Fourier Analysis Overview (0B)

• CTFS:	Continuous	Time	Fourier	Series
• CTFT:	Continuous	Time	Fourier	Transform
• DTFS:	Discrete	Time	Fourier	Series
• DTFT:	Discrete	Time	Fourier	Transform
• DFT:	Discrete		Fourier	Transform

Copyright (c) 2009 - 2016 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Fourier Analysis Methods – Frequency View

	Discrete Frequency	Continuous Frequency
Continuous Time	**CTFS** C_k $e^{-jk\omega_0 t}$ $k\omega_0$ Aperiodic in freq	**CTFT** $X(j\omega)$ $e^{-j\omega t}$ ω Aperiodic in freq
Discrete Time	**DTFS / DFT** γ_k $x[n]$ $e^{-jk\hat{\omega}_0 n}$ $k\hat{\omega}_0$ **Periodic** in freq	**DTFT** $X(j\hat{\omega})$ $x[n]$ $e^{-j\hat{\omega} n}$ $\hat{\omega}$ **Periodic** in freq
	Normalized Discrete Frequency	**Normalized Continuous Frequency**

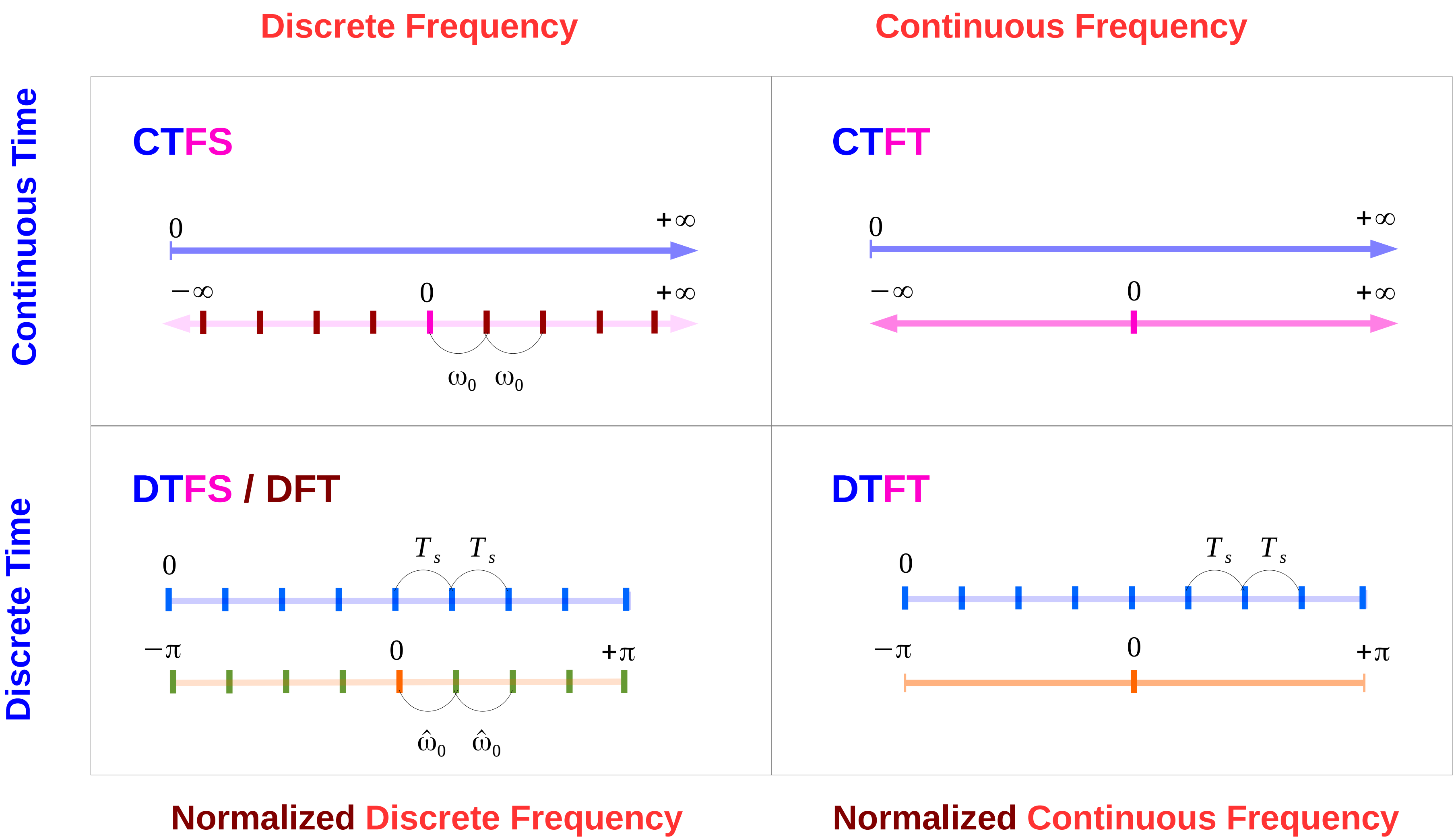
Fourier Analysis Methods – Time View

	Discrete Frequency	Continuous Frequency
Continuous Time	**CTFS** $x(t)$ **Periodic** in time → $k\,\omega_0$	**CTFT** $x(t)$ Aperiodic in time
Discrete Time	**DTFS / DFT** $x[n]$ **Periodic** in time → $k\,\hat{\omega}_0$	**DTFT** $x[n]$ Aperiodic in time
	Normalized Discrete Frequency	**Normalized Continuous Frequency**

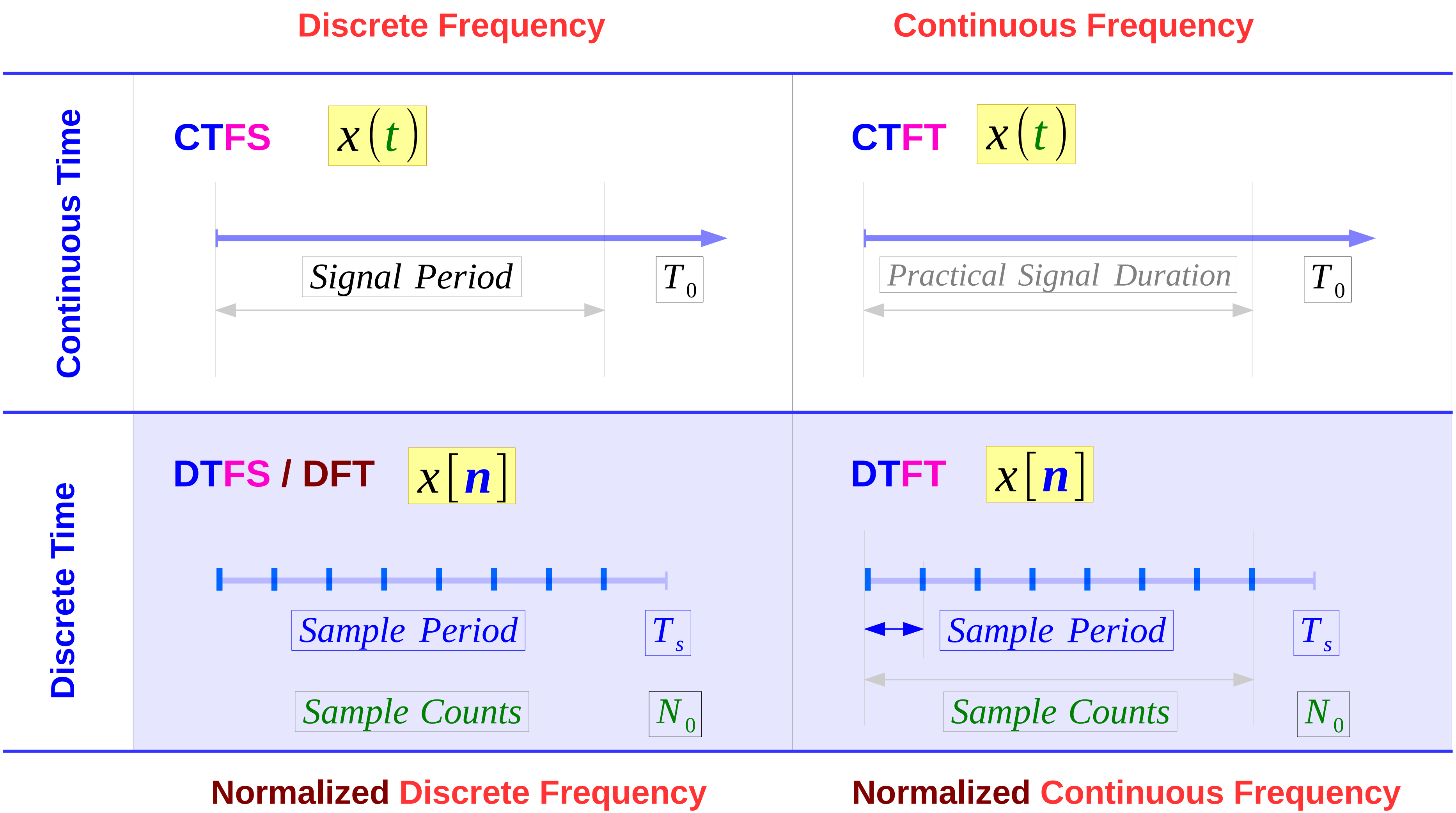
Fourier Analysis Methods

	Discrete Frequency	Continuous Frequency
Continuous Time	**CTFS** C_k $x(t)$ **Periodic** in time $e^{-jk\omega_0 t}$ $k\omega_0$ Aperiodic in freq	**CTFT** $X(j\omega)$ $x(t)$ Aperiodic in time $e^{-j\omega t}$ ω Aperiodic in freq
Discrete Time	**DTFS / DFT** γ_k $x[n]$ **Periodic** in time $e^{-jk\hat{\omega}_0 n}$ $k\hat{\omega}_0$ **Periodic** in freq	**DTFT** $X(j\hat{\omega})$ $x[n]$ Aperiodic in time $e^{-j\hat{\omega} n}$ $\hat{\omega}$ **Periodic** in freq
	Normalized Discrete Frequency	**Normalized Continuous Frequency**

Time and Frequency Domain Resolutions



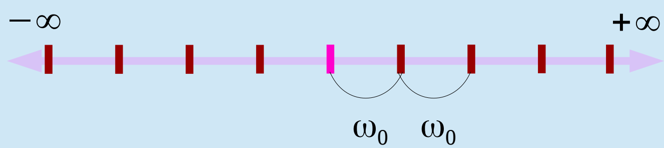
Time Domain Resolutions



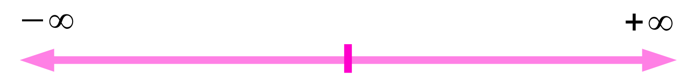
Frequency Domain Resoltuions

	Discrete Frequency	Continuous Frequency
Continuous Time	CTFS C_k	CTFT $X(j\omega)$
Discrete Time	DTFS / DFT $\gamma_k \,/\, X[k]$	DTFT $X(j\hat{\omega})$
	Normalized Discrete Frequency	Normalized Continuous Frequency

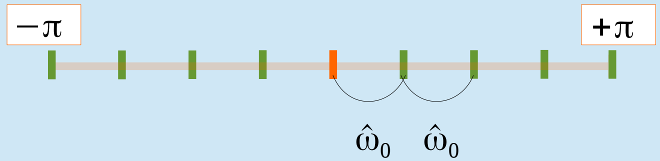
CTFS C_k — *Frequency Resolution*



CTFT $X(j\omega)$

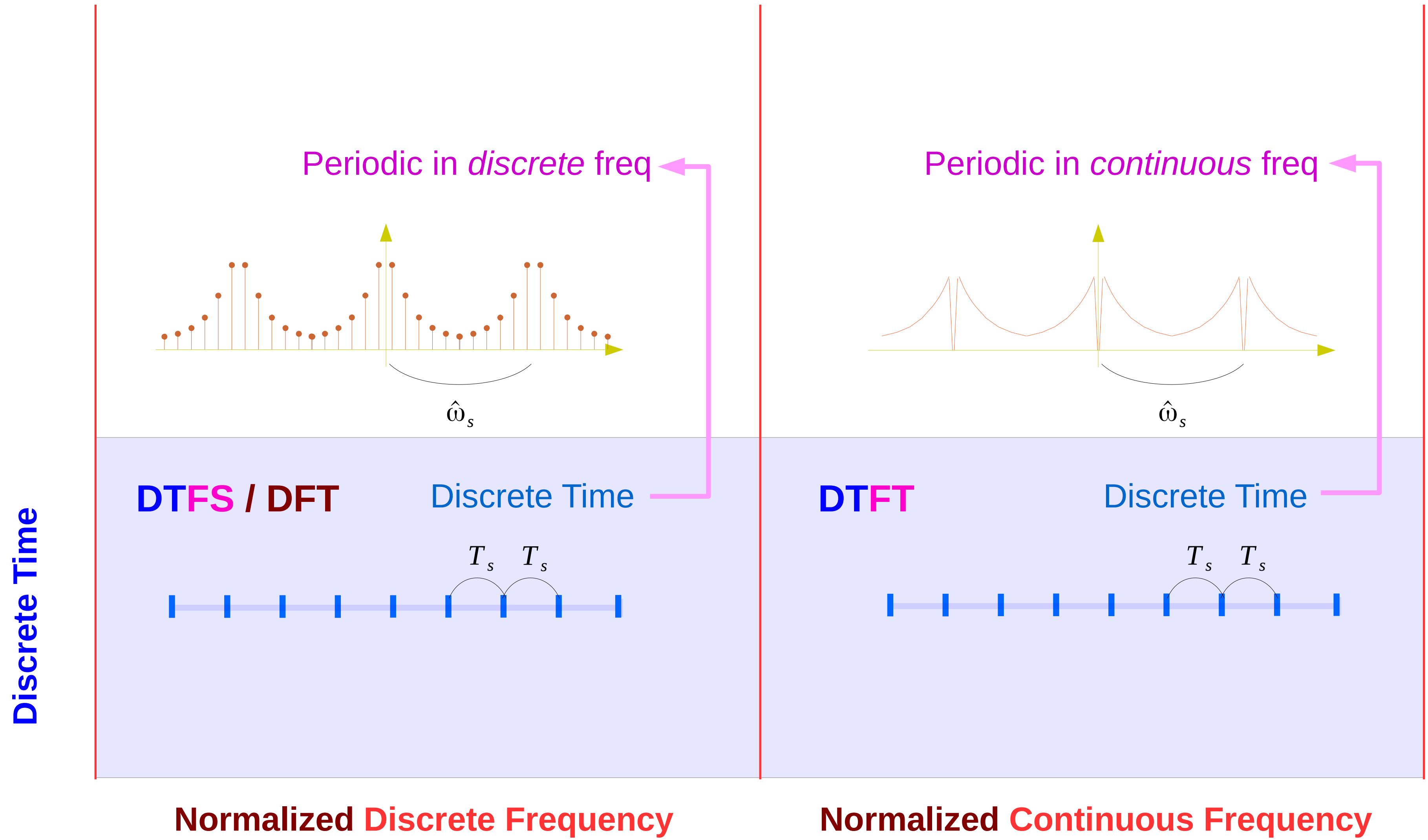


DTFS / DFT $\gamma_k \,/\, X[k]$ — *Frequency Resolution*



DTFT $X(j\hat{\omega})$

Discrete Time and Periodic Frequency

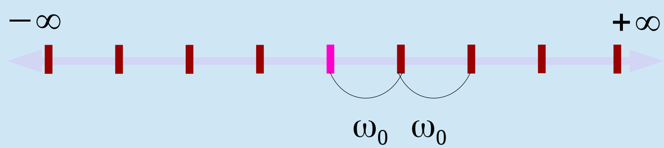


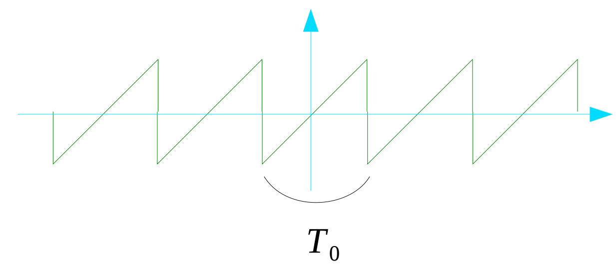
Periodic Time and Discrete Frequency

Discrete Frequency

Continuous Time

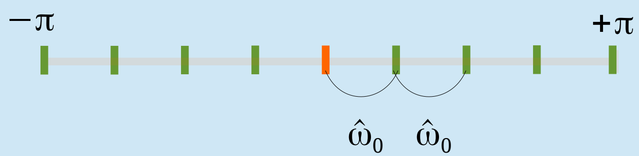
CTFS — Discrete Frequency → Periodic in *continuous* time

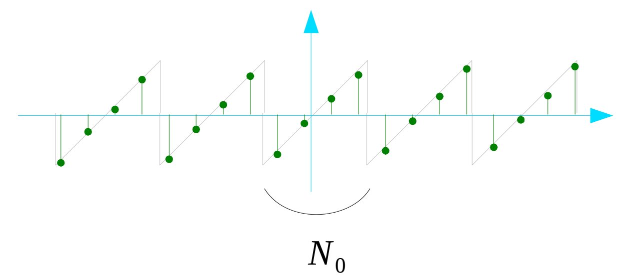




Discrete Time

DTFS / DFT — Discrete Frequency → Periodic in *discrete* time





Normalized Discrete Frequency

Discrete Time Resolution T_s

ω_s: *replication frequency*
$\hat{\omega}_s$: *normalized replication frequency*

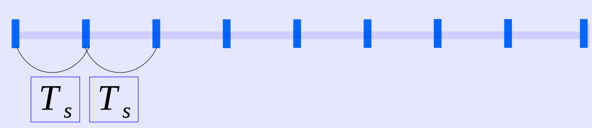
$$\omega_s = \frac{2\pi}{T_s} \quad \Longleftrightarrow \quad \hat{\omega}_s = \frac{2\pi}{1}$$

ω_s: *replication frequency*
$\hat{\omega}_s$: *normalized replication frequency*

$$\omega_s = \frac{2\pi}{T_s} \quad \Longleftrightarrow \quad \hat{\omega}_s = \frac{2\pi}{1}$$

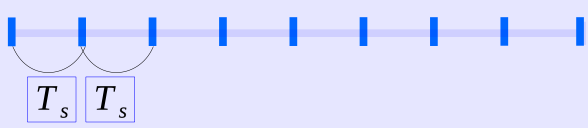
Discrete Time

DTFS / DFT



Discrete Time
Periodic in *discrete* freq

DTFT



Discrete Time
Periodic in *continuous* freq

Normalized Discrete Frequency | **Normalized Continuous Frequency**

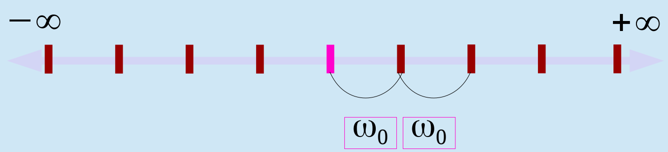
Discrete Frequency Resolutions $\omega_0 , \ \hat{\omega}_0$

Discrete Frequency

Continuous Time

CTFS

Periodic in *continuous* time
Discrete Frequency



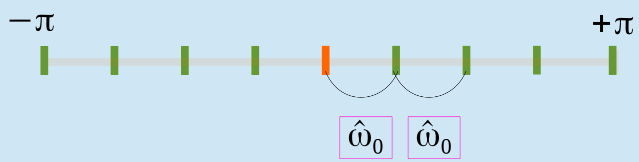
$$\omega_0 = \frac{2\pi}{T_0}$$

$period: T_0 \ seconds$

Discrete Time

DTFS / DFT

Periodic in *discrete* time
Discrete Frequency



$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$period: N_0 \ samples$

Normalized Discrete Frequency

Normalized Frequency

$$\hat{\omega}_s = \frac{2\pi}{1} = \left(\frac{2\pi}{T_s}\right) T_s$$

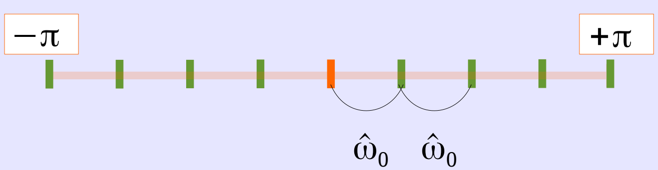
$$\hat{\omega}_0 = \frac{2\pi}{N_0} = \left(\frac{2\pi}{T_0}\right) T_s$$

$$\hat{\omega}_s = \frac{2\pi}{1} = \left(\frac{2\pi}{T_s}\right) T_s$$

$\hat{\omega}$ *continuous variable*

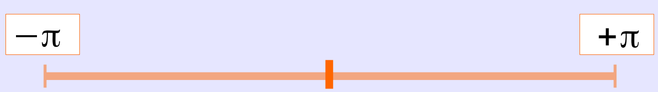
Discrete Time

DTFS / DFT $\gamma_k \; / \; X[k]$



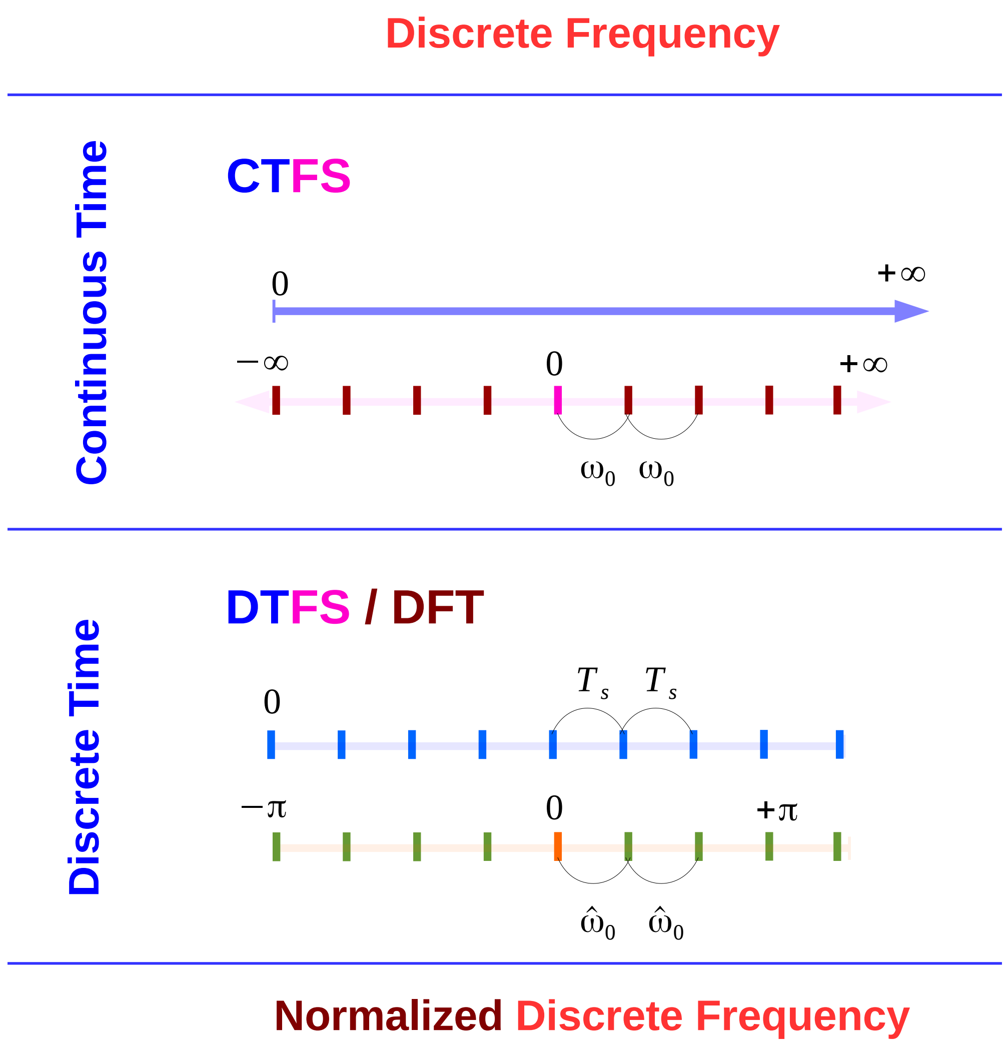
Normalized Discrete Frequency

DTFT $X(j\hat{\omega})$



Normalized Continuous Frequency

Normalized by $1/T_s$



frequency resolution

$$\omega_0 = \frac{2\pi}{T_0}$$

$$T_0 = N_0 T_s$$

$$\omega_0 = \frac{2\pi}{N_0 T_s}$$

$\cdot T_s$ $\qquad$ $\cdot f_s$

$$\frac{\omega_0}{1/T_s} = \frac{2\pi}{N_0 \cancel{T_s}} \cancel{T_s}$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

normalized frequency resolution

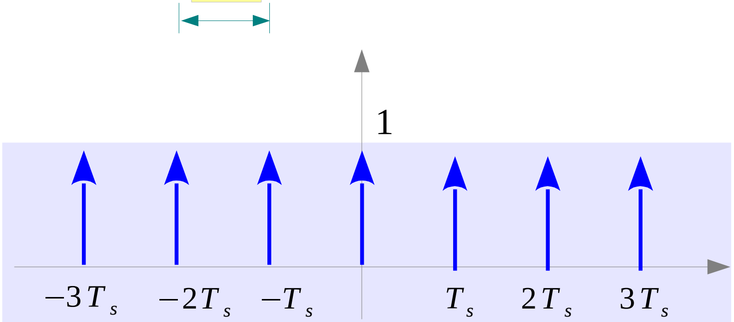
CTFT pair of an impulse train

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

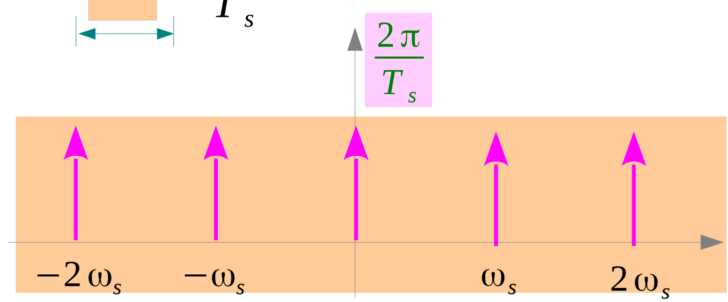
$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

$$\frac{2\pi}{T_s}$$

T_s



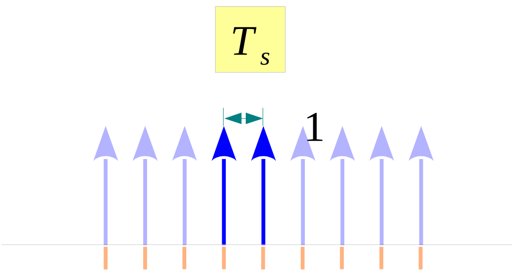
$$\omega_s = \frac{2\pi}{T_s}$$



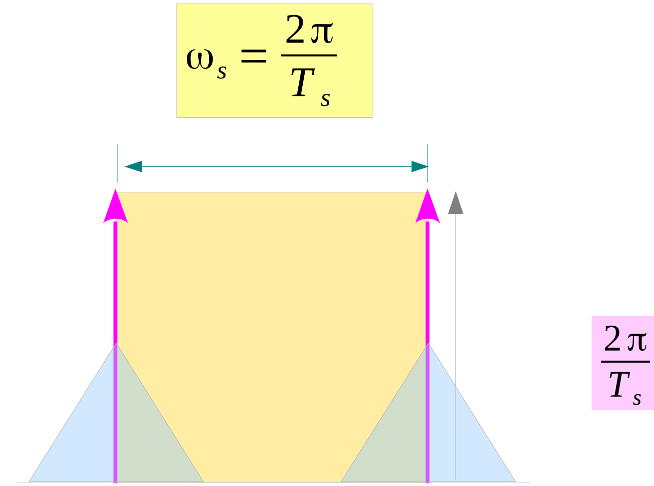
Sampling

Replication

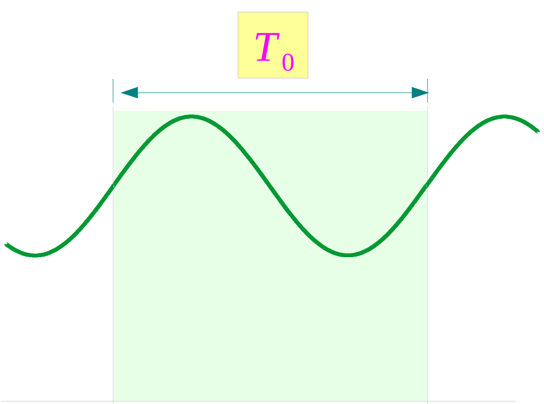
Sampling and Replicating



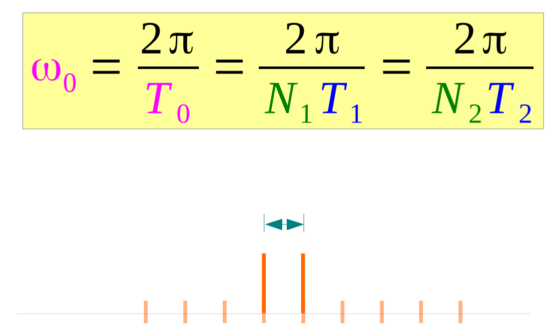
Sampling



Replication



Period

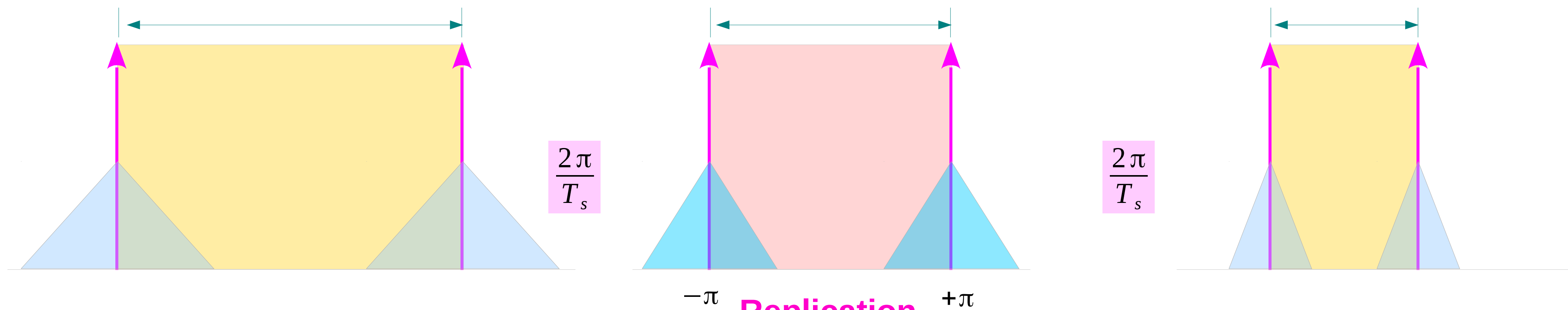


Resolution

Normalization

Relatively scaled figures

$\omega_s = \frac{2\pi}{T_1}$ $\qquad$ $\hat{\omega}_s = \frac{2\pi}{1}$ $\qquad$ $\omega_s = \frac{2\pi}{T_2}$



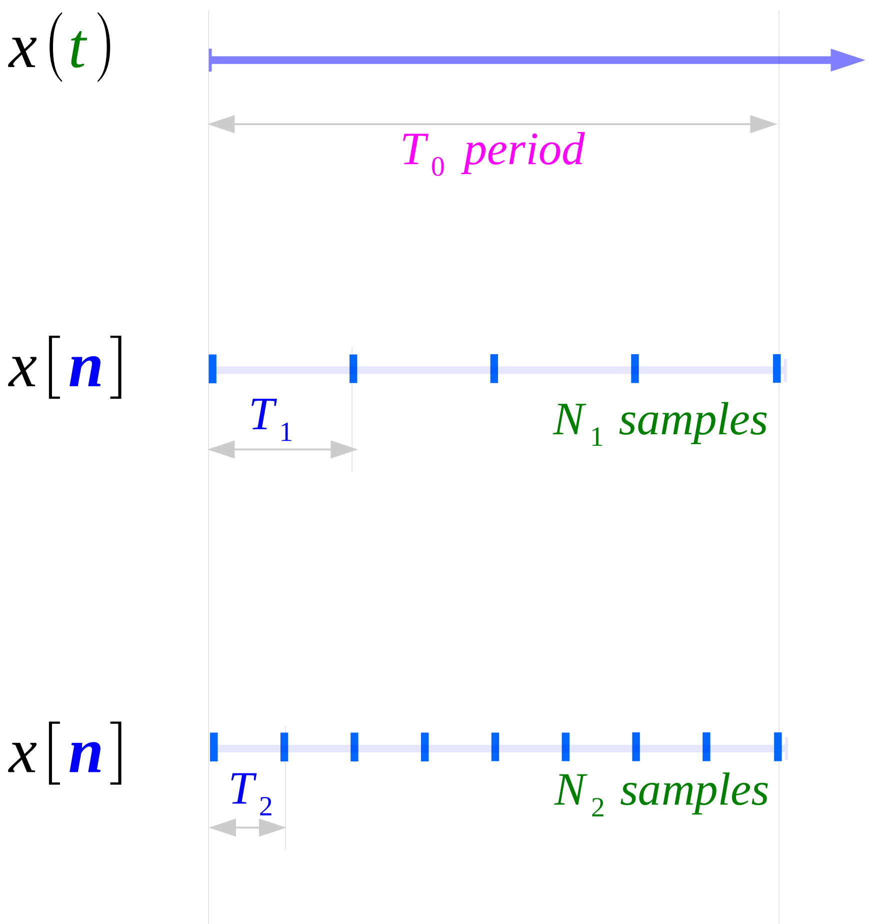
Replication

$T_s = small\ T_1$ → fine $\qquad$ coarse ← $T_s = large\ T_2$

$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N_1 T_1}$ ⇒ $\hat{\omega}_0 = \frac{2\pi}{N_1}$ $\qquad$ $\hat{\omega}_0 = \frac{2\pi}{N_2}$ ⇐ $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N_2 T_2}$

Resolution

Sampling Period and the Number of Samples



$\textit{fundamental period } T_0 = T_1 N_1 = T_2 N_2$

$\textit{sampling period } T_s : \quad T_1 \qquad (T_1 > T_2)$

$\textit{number of samples} : \quad N_1 \qquad (N_1 < N_2)$

$\textit{sampling period } T_s : \quad T_2 \qquad (T_1 > T_2)$

$\textit{number of samples} : \quad N_2 \qquad (N_1 < N_2)$

Frequency Replication and Resolution (1)

fundamental period T_0 ⟷ *frequency resolution* ω_0

$$T_0 = T_1 N_1 = T_2 N_2$$

$$\omega_0 = \frac{2\pi}{T_0} = \omega_{01} = \frac{2\pi}{N_1 T_1} = \omega_{02} = \frac{2\pi}{N_2 T_2}$$

$$N_1 < N_2$$

$$\hat{\omega}_{01} = \frac{2\pi}{N_1} \quad > \quad \hat{\omega}_{02} = \frac{2\pi}{N_2}$$

$$\hat{\omega}_{01} = \omega_{01} T_1 \quad > \quad \hat{\omega}_{02} = \omega_{02} T_2$$

sampling period T_s: ⟷ *replication period* ω_1, ω_2:

$$T_1 > T_2$$

$$\omega_1 = \frac{2\pi}{T_1} \quad < \quad \omega_2 = \frac{2\pi}{T_2}$$

$$\hat{\omega}_1 = 2\pi \quad = \quad \hat{\omega}_2 = 2\pi$$

$$\hat{\omega}_1 = \omega_1 T_1 \quad = \quad \hat{\omega}_2 = \omega_2 T_2$$

coarse fine

Frequency Replication and Resolution (2)

$T_1 > T_2$

$N_1 < N_2$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\omega = \frac{\hat{\omega}}{T_s}$$

$$\omega T_s = \hat{\omega}$$

replication frequency

$$\omega_1 = \frac{2\pi}{T_1} \qquad \hat{\omega}_1 = 2\pi$$

$$\wedge \qquad \qquad ||$$

$$\omega_2 = \frac{2\pi}{T_2} \qquad \hat{\omega}_2 = 2\pi$$

large bandwidth

frequency resolutions

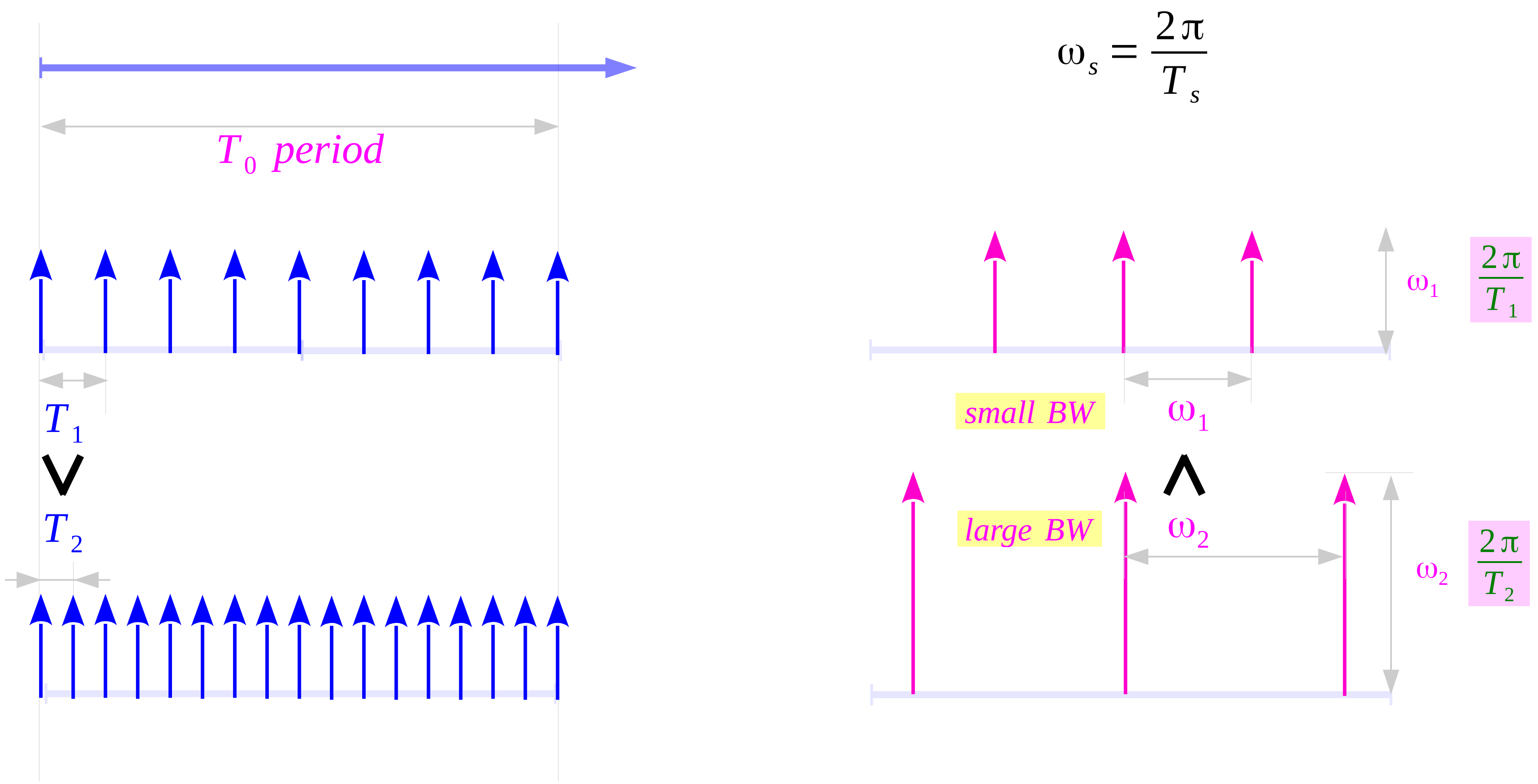
$$\omega_{01} = \frac{2\pi}{N_1 T_1} \qquad \hat{\omega}_{01} = \frac{2\pi}{N_1} \qquad \hat{\omega}_{01} = \omega_{01} T_1$$

$$|| \qquad \qquad \vee$$

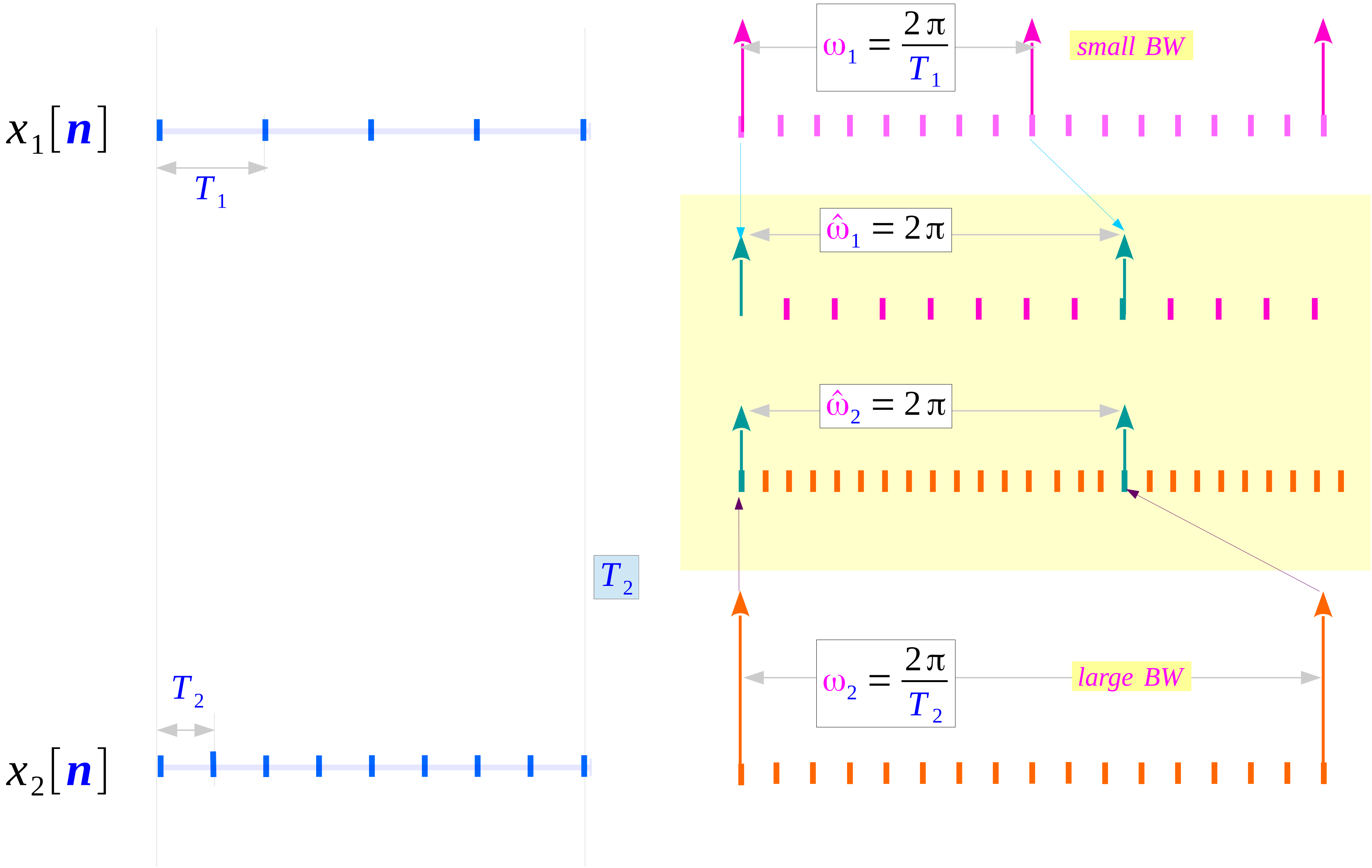
$$\omega_{02} = \frac{2\pi}{N_2 T_2} \qquad \hat{\omega}_{02} = \frac{2\pi}{N_2} \qquad \hat{\omega}_{02} = \omega_{02} T_2$$

fine resolution

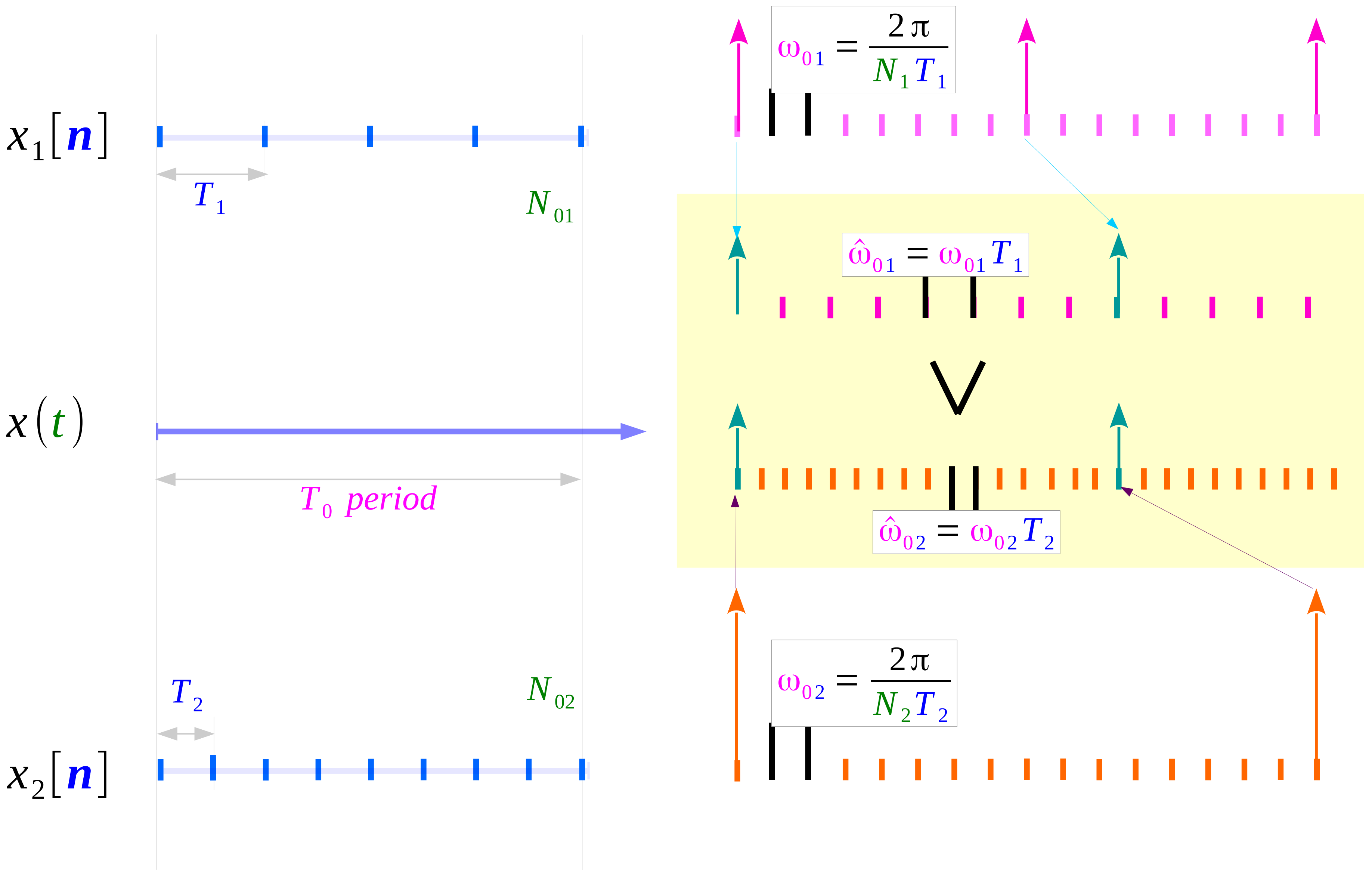
Sampling Period and Replication Period



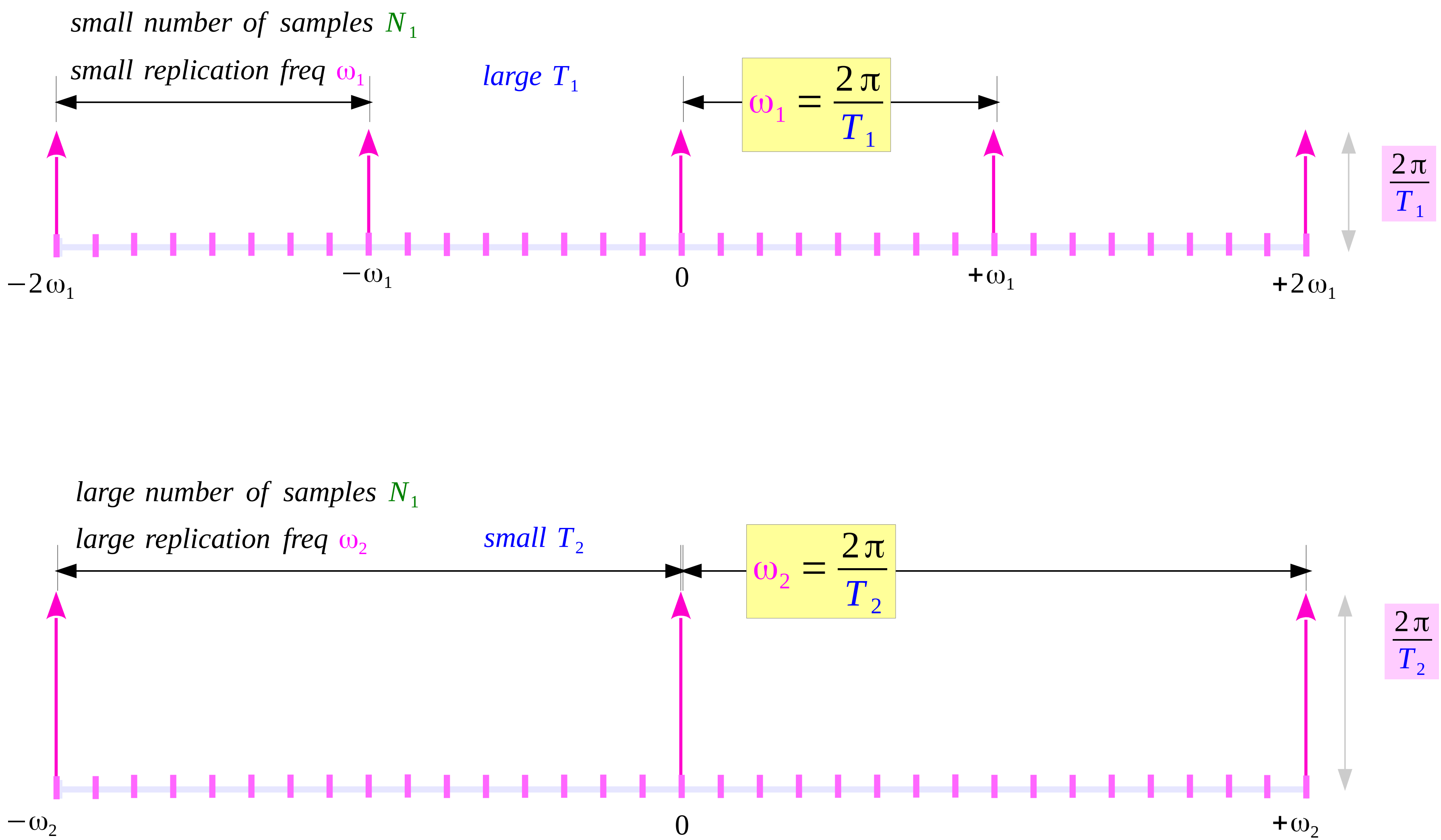
T_1 & T_2 periods, ω_1 & ω_2 replication frequencies



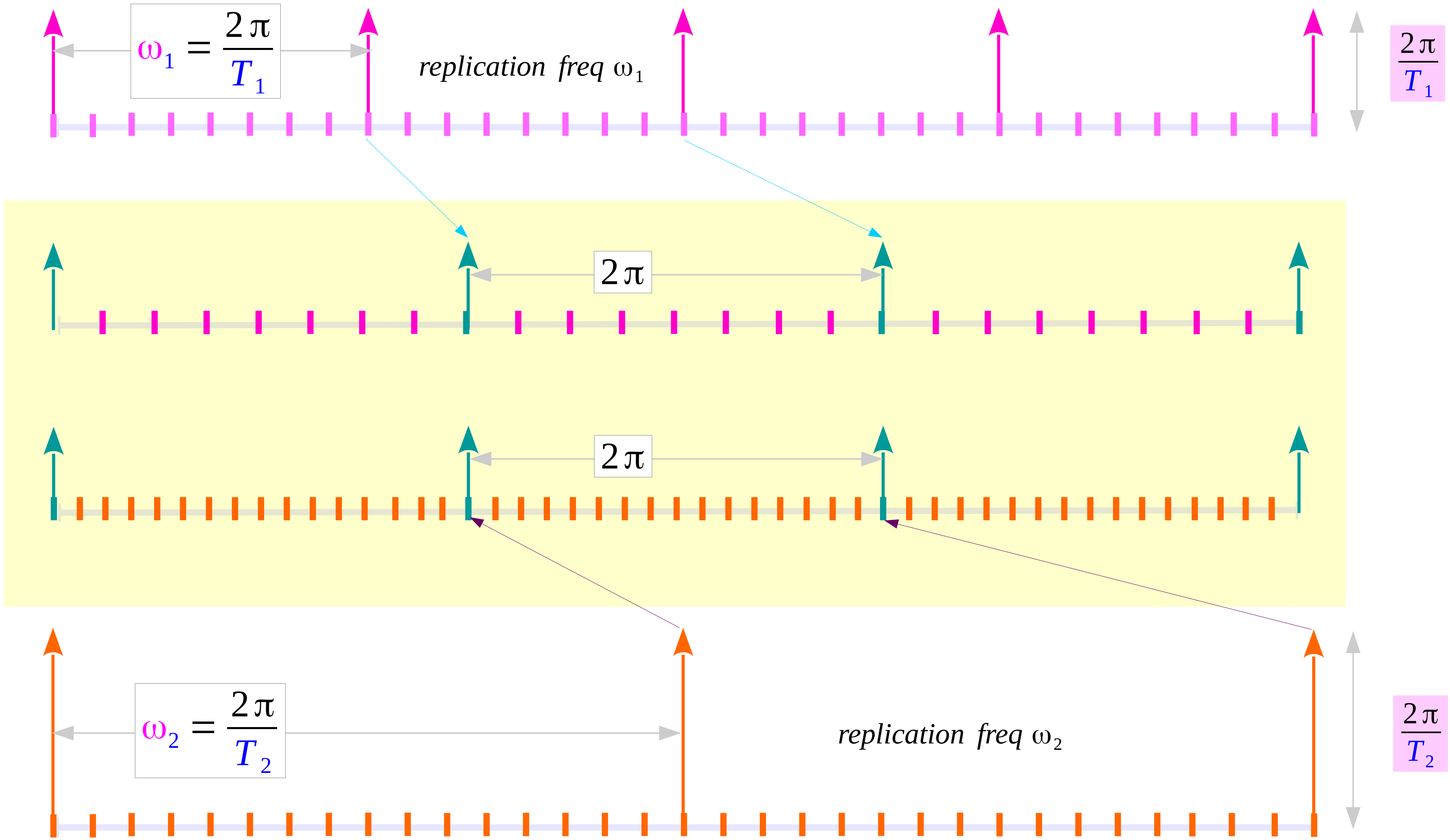
T_1 & T_2 periods, ω_{01} & ω_{02} frequency resolutions



Replication Frequency



Normalized Replication Frequencies

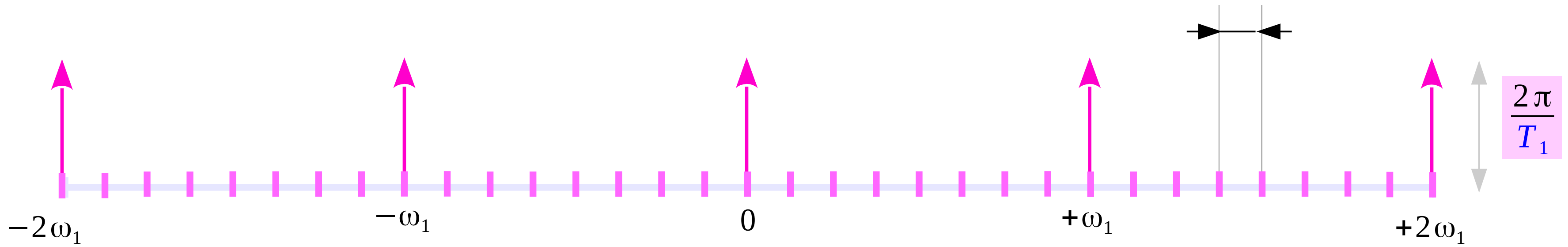


Frequency Resolution

the same *signal period* T_0

the same *frequency resolution* ω_0

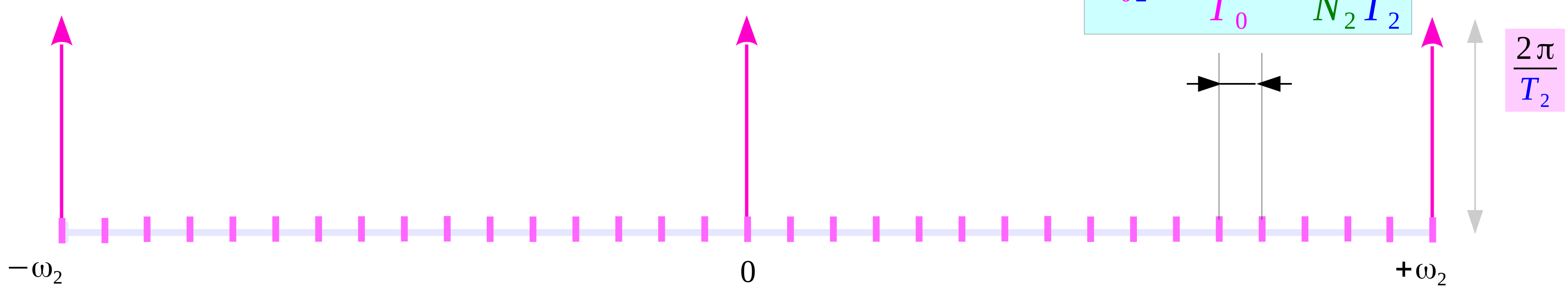
$$\omega_{01} = \frac{2\pi}{T_0} = \frac{2\pi}{N_1 T_1}$$



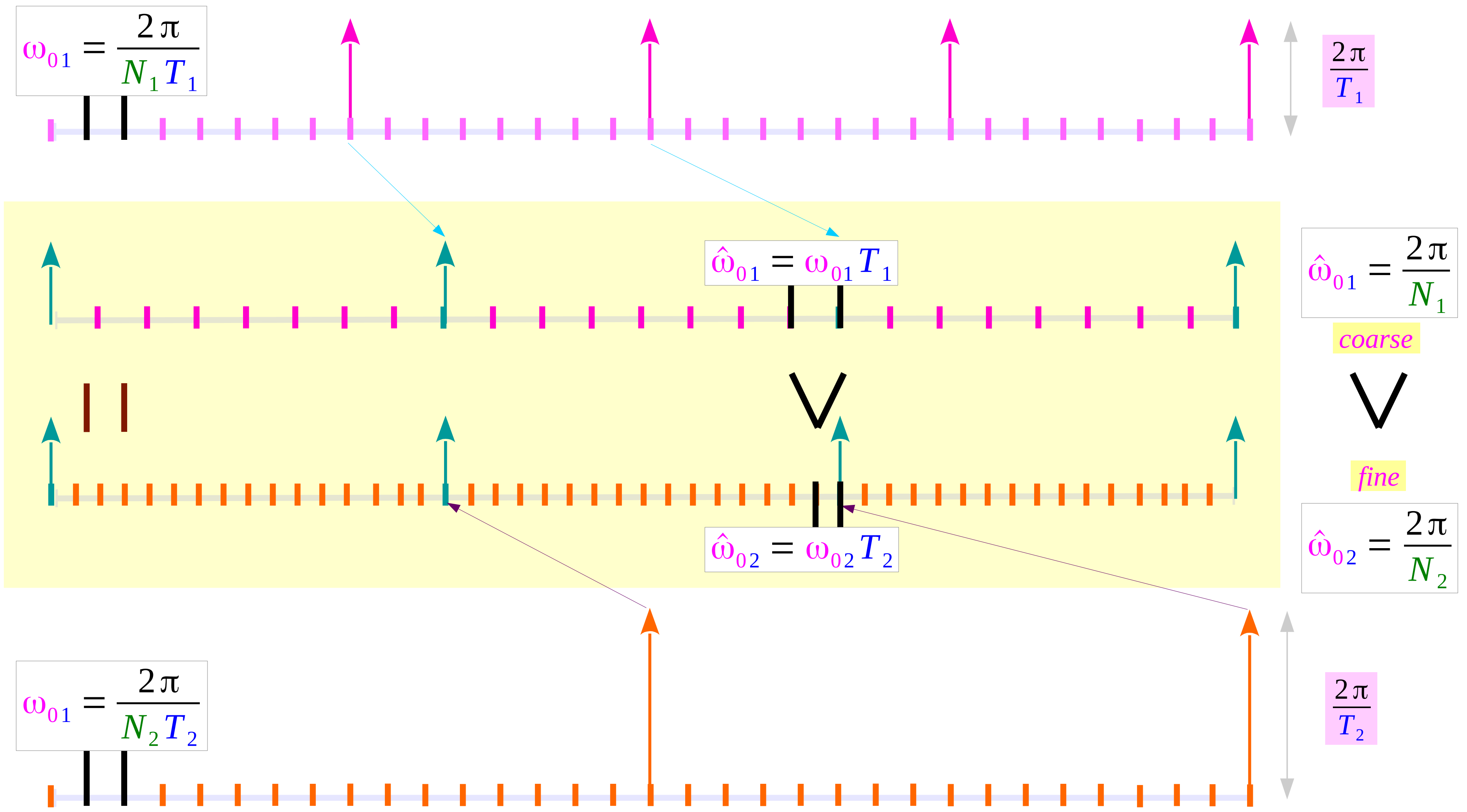
the same *signal period* T_0

the same *frequency resolution* ω_0

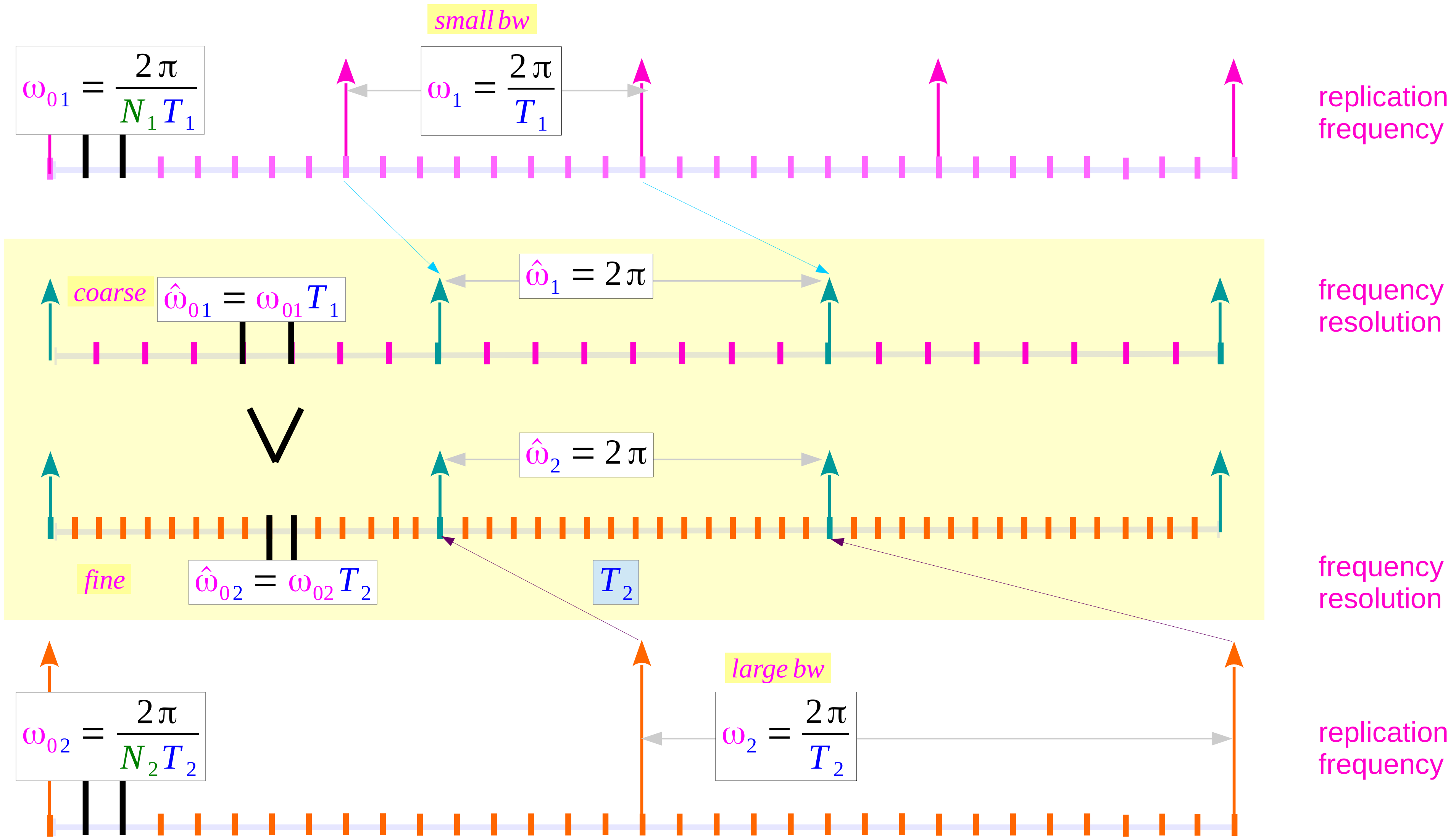
$$\omega_{02} = \frac{2\pi}{T_0} = \frac{2\pi}{N_2 T_2}$$



Normalized Frequency Resolutions

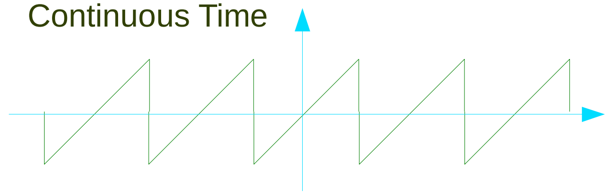


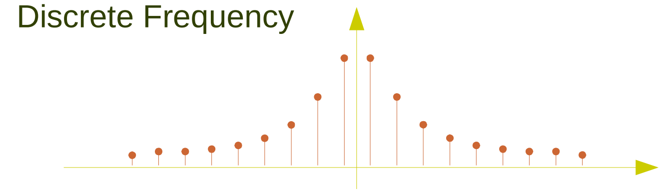
Normalized ω_0 & ω_s



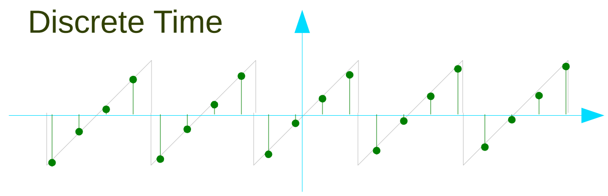
Types of Fourier Transforms

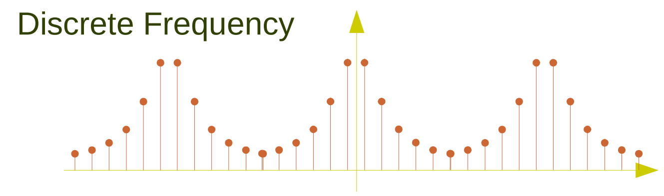
Continuous Time
Fourier Series
CTFS



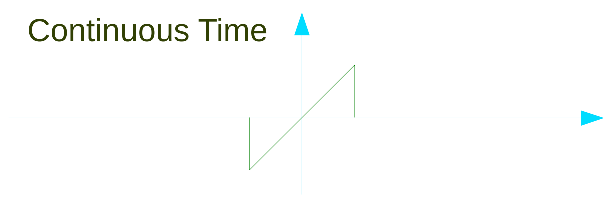


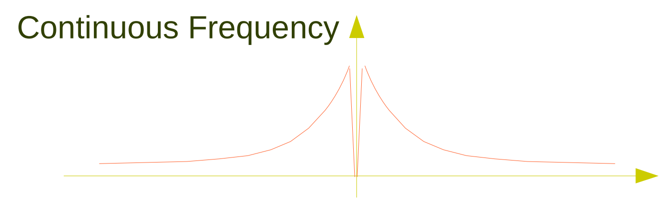
Discrete Time
Fourier Series
DTFS / DFT



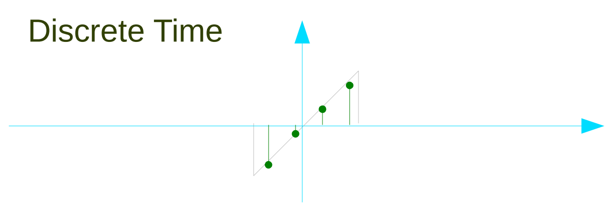


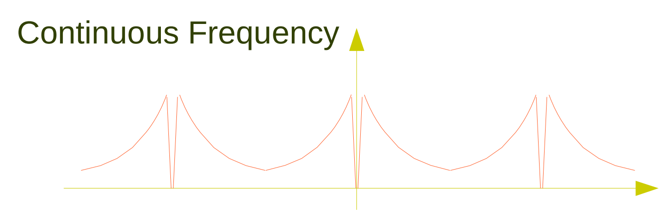
Continuous Time
Fourier Transform
CTFT





Discrete Time
Fourier Transform
DTFT

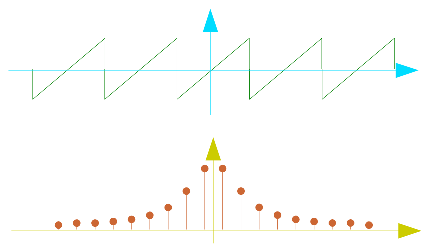




1. CTFS

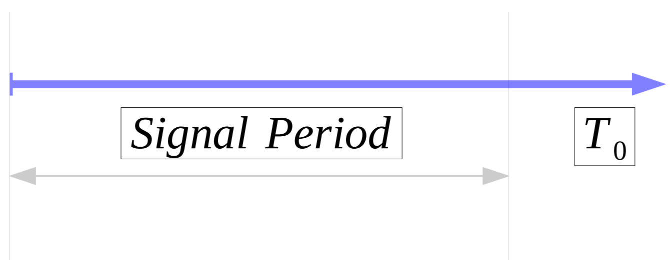
CT **Continuous Time**

FS **Discrete Frequency**

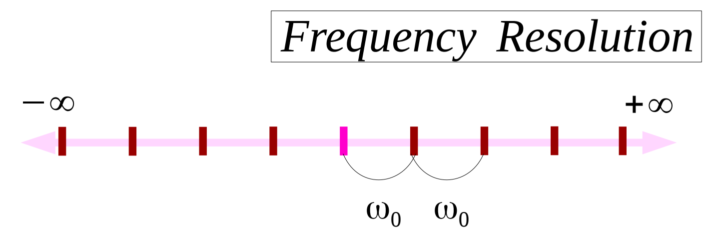


$$\omega_0 = \frac{2\pi}{T_0}$$

$x(t)$



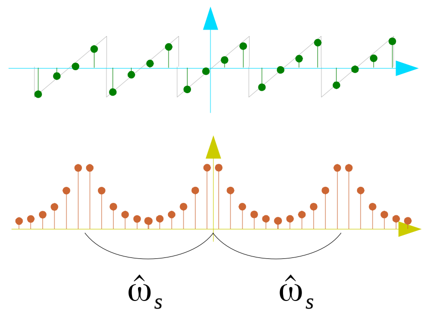
C_k



2. DTFS / DFT

DT **Discrete Time**

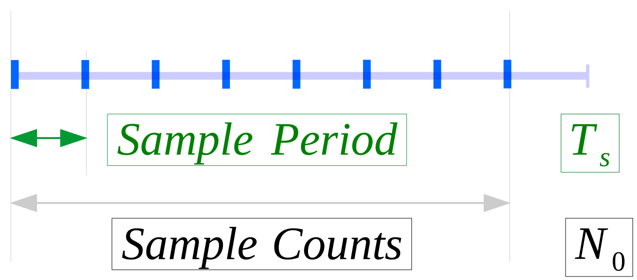
FS **Discrete Frequency (Normalized)**



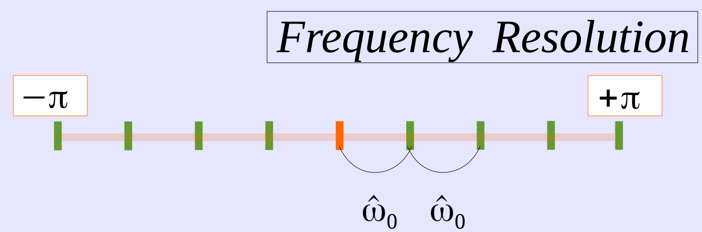
$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$$\hat{\omega}_s = \frac{2\pi}{1} \quad \Longleftrightarrow \quad \omega_s = \frac{2\pi}{T_s}$$

$x[n]$

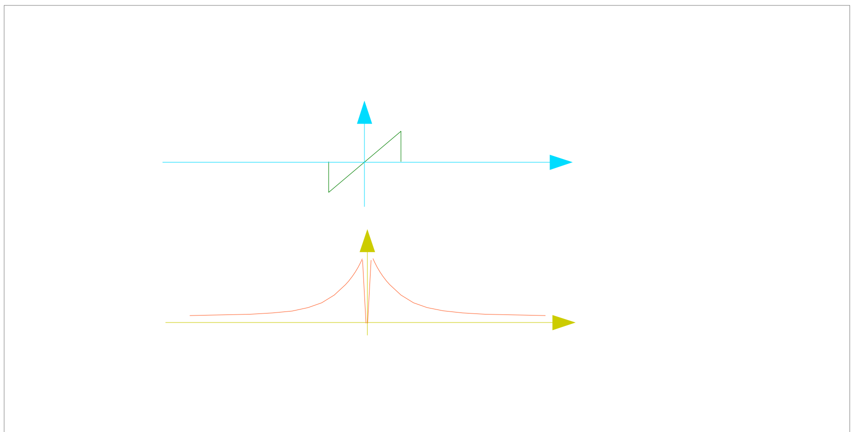


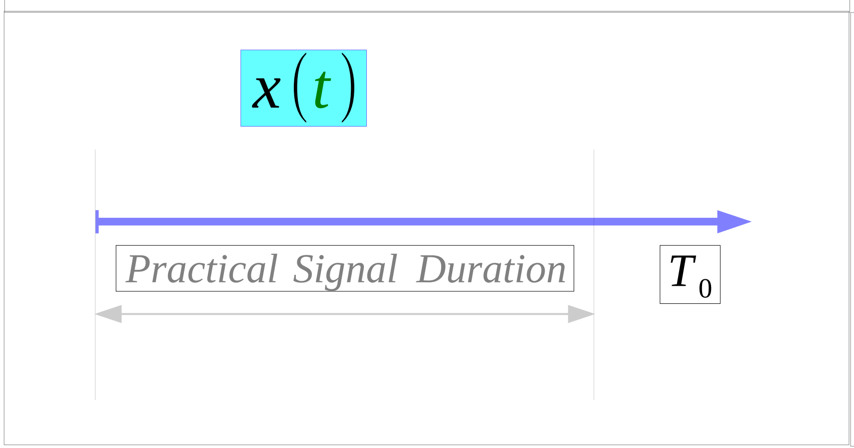
γ_k / $X[k]$

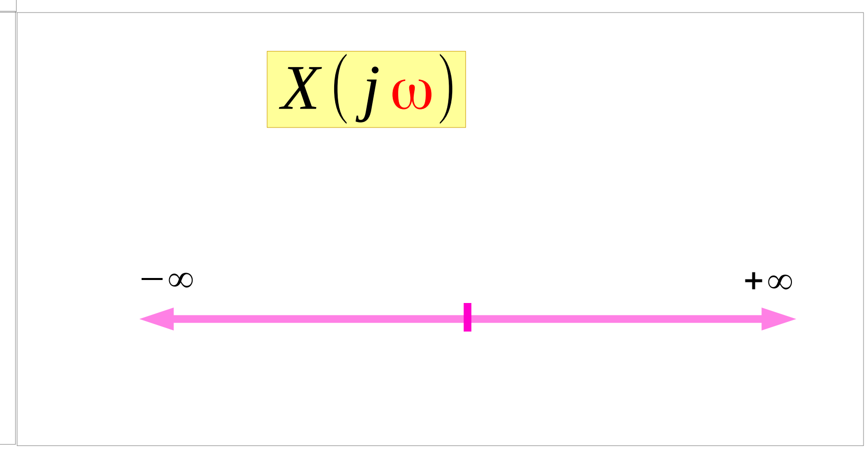


CT **Continuous Time**

FT **Continuous Frequency**



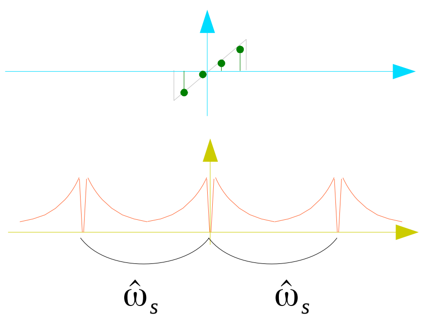




4. DTFT

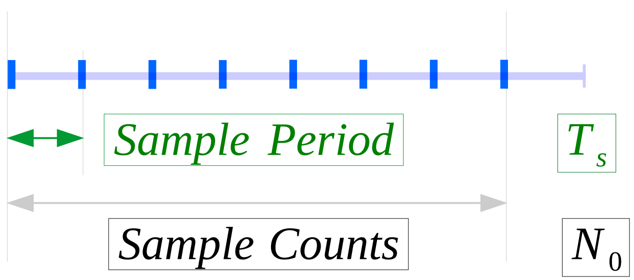
DT **Discrete Time**

FT **Continuous Frequency (Normalized)**

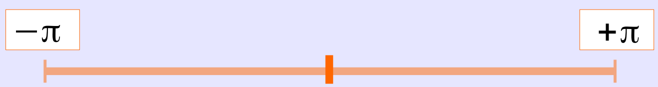


$$\hat{\omega}_s = \frac{2\pi}{1} \quad \Longleftrightarrow \quad \omega_s = \frac{2\pi}{T_s}$$

$x[n]$



$X(j\hat{\omega})$



Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

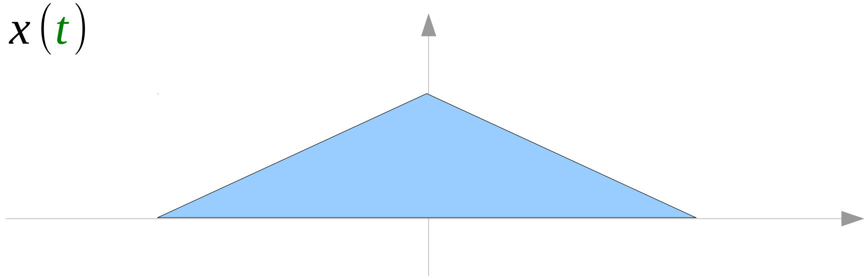
Continuous Time Fourier **Transform**

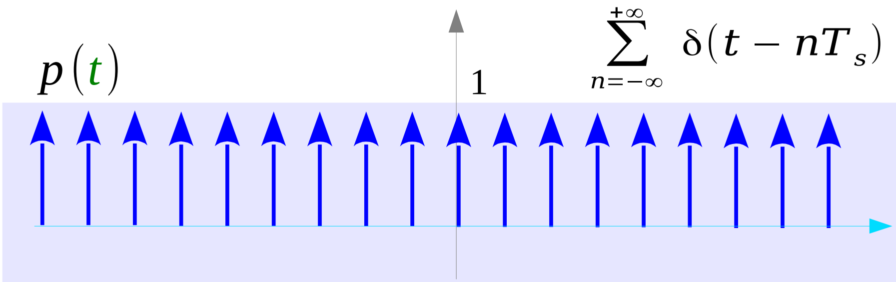
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier **Transform**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

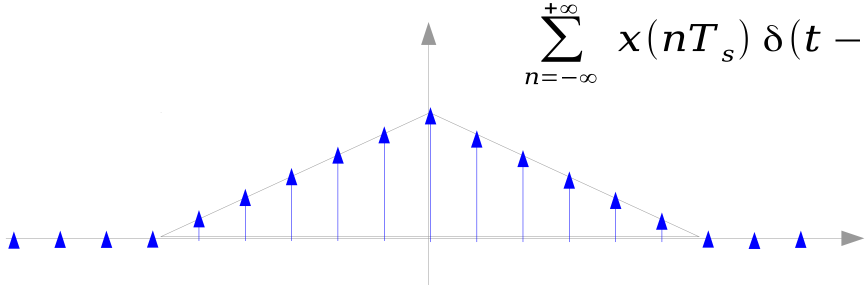
Multiplication with an Impulse Train

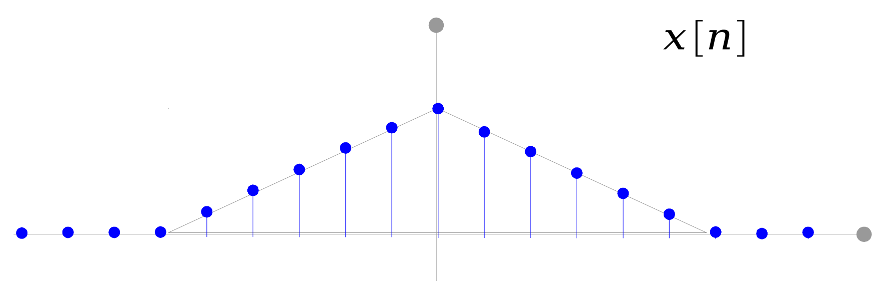




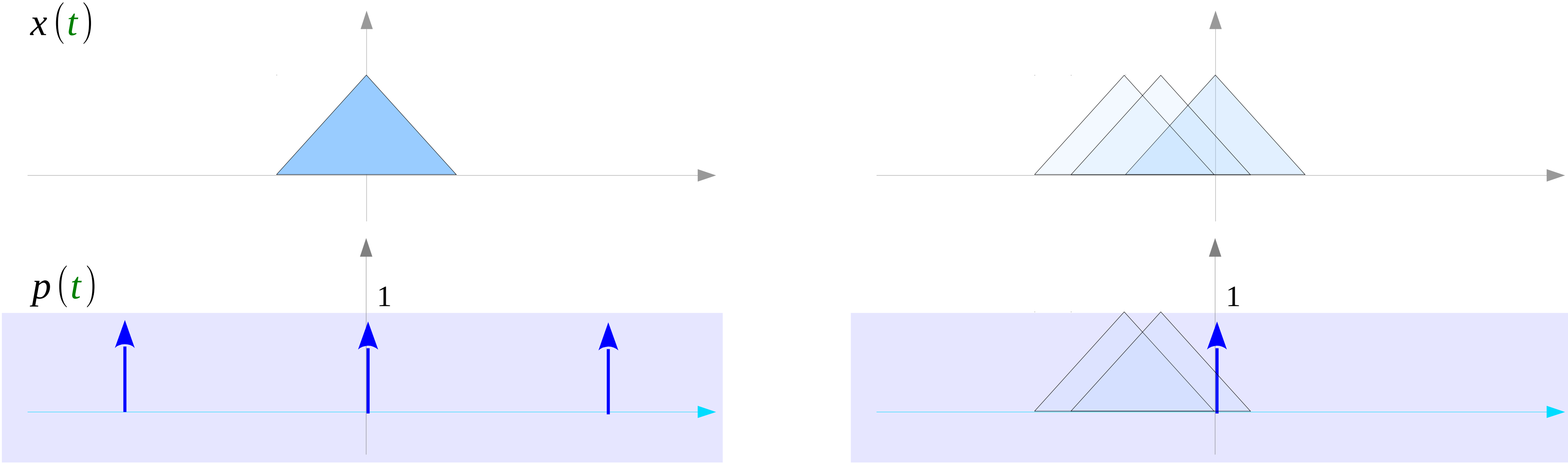
$x(t) \cdot p(t)$

Multiplication with a dense impulse train

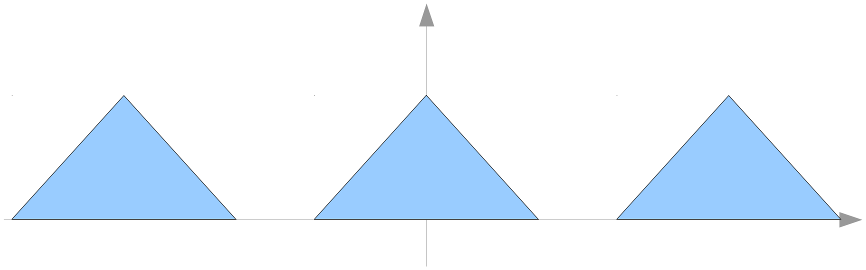




Convolution with an Impulse Train



$x(t) * p(t)$ **Multiplication with a sparse impulse train**



Convolution & Multiplication Properties

$$x(t) * y(t) \Leftrightarrow X(j\omega) \cdot Y(j\omega)$$

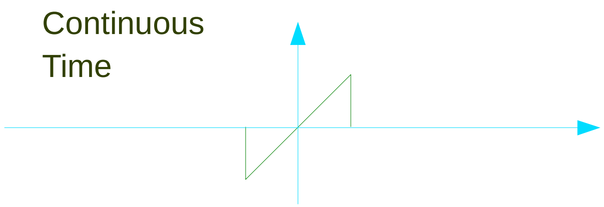
$$x(t) \cdot y(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

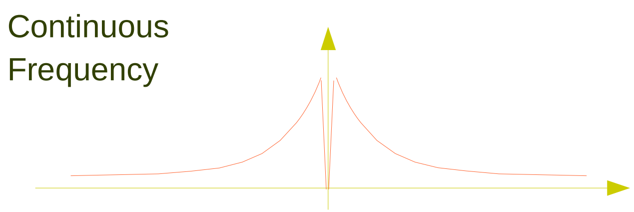
$$x(t) * y(t) \Leftrightarrow X(f) \cdot Y(f)$$

$$x(t) \cdot y(t) \Leftrightarrow X(f) * Y(f)$$

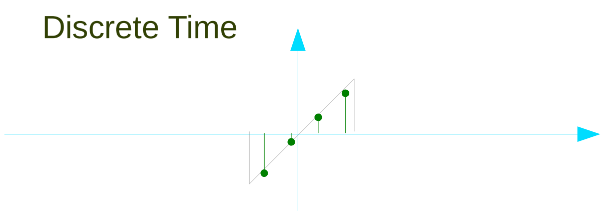
Types of Fourier Transforms

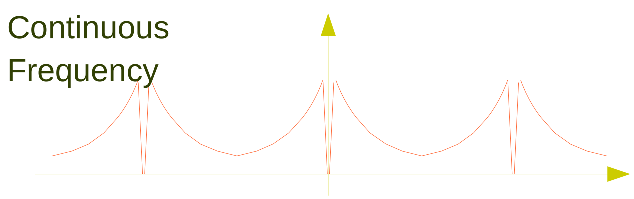
Continuous Time Fourier Transform
CTFT



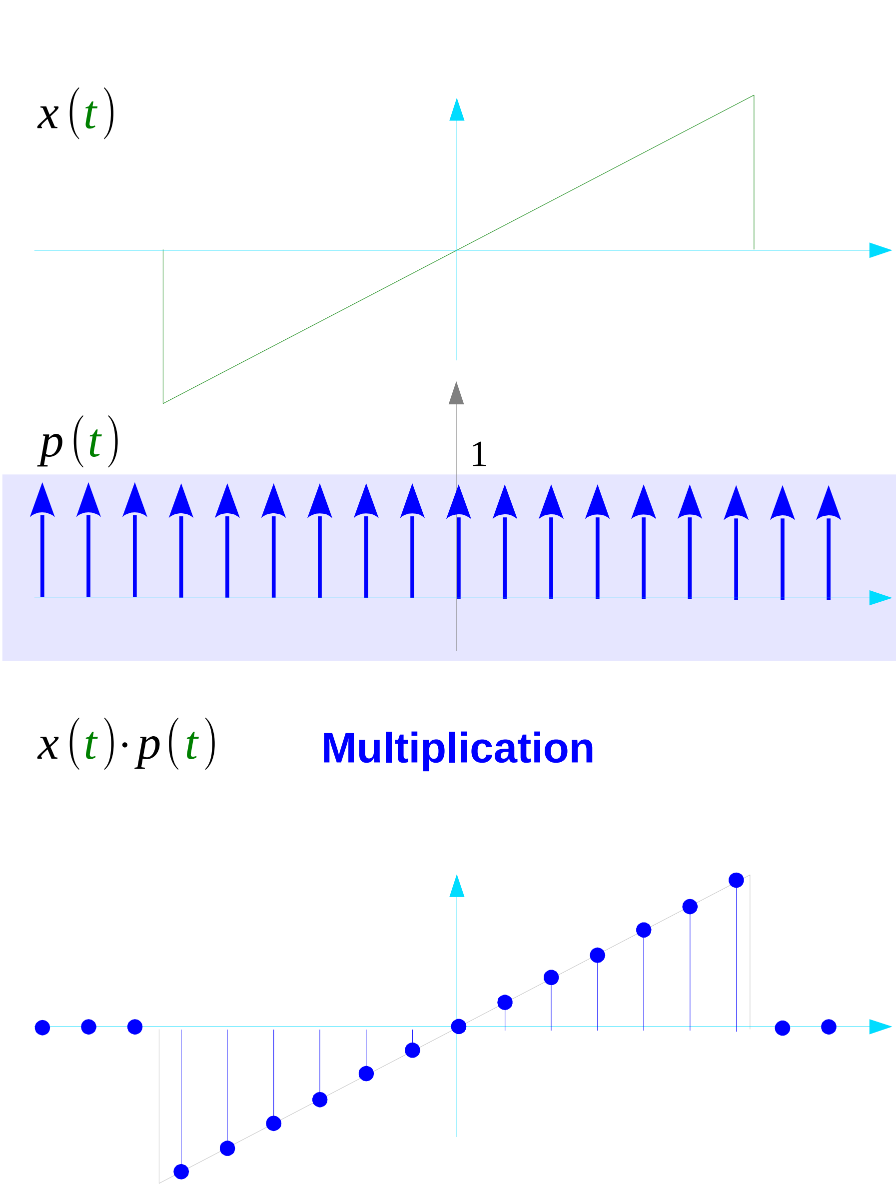


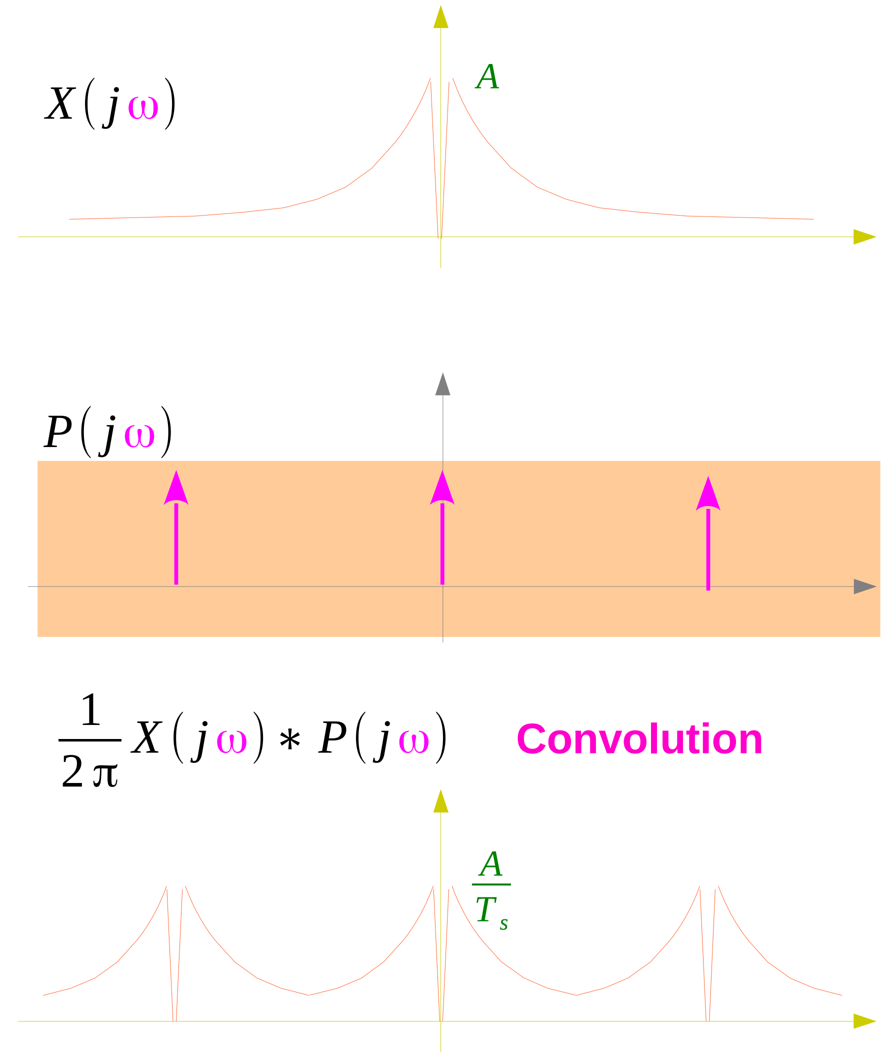
Discrete Time Fourier Transform
DTFT



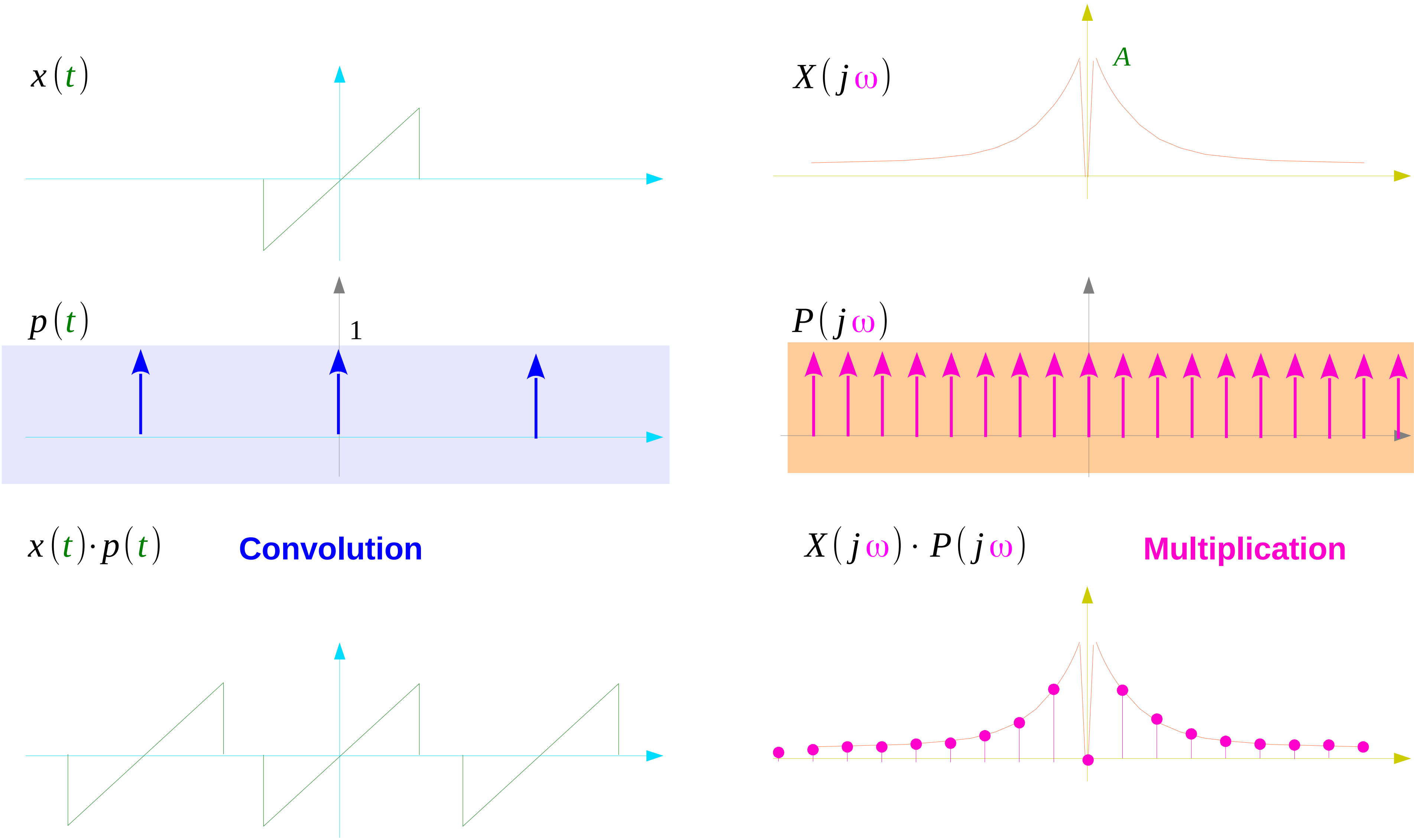


Multiplication & Convolution





Convolution & Multiplication



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M.J. Roberts, Fundamentals of Signals and Systems
[4] S.J. Orfanidis, Introduction to Signal Processing
[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings