n-1} u(-n)$
$a^{n-1} u(n-1)$

(6)

**/a**

$-(\frac{1}{a})^n u(-n)$
$(\frac{1}{a})^n u(n-1)$

$-(\frac{1}{a})^{n-1} u(-n)$
$(\frac{1}{a})^{n-1} u(n-1)$

Comp.ROC

(3)

**/a**

$- a^n u(-n)$
$a^n u(n-1)$

$- a^{n-1} u(-n)$
$a^{n-1} u(n-1)$

(4)

**\*a**

$-(\frac{1}{a})^n u(-n)$
$(\frac{1}{a})^n u(n-1)$

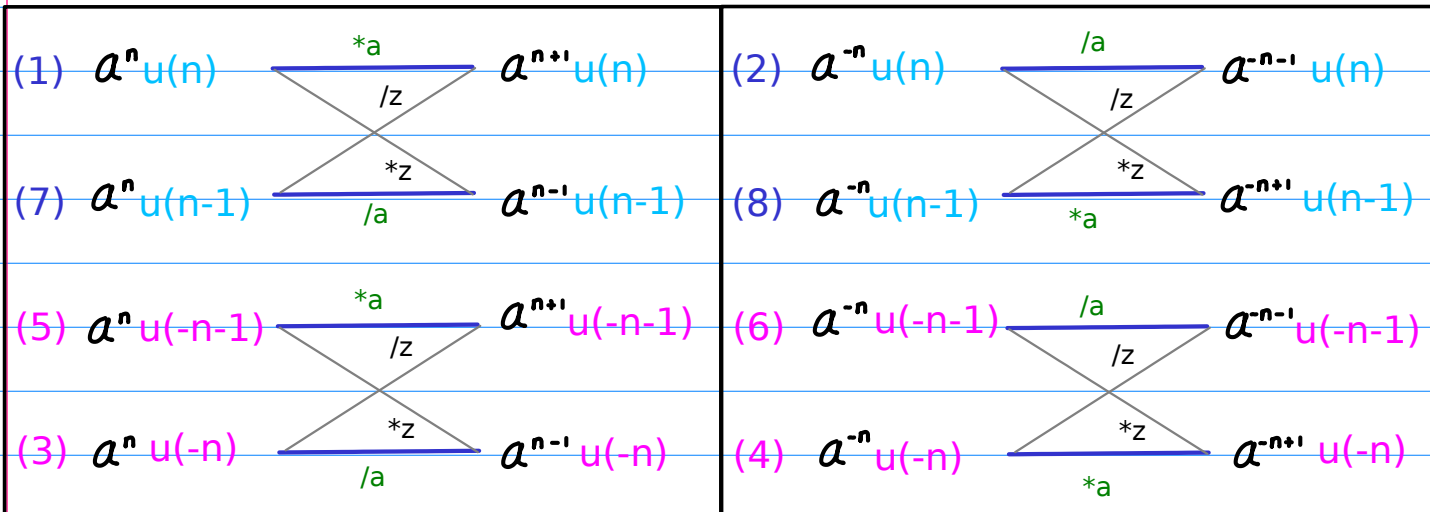
$-(\frac{1}{a})^{n-1} u(-n)$
$(\frac{1}{a})^{n-1} u(n-1)$

(7)

**/a**

(8)

**\*a**



# Scale by **a**

## 3. Sequence values

complementary  
pair  
ordering

(1)

**\*a**

(2)

**/a**

$(a^0, a^1, a^2, \dots)$	$(a^1, a^2, a^3, \dots)$
$-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$

$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
$-(\dots, a^3, a^2, a^1)$	$-(\dots, a^2, a^1, a^0)$

Comp.ROC

(5)

**\*a**

(6)

**/a**

(3)

**/a**

(4)

**\*a**

$-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$
$(a^1, a^2, a^3, \dots)$	$(a^0, a^1, a^2, \dots)$

$-(\dots, a^2, a^1, a^0)$	$-(\dots, a^3, a^2, a^1)$
$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$

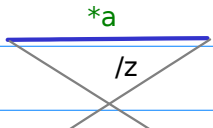
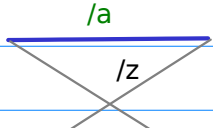
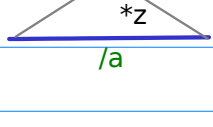
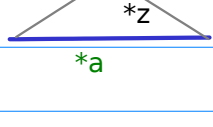
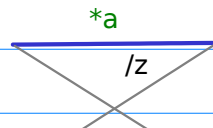
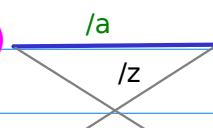
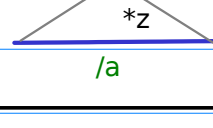
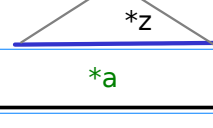
Comp.ROC

(7)

**/a**

(8)

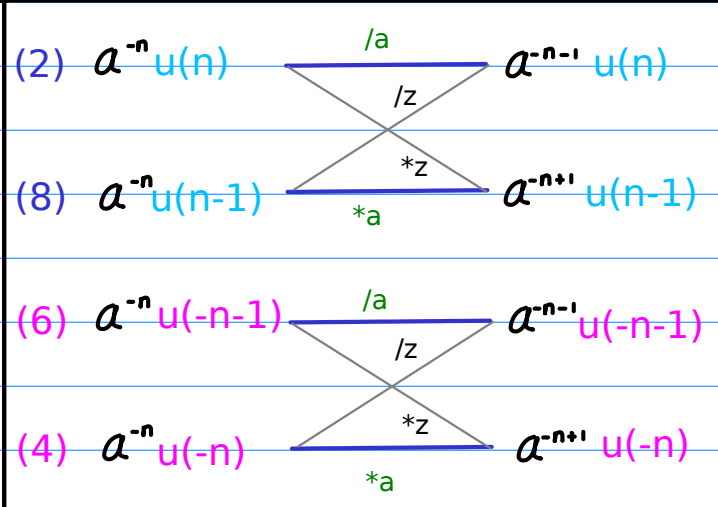
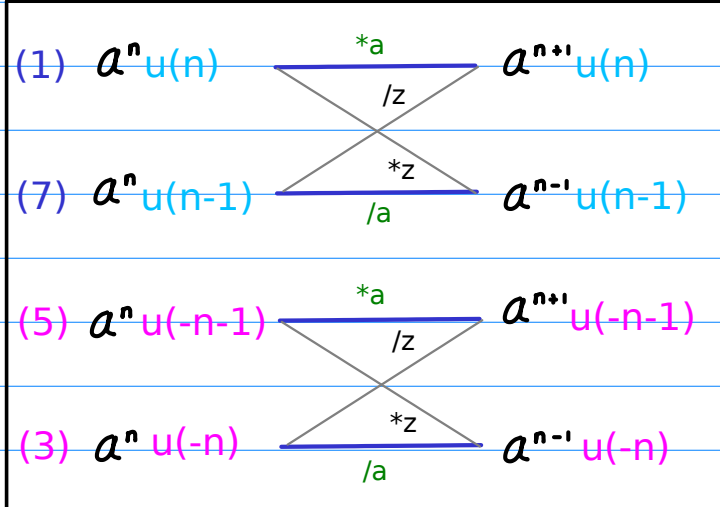
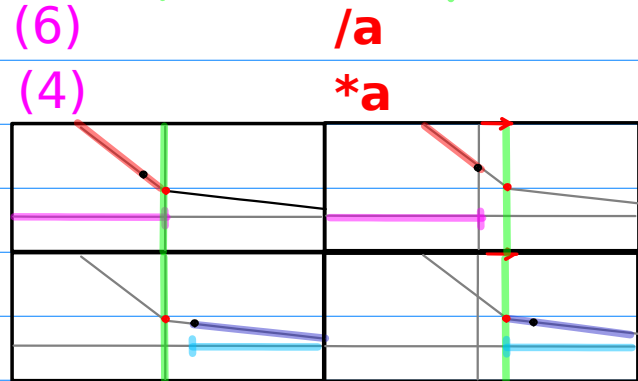
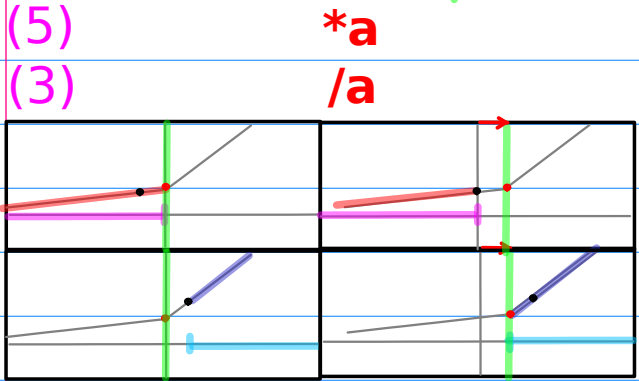
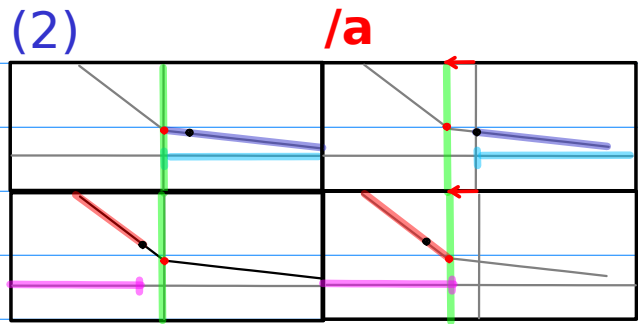
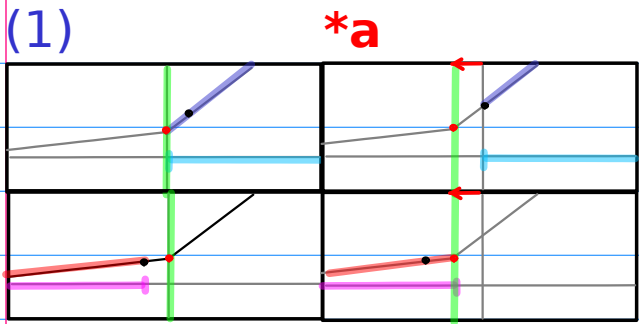
**\*a**

(1) $a^n u(n)$		$a^{n+1} u(n)$	(2) $a^{-n} u(n)$		$a^{-n-1} u(n)$
(7) $a^n u(n-1)$		$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$		$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$		$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$		$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$		$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$		$a^{-n+1} u(-n)$

# Scale by **a**

## 4. Graphs

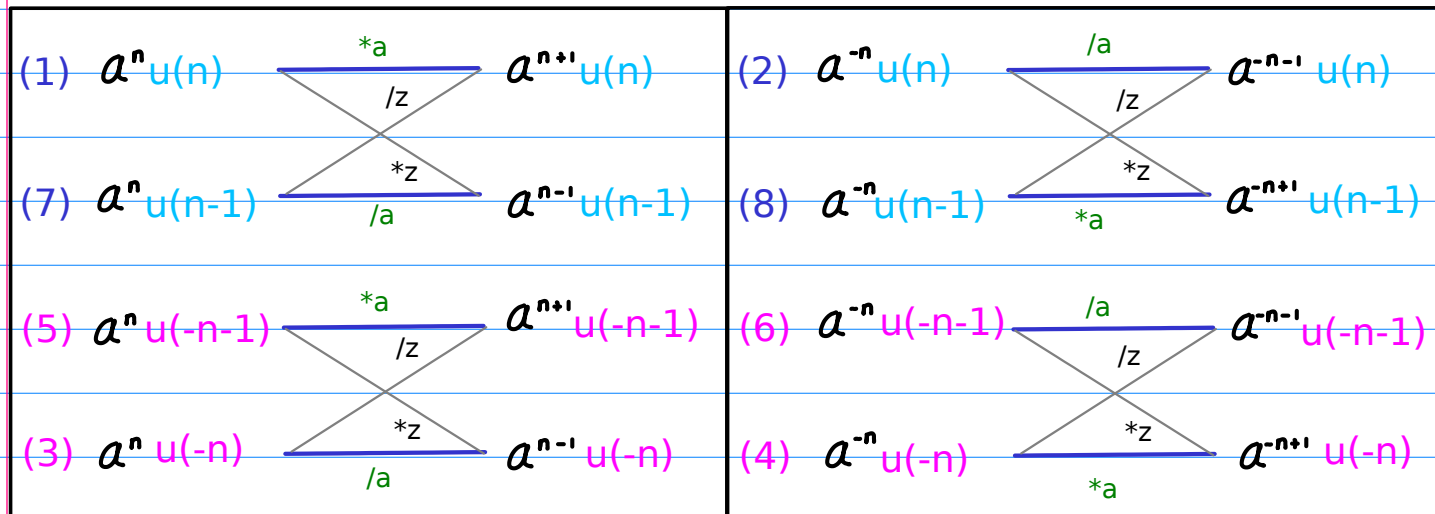
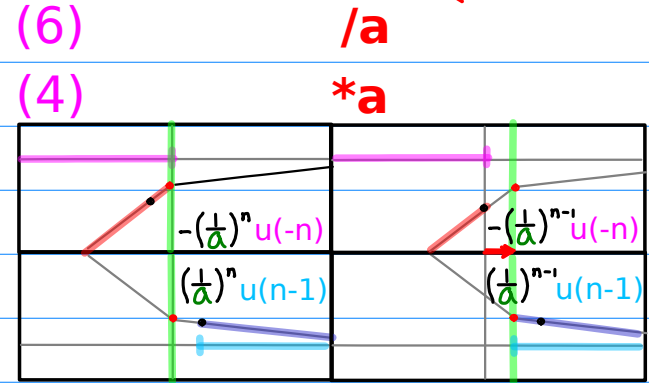
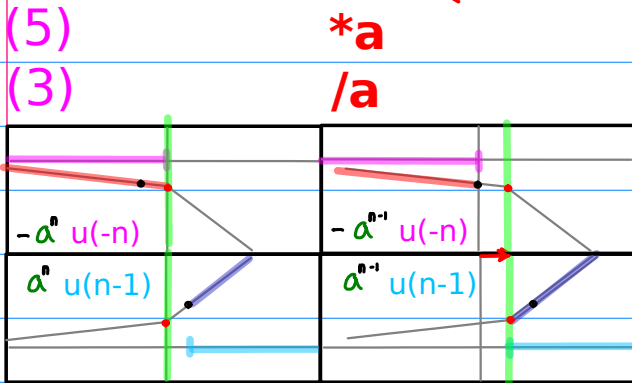
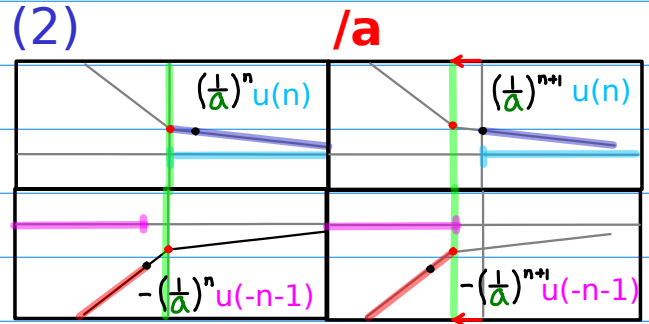
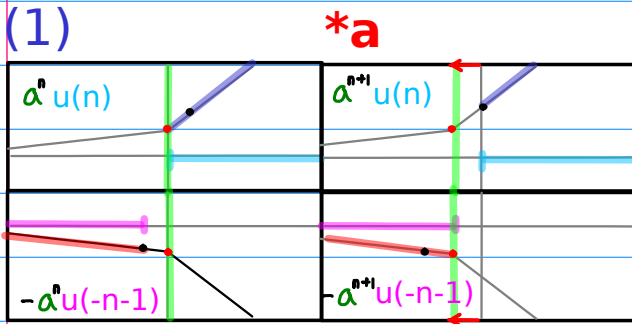
complementary  
pair  
ordering



# Scale by **a**

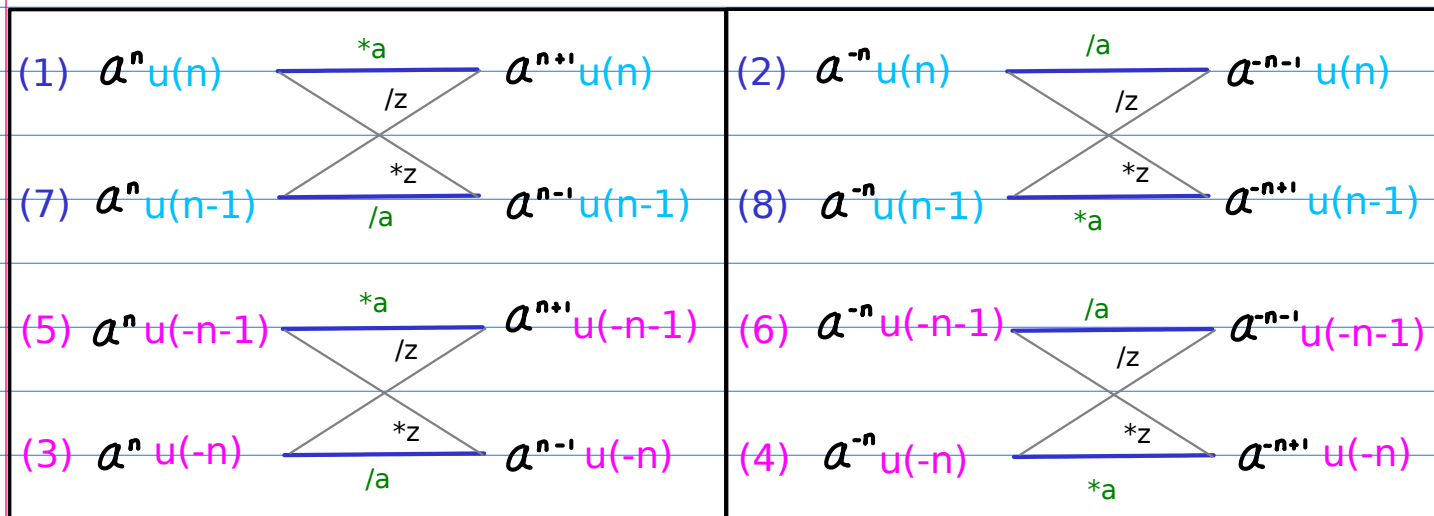
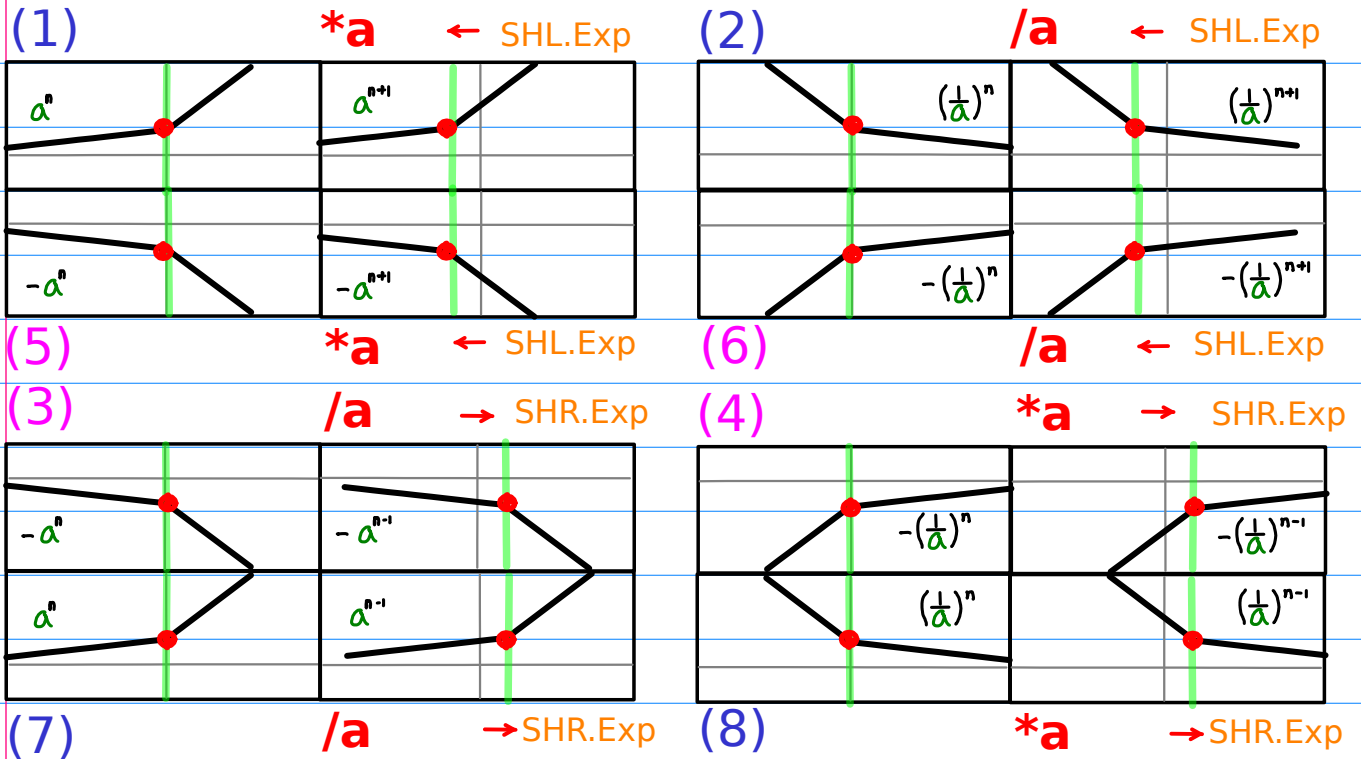
## 5. Graphs - signs

complementary  
pair  
ordering



# Scale by **a**

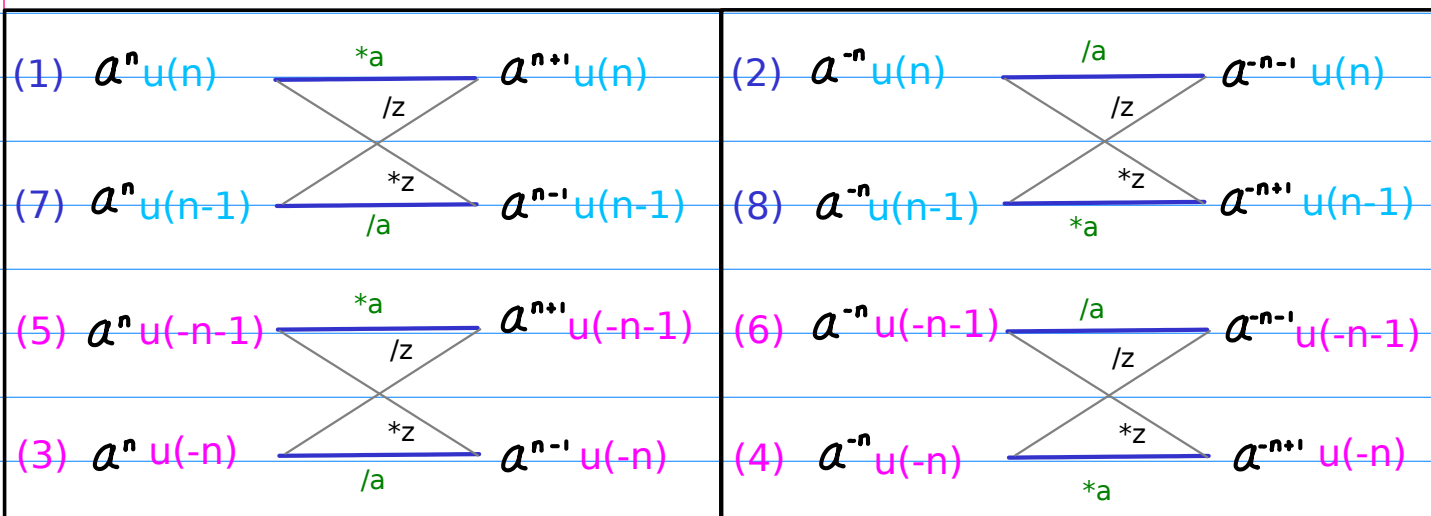
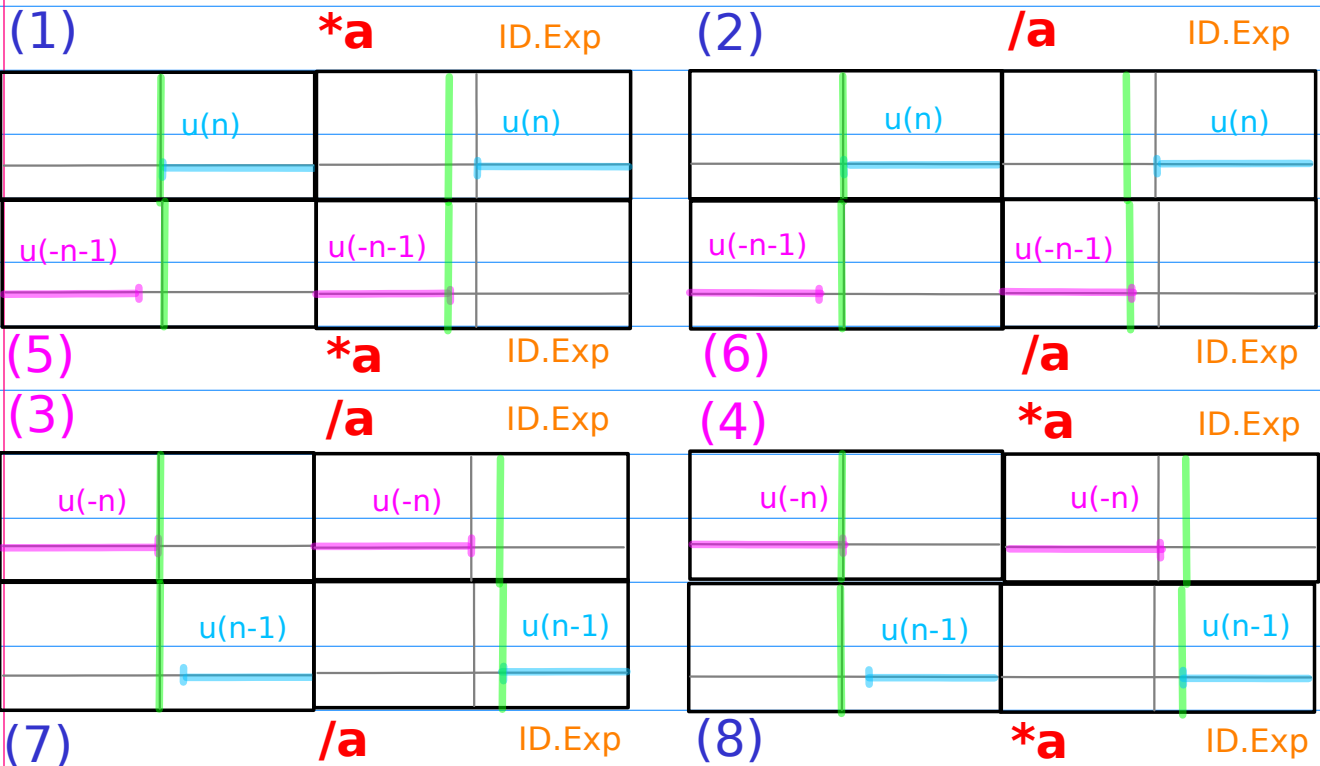
## 6. Graphs - Exponents



# Scale by **a**

## 7. Graphs - Ranges

complementary  
pair  
ordering



# Scale by $z$

## 1. Geometric Series

(1)

$*z$

$$\frac{1}{1-az} \quad |z| < a^{-1}$$

$$-\frac{a^i z^i}{1-a^i z^i} \quad |z| > a^{-1}$$

$$\frac{z}{1-az} \quad |z| < a^{-1}$$

$$-\frac{a^i}{1-a^i z^i} \quad |z| > a^{-1}$$

(2)

$*z$

$$\frac{1}{1-a^{-1}z} \quad |z| < a$$

$$-\frac{a z^i}{1-a z^i} \quad |z| > a$$

$$\frac{z}{1-a^{-1}z} \quad |z| < a$$

$$-\frac{a}{1-a z^i} \quad |z| > a$$

Comp.ROC

(5)

$*z$

(3)

$/z$

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$$\frac{az}{1-az} \quad |z| < a^{-1}$$

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$$\frac{a}{1-az} \quad |z| < a^{-1}$$

(6)

$*z$

(4)

$/z$

$$-\frac{1}{1-az^{-1}} \quad |z| > a$$

$$\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$$

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z| > a$$

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z| < a$$

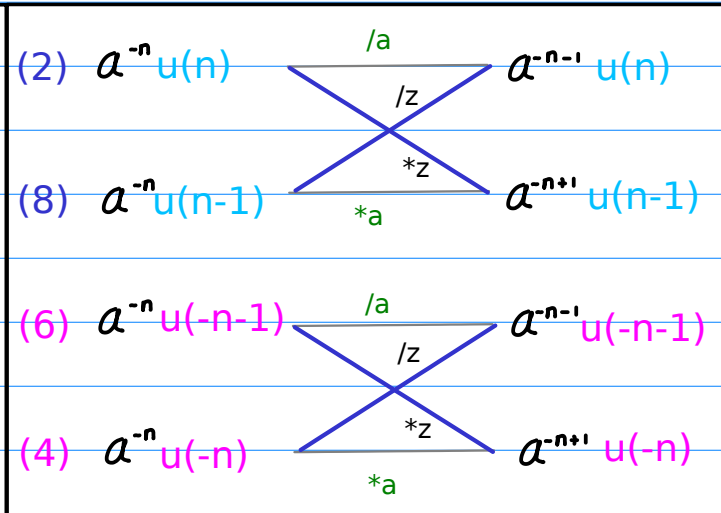
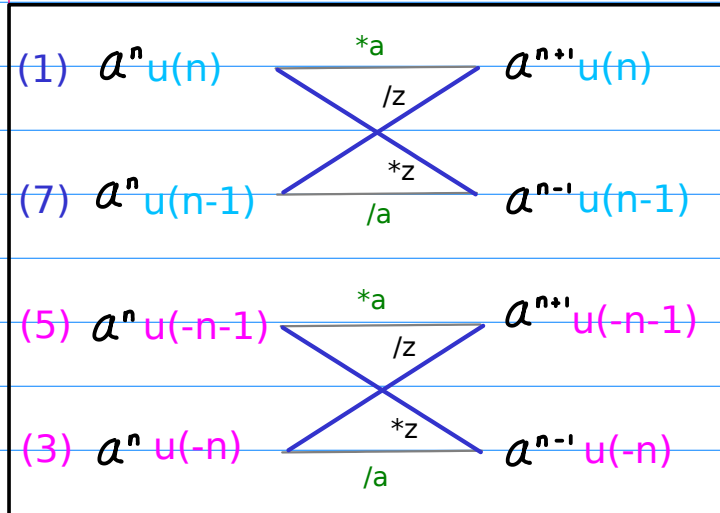
Comp.ROC

(7)

$/z$

(8)

$/z$





# Scale by $z$

## 2. Sequences

complementary pair ordering

(1)

$*z$

$a^n u(n)$	$a^{n-1} u(n-1)$
$-a^n u(-n-1)$	$-a^{n-1} u(-n)$

(2)

$*z$

$(\frac{1}{a})^n u(n)$	$(\frac{1}{a})^{n-1} u(n-1)$
$-(\frac{1}{a})^n u(-n-1)$	$-(\frac{1}{a})^{n-1} u(-n)$

(5)

$*z$

$-a^n u(-n)$	$-a^{n+1} u(-n-1)$
$a^n u(n-1)$	$a^{n+1} u(n)$

(6)

$*z$

$-(\frac{1}{a})^n u(-n)$	$-(\frac{1}{a})^{n+1} u(-n-1)$
$(\frac{1}{a})^n u(n-1)$	$(\frac{1}{a})^{n+1} u(n)$

(3)

$/z$

$-a^n u(-n)$	$-a^{n+1} u(-n-1)$
$a^n u(n-1)$	$a^{n+1} u(n)$

(4)

$/z$

$-(\frac{1}{a})^n u(-n)$	$-(\frac{1}{a})^{n+1} u(-n-1)$
$(\frac{1}{a})^n u(n-1)$	$(\frac{1}{a})^{n+1} u(n)$

(7)

$/z$

(1) $a^n u(n)$	$*a$	$a^{n+1} u(n)$
(7) $a^n u(n-1)$	$/z$	$a^{n-1} u(n-1)$
(5) $a^n u(-n-1)$	$*z$	$a^{n+1} u(-n-1)$
(3) $a^n u(-n)$	$/a$	$a^{n-1} u(-n)$

(8)

$/z$

(2) $a^{-n} u(n)$	$/a$	$a^{-n-1} u(n)$
(8) $a^{-n} u(n-1)$	$/z$	$a^{-n-1} u(n-1)$
(6) $a^{-n} u(-n-1)$	$*z$	$a^{-n+1} u(-n-1)$
(4) $a^{-n} u(-n)$	$*a$	$a^{-n+1} u(-n)$

# Scale by $z$

## 3. Sequence values

complementary pair ordering

(1)

$*z$

$(a^0, a^1, a^2, \dots)$	$(a^0, a^1, a^2, \dots)$
$-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\dots, \frac{1}{a^2}, \frac{1}{a^2}, \frac{1}{a^1})$

(2)

$*z$

$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
$-(\dots, a^3, a^2, a^1)$	$-(\dots, a^3, a^2, a^1)$

Comp.ROC

(5)

$*z$

$-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\dots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$
$(a^1, a^2, a^3, \dots)$	$(a^1, a^2, a^3, \dots)$

(6)

$*z$

$-(\dots, a^2, a^1, a^0)$	$-(\dots, a^2, a^1, a^0)$
$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$

Comp.ROC

(3)

$/z$

(7)

$/z$

(4)

$/z$

(8)

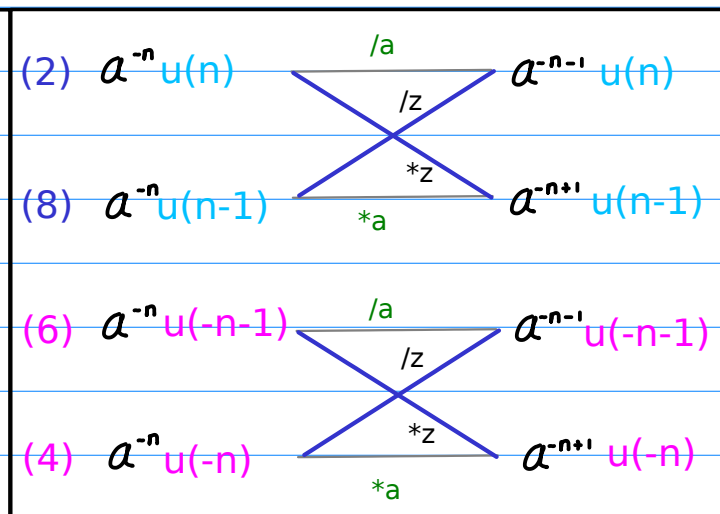
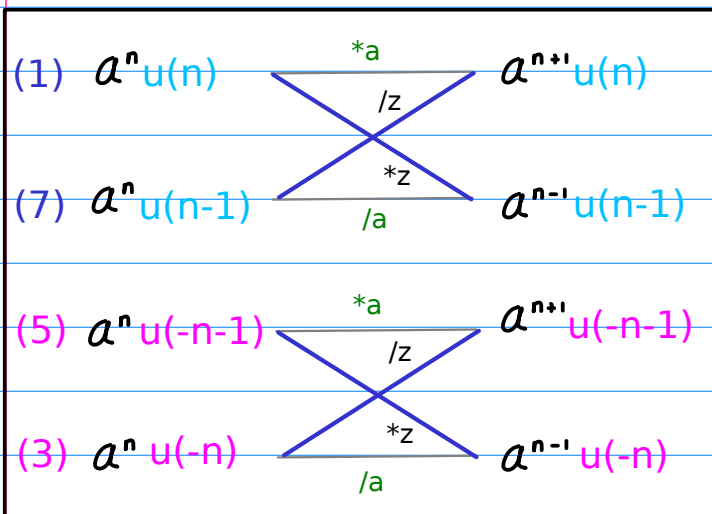
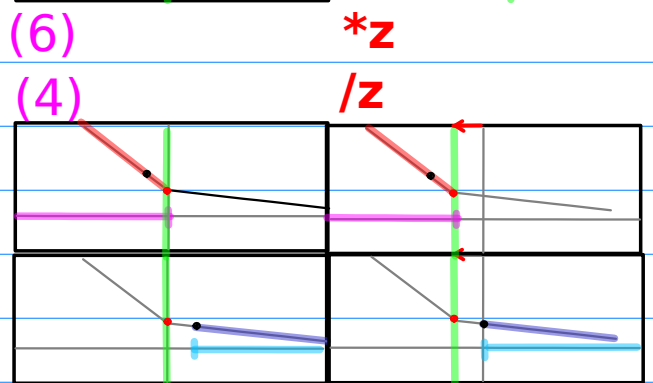
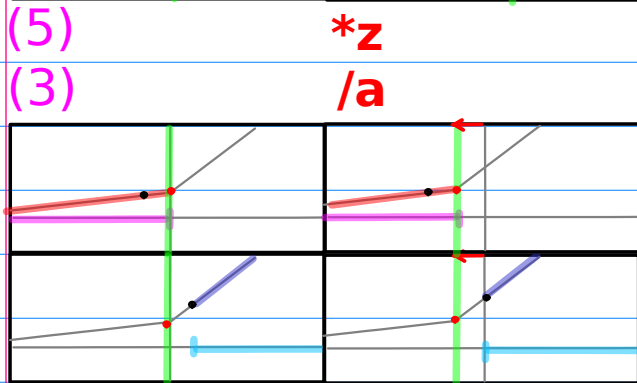
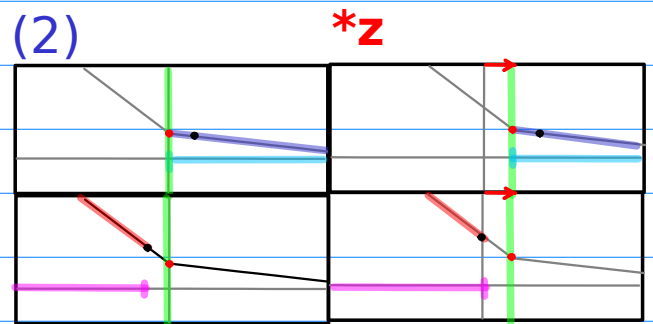
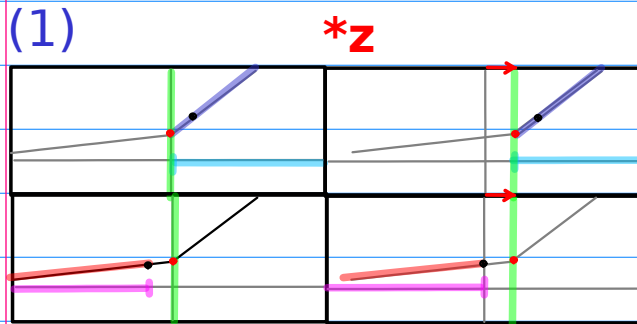
$/z$

(1) $a^n u(n)$	$*a$	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	$/a$	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	$/z$	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	$/z$	$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	$*z$	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	$*a$	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	$/a$	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	$/z$	$a^{-n+1} u(-n)$

# Scale by $z$

## 4. Graphs

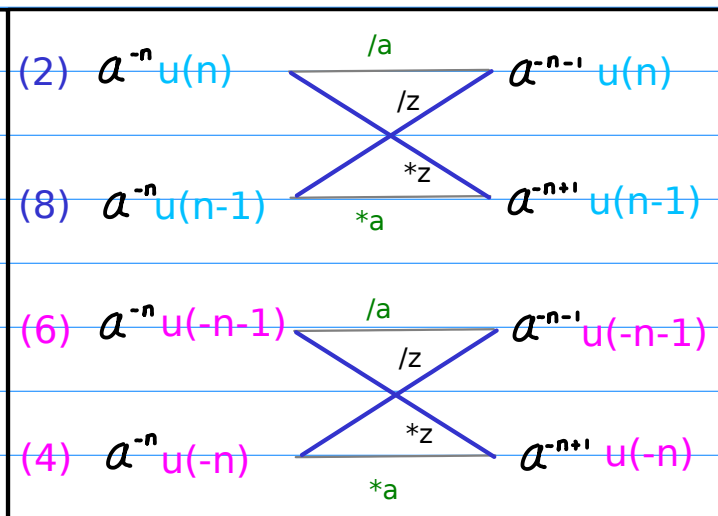
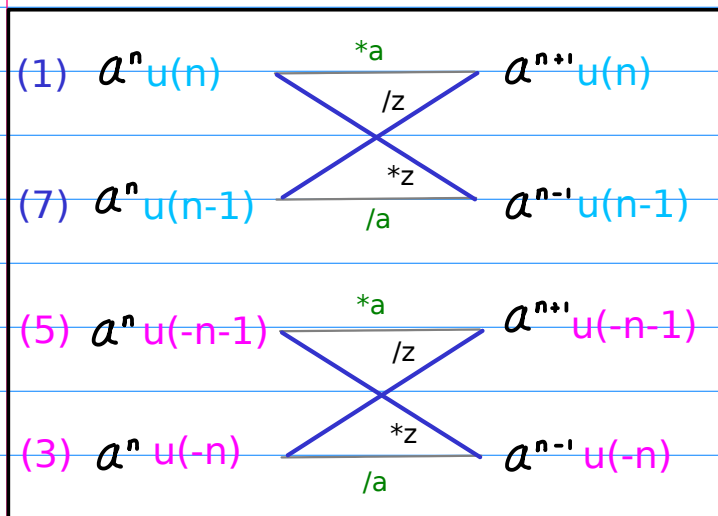
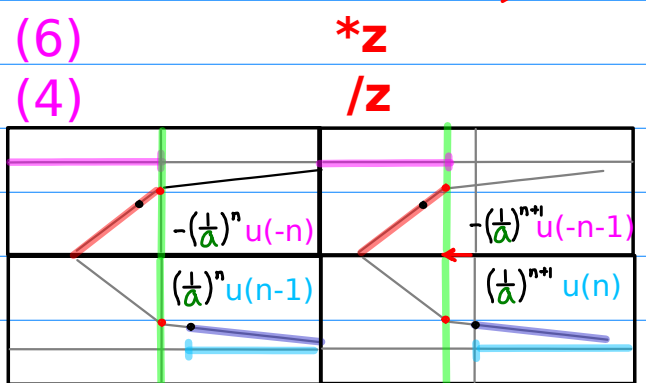
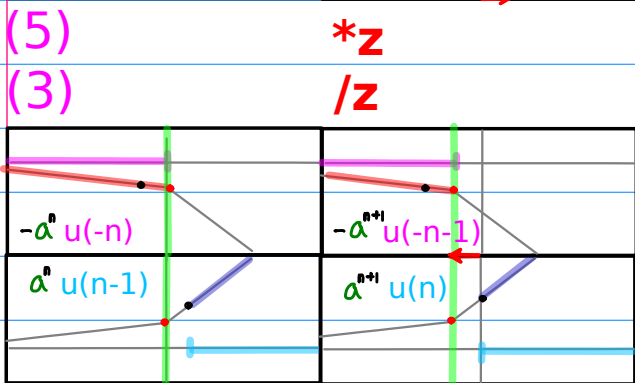
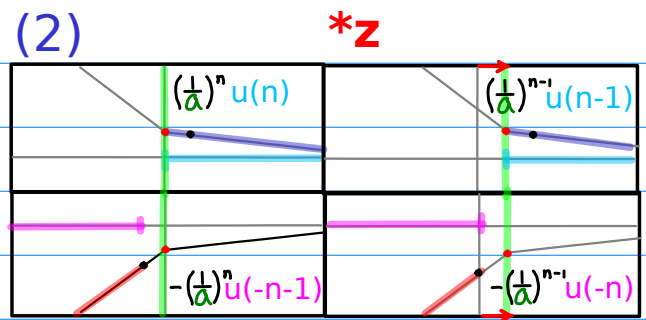
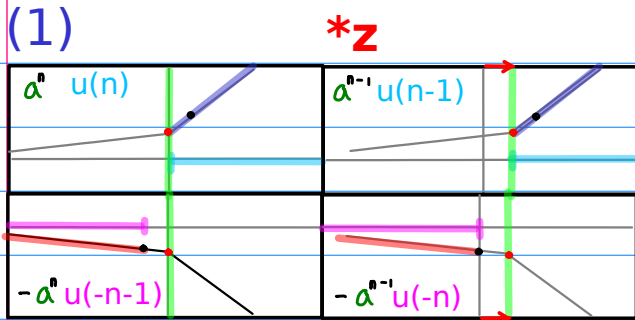
complementary pair ordering



# Scale by $z$

## 5. Graphs - signs

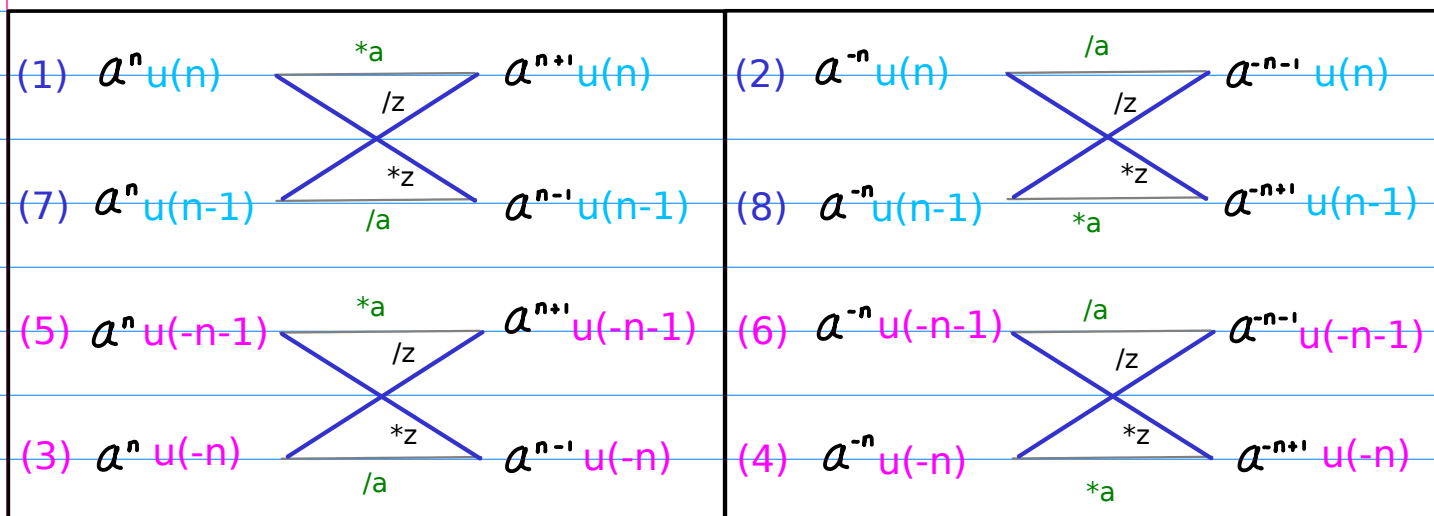
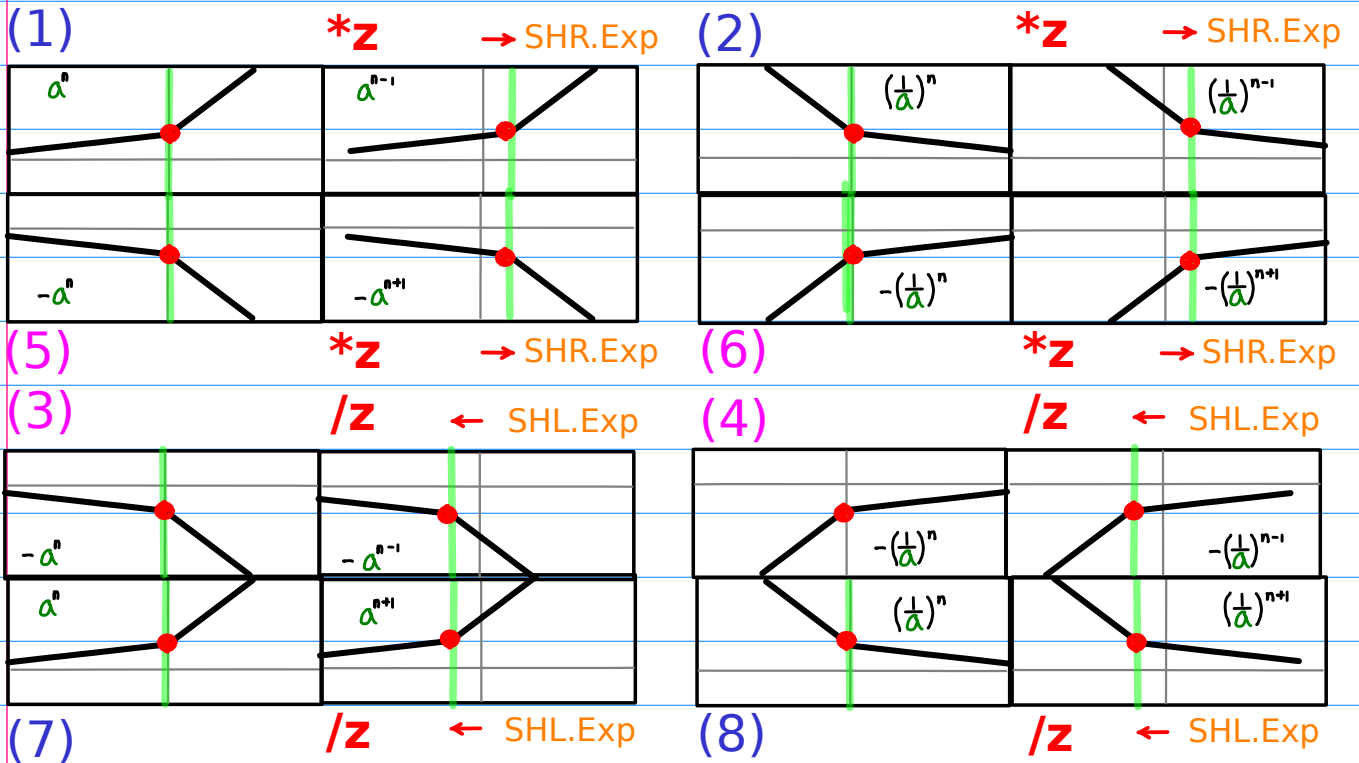
complementary pair ordering



# Scale by $z$

## 6. Graphs - Exponents

complementary pair ordering

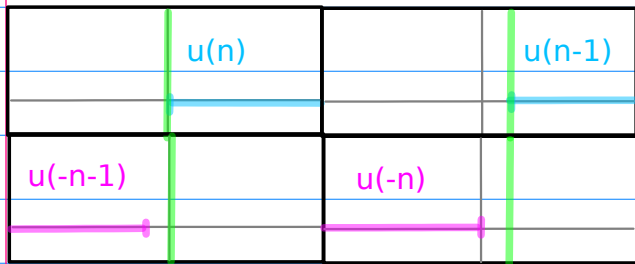


# Scale by $z$

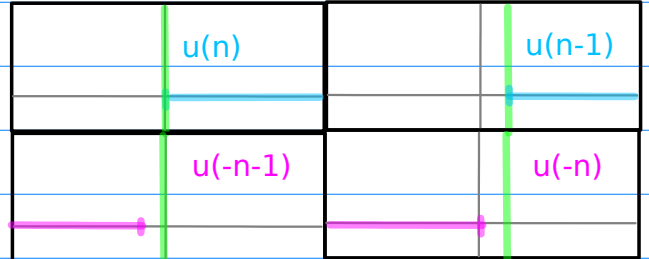
## 7. Graphs - Ranges

complementary pair ordering

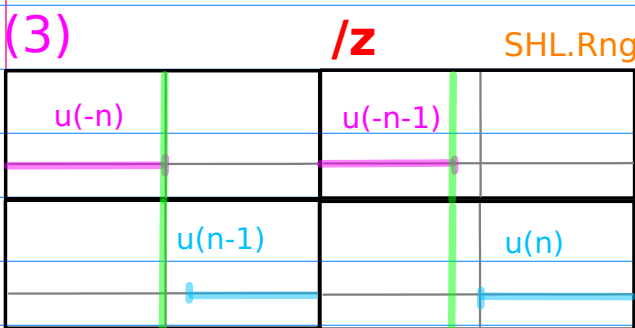
(1)  $*z$  SHR.Rng



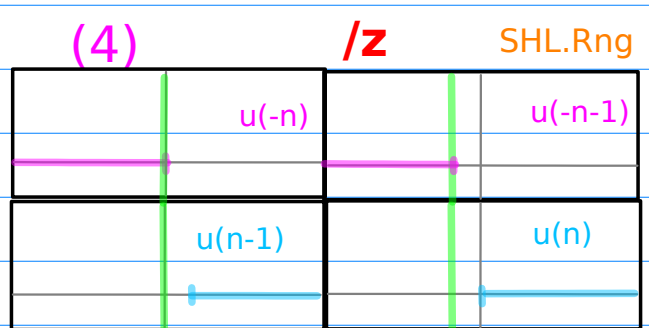
(2)  $*z$  SHR.Rng



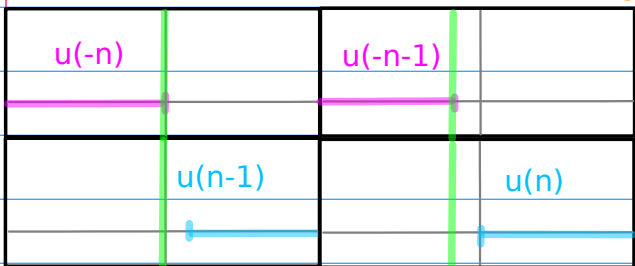
(5)  $*z$  SHR.Rng



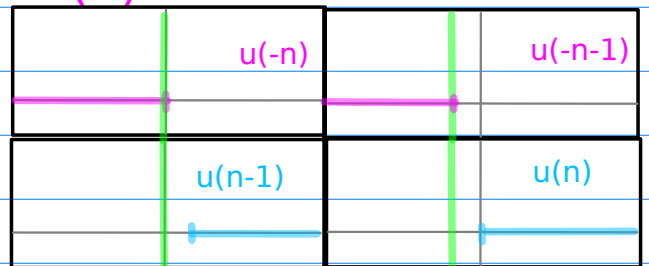
(6)  $*z$  SHR.Rng



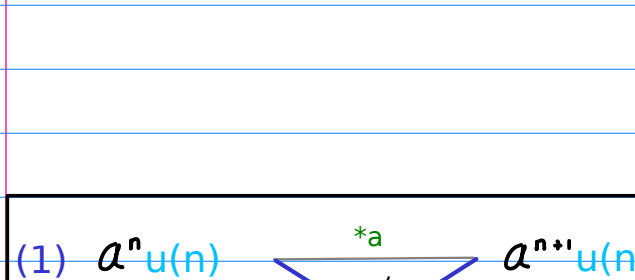
(3)  $/z$  SHL.Rng



(4)  $/z$  SHL.Rng



(7)  $/z$  SHL.Rng



(8)  $/z$  SHL.Rng

