

# Temporal Characteristics of Random Processes

Young W Lim

Nov 15, 2022

Copyright (c) 2021 - 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.



Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

- 1 Random Variables
- 2 Random Processes
- 3 Stochastic Process

# Random Variable Definition

## A random variable

a **function** over a **sample space**  $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome**  $s$  of an **experiment**

# Random Variable Definition

## A random variable

- a **random variable** : a capital letter  $X$
- a particular value : a lowercase letter  $x$
- a **sample space**  $S = \{s_1, s_2, s_3, \dots, s_n\}$
- an **outcome** (an element of  $S$ ) :  $s$

$$s \rightarrow X(s)$$

$$x = X(s)$$

$$s \rightarrow x$$

# Understanding Random Variables (1)

random variables are used to quantify outcomes of a random occurrence, and therefore, can take on many values.

Random variables are required to be measurable and are typically real numbers.

For example, the letter  $X$  may be designated to represent the sum of the resulting numbers after three dice are rolled.

In this case,  $X$  could be 3 ( $1 + 1 + 1$ ), 18 ( $6 + 6 + 6$ ), or somewhere between 3 and 18, since the highest number of a die is 6 and the lowest number is 1.

<https://www.investopedia.com/terms/r/random-variable.asp>

## Understanding Random Variables (2)

A random variable is different from an algebraic variable. The variable in an algebraic equation is an unknown value that can be calculated.

The equation  $10 + x = 13$  shows that we can calculate the specific value for  $x$  which is 3.

On the other hand, a random variable has a set of values, and any of those values could be the resulting outcome as seen in the example of the dice above.

<https://www.investopedia.com/terms/r/random-variable.asp>

## Understanding Random Variables (3)

A random variable is different from an algebraic variable. The variable in an algebraic equation is an unknown value that can be calculated.

The equation  $10 + x = 13$  shows that we can calculate the specific value for  $x$  which is 3.

On the other hand, a random variable has a set of values, and any of those values could be the resulting outcome as seen in the example of the dice above.

<https://www.investopedia.com/terms/r/random-variable.asp>



## Understanding Random Variables (4)

A typical example of a random variable is the outcome of a coin toss. Consider a probability distribution in which the outcomes of a random event are not equally likely to happen. If the random variable  $Y$  is the number of heads we get from tossing two coins, then  $Y$  could be 0, 1, or 2. This means that we could have no heads, one head, or both heads on a two-coin toss.

<https://www.investopedia.com/terms/r/random-variable.asp>

# Formal definition of a random variable

A random variable  $X$  is a measurable function  $X: \Omega \rightarrow E$  from a set of possible outcomes  $\Omega$  to a measurable space  $E$ .

The technical axiomatic definition requires  $\Omega$  to be a sample space of a probability triple  $(\Omega, \mathcal{F}, P)$

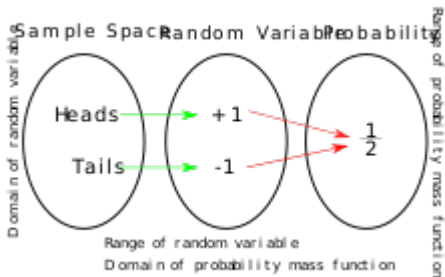
A random variable is often denoted by capital roman letters such as  $X, Y, Z, T$ .

The probability that  $X$  takes on a value in a measurable set  $S \subseteq E$  is written as

$$P(X \in S) = P(\{\omega \in \Omega \mid X(\omega) \in S\})$$

[https://en.wikipedia.org/wiki/Random\\_variable](https://en.wikipedia.org/wiki/Random_variable)

# Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

[https://en.wikipedia.org/wiki/Random\\_variable](https://en.wikipedia.org/wiki/Random_variable)

# Probability Space (1)

In probability theory, a probability space or a probability triple  $(\Omega, \mathcal{F}, P)$  is a mathematical construct that provides a formal model of a random process or "experiment".

For example, one can define a probability space which models the throwing of a die

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

## Probability Space (2)

A probability space consists of three elements

A sample space,  $\Omega$  , which is the set of all possible outcomes.

An event space, which is a set of events  $\mathcal{F}$  ,

an event being a set of outcomes in the sample space.

A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

## Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be  $\{1, 2, 3, 4, 5, 6\}$ . For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as  $\{5\}$  ("the die lands on 5"), as well as complex events such as  $\{2, 4, 6\}$  ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example,  $\{5\}$  would be mapped to  $1/6$ , and  $\{2, 4, 6\}$  would be mapped to  $3/6 = 1/2$ .

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

## Random Process (1)

## A random process

a function of both **outcome**  $s$  and **time**  $t$

$$X(t, s)$$

assigning a **time function** to every **outcome**  $s_i$

$$s_i \rightarrow x(t, s_i)$$

## Random Process (2)

## A random process

the family of such **time functions** is called a **random process**

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$



## Random Process (3)

We have seen that a random variable  $X$  is a rule which assigns a number to every outcome  $e$  of an experiment.

The random variable is a function  $X(e)$  that maps the set of experiment outcomes to the set of numbers.

A random process is a rule that maps every outcome  $e$  of an experiment to a function  $X(t, e)$ .

A random process is usually conceived of as a function of time, but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

The function  $X(u, v, e)$  would be a function whose value depended on the location  $(u, v)$  and the outcome  $e$ .

# Ensemble of time functions

## Time functions

A random process  $X(t, s)$  represents a family or ensemble of **time functions**

$X(t, s)$  represents

- a **single time function**  $x(t, s)$
- when  $t$  is a variable and  $s$  is fixed at an outcome

$x(t, s)$  represents

- a **sample function**,
- an ensemble member,
- a realization of the process

## Short-form notation for time functions

The short-form notation  $x(t)$ 

to represent a specific waveform of a random process  $X(t)$   
for a given outcome  $s_j$

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

## Random Process Example

## Example

$$X(t, s_1) = x_1(t)$$

$$s_1 \rightarrow x_1(t)$$

$$X(t, s_2) = x_2(t)$$

$$s_2 \rightarrow x_2(t)$$

...

...

$$X(t, s_n) = x_n(t)$$

$$s_n \rightarrow x_n(t)$$

$S = \{s_1, s_2, s_3, \dots, s_n\}$  a sample space

$X(t) = \{x_1(t), x_2(t), x_3(t), \dots, x_n(t)\}$  a random process

## Random variables with time

a **random process**  $X(t, s)$  represents a **single time function** when  $t$  is a variable and  $s$  is fixed at an outcome

a random process  $X(t, s)$  represents a **single random variable** when both  $t$  and  $s$  are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \quad \text{random variable}$$

$$X(t, s) = X(t) \quad \text{random process}$$

# An alphabet

the **alphabet** of  $X(t)$

the set of its possible values

- the values of **time**  $t$  for which a **random process** is defined
- the **alphabet** of the random variable  $X = X(t)$  at time  $t$

# Classification of Random Processes

## (1) Types of time and alphabet

- the values of **time**  $t$  for which a **random process** is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable  $X = X(t)$  at time  $t$ 
  - continuous alphabet
  - discrete alphabet

# Classification of Random Processes

(2) types of the random variable  $X(t)$  and the time  $t$

- a continuous **alphabet** continuous **time** random process
  - $X(t)$  has continuous values and  $t$  has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$  has discrete values and  $t$  has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$  has continuous values and  $t$  has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$  has discrete values and  $t$  has discrete values



# Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic** if **future values** of any sample function cannot be predicted exactly from **observed past values**
- A process is **deterministic** if **future values** of any sample function can be predicted from **observed past values**

## Deterministic Random Process Example (1)

$$X(t) = A \cos(\omega_0 t + \Theta)$$

$A$ ,  $\Theta$ , or  $\omega_0$  (or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

## Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i}t + \Theta_i)$$

the knowledge of the sample function  
prior to any time instance fully allows  
the prediction of the sample function's future values  
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \quad \implies \quad x_i(t) \quad t > 0$$

# Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome $\theta_k$
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$ , of the outcome $\theta_k$

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Functions and variables of a random process  $X(t, \theta)$  (2)

- $X(t, \theta)$  is a **family of functions**. Imagine a giant strip chart recording in which each pen is identified with a different  $\theta$ . This family of functions is traditionally called an **ensemble**.
- A **single function**  $X(t, \theta_k)$  is selected by the **outcome**  $\theta_k$ . This is just a **time function** that we could call  $X_k(t)$ . Different **outcomes** give us different **time functions**.
- If  $t$  is fixed, say  $t = t_1$ , then  $X(t_1, \theta)$  is a **random variable**. Its value depends on the **outcome**  $\theta$ .
- If both  $t_1$  and  $\theta_k$  are given then  $X(t_1, \theta_k)$  is just a **number**.

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

# Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>  
<https://en.wiktionary.org/wiki/stochastic>

## Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is  $n$ -dimensional **Euclidean space**  $\mathbb{R}^n$  or a manifold

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)



## Stochastic Process (4)

A **stochastic process** can be denoted, by  $\{X(t)\}_{t \in T}$ ,  $\{X_t\}_{t \in T}$ ,  $\{X(t)\}$ ,  $\{X_t\}$  or simply as  $X$  or  $X(t)$ , although  $X(t)$  is regarded as an abuse of function notation.

For example,  $X(t)$  or  $X_t$  are used to refer to the **random variable** with the **index**  $t$ , and not the entire **stochastic process**.

If the **index set** is  $T = [0, \infty)$ , then one can write, for example,  $(X_t, t \geq 0)$  to denote the **stochastic process**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space**  $(\Omega, \mathcal{F}, P)$ ,

- $\Omega$  is a **sample space**,
- $\mathcal{F}$  is a  $\sigma$ -**algebra**,
- $P$  is a **probability measure**;
- the **random variables**, indexed by some set  $T$ ,
- all take values in the same **mathematical space**  $S$ , which must be **measurable** with respect to some  $\sigma$ -algebra  $\Sigma$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process Definition (2)

In other words, for a given **probability space**  $(\Omega, \mathcal{F}, P)$  and a **measurable space**  $(S, \Sigma)$ , a **stochastic process** is a **collection** of  $S$ -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point  $t \in T$  had the meaning of time, so  $X(t)$  is a **random variable** representing a value observed at time  $t$ .

A **stochastic process** can also be written as  $\{X(t, \omega) : t \in T\}$  to reflect that it is actually a function of two variables,  $t \in T$  and  $\omega \in \Omega$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a  $S^T$ -valued random variable, where  $S^T$  is the space of all the possible functions from the set  $T$  into the space  $S$ .

However this alternative definition as a "function-valued random variable" in general requires additional regularity assumptions to be well-defined.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Index set

The set  $T$  is called the index set or parameter set of the stochastic process.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set  $T$  the interpretation of time.

In addition to these sets, the index set  $T$  can be another set with a total order or a more general set, such as the Cartesian plane  $R^2$  or  $n$ -dimensional Euclidean space, where an element  $t \in T$  can represent a point in space.

That said, many results and theorems are only possible for stochastic processes with a totally ordered index set.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# State space

The mathematical space  $S$  of a stochastic process is called its state space. This mathematical space can be defined using integers, real lines,  $n$ -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces. The state space is defined using elements that reflect the different values that the stochastic process can take.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Sample function (1)

A sample function is a single outcome of a stochastic process, so it is formed by taking a single possible value of each random variable of the stochastic process. More precisely, if  $\{X(t, \omega) : t \in T\}$  is a stochastic process, then for any point  $\omega \in \Omega$ , the mapping  $X(\cdot, \omega) : T \rightarrow S$ , is called a sample function, a realization, or, particularly when  $T$  is interpreted as time, a sample path of the stochastic process  $\{X(t, \omega) : t \in T\}$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)



## Sample function (2)

This means that for a fixed  $\omega \in \Omega$ , there exists a sample function that maps the index set  $T$  to the state space  $S$ .]

Other names for a sample function of a stochastic process include trajectory, path function or path

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

