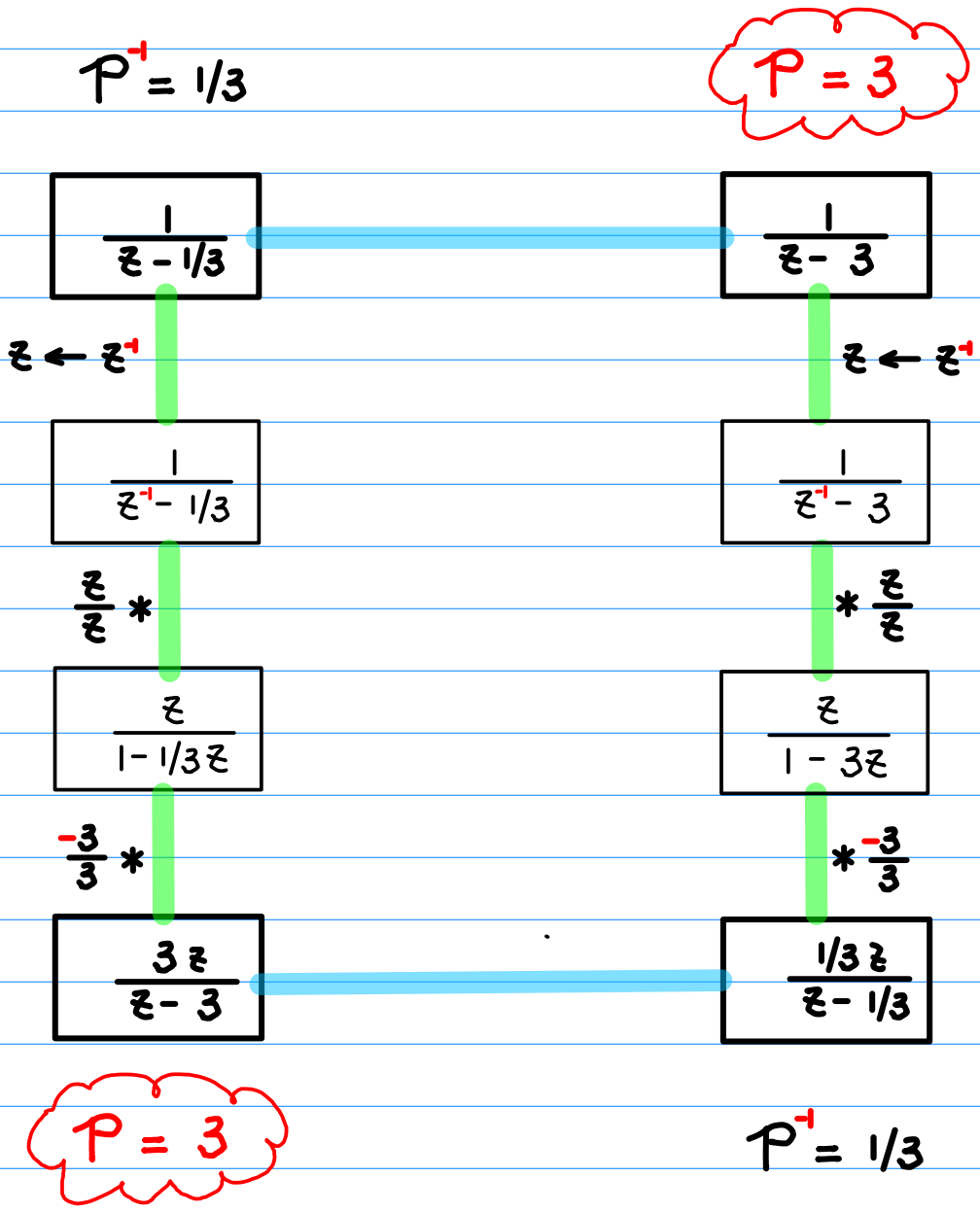
Laurent Series and z-Transform - Geometric Series Double Pole Examples B

20220630 Thr

Reciprocal Pole Relation (p=3)



Reciprocal Pole Partial Fractions (p=3)

$p^{-1} = 1/3$

$p = 3$

$$\frac{3}{8}\frac{-1}{(z-1/3)(z-3)} = \left(\frac{1}{z-1/3} - \frac{1}{z-3}\right)$$

$z \leftarrow z^{-1}$

$$\frac{3}{8}\frac{-1}{(z^{-1}-1/3)(z^{-1}-3)} = \left(\frac{1}{z^{-1}-1/3} - \frac{1}{z^{-1}-3}\right)$$

$* \frac{z}{z}$

$-1, z^{-1}$

$$\frac{3}{8}\frac{-z^2}{(1-1/3z)(1-3z)} = \left(\frac{z}{(1-1/3z)} - \frac{z}{(1-3z)}\right)$$

$* \frac{3}{3}$

$$\frac{3}{8}\frac{-z^2 \cdot 3 \cdot 1/3}{(3-z)(1/3-z)} = \left(\frac{3z}{(3-z)} - \frac{1/3\,z}{(1/3-z)}\right)$$

$* -1$

$$\frac{3}{8}\frac{-z^2}{(z-1/3)(z-3)} = \left(\frac{1/3\,z}{z-1/3} - \frac{3z}{z-3}\right)$$

Reciprocal Pole Relation : Partial Fractions and Geometric Power Series

$p^{-1} = 1/3$

$p = 3$

$$\frac{3}{8}\,\frac{-1}{(z-1/3)(z-3)} = \left(\frac{1}{z-1/3} - \frac{1}{z-3}\right)$$

$-1, z^{-1}$

$$\frac{3}{8}\,\frac{-z^2}{(z-1/3)(z-3)} = \left(\frac{1/3\,z}{z-1/3} - \frac{3z}{z-3}\right)$$

$\frac{1}{z-1/3}$

/(1/3) /z

$-\frac{3}{1-3z}$ $\frac{z^{-1}}{1-1/3z^{-1}}$

-(1') (5')

$-3^{n+1}\,u(n)$ $3^{n+1}\,u(-n-1)$

$\frac{1}{z-3}$

/3 /z

$-\frac{1/3}{1-1/3z}$ $\frac{z^{-1}}{1-3z^{-1}}$

-(2') (6')

$-\left(\frac{1}{3}\right)^{n+1}u(n)$ $\left(\frac{1}{3}\right)^{n+1}u(-n-1)$

$\frac{1/3\,z}{z-1/3}$

/(1/3) /z

$-\frac{z}{1-3z}$ $\frac{1/3}{1-1/3z^{-1}}$

-(7') (3')

$-3^{n-1}\,u(n-1)$ $3^{n-1}\,u(-n)$

$\frac{3z}{z-3}$

/3 /z

$-\frac{z}{1-1/3z}$ $\frac{3}{1-3z^{-1}}$

-(8') (4')

$-\left(\frac{1}{3}\right)^{n-1}u(n-1)$ $\left(\frac{1}{3}\right)^{n-1}u(-n)$

Reciprocal poles in Simple Pole Forms (p=3)

$p^{-1} = 1/3$

$p = 3$

$p^{-1} = 1/3$

$$\frac{1}{z - 1/3}$$

$$\frac{1}{z^{-1} - 1/3}$$

$$\frac{z}{1 - 1/3z}$$

$$-\frac{3z}{z - 3}$$

$p = 3$

$$\frac{1}{z - 3}$$

$$\frac{1}{z^{-1} - 3}$$

$$\frac{z}{1 - 3z}$$

$$-\frac{1/3z}{z - 1/3}$$

$$\frac{3}{8}\,\frac{-1}{(z - 1/3)(z - 3)} = \left(\frac{1}{z - 1/3} - \frac{1}{z - 3}\right)$$

$-1, z^{-1}$

$$\frac{1}{z - 1/3}$$

$$\frac{1}{z - 3}$$

$-1, z^{-1}$

$-1, z^{-1}$

$$\frac{1/3z}{z - 1/3}$$

$$\frac{3z}{z - 3}$$

$$\frac{3}{8}\,\frac{-z^2}{(z - 1/3)(z - 3)} = \left(\frac{1/3z}{z - 1/3} - \frac{3z}{z - 3}\right)$$

Reciprocal poles in Geometric Series Forms (p=2)

Causal

$p^{-1} = 1/3$

$p = 3$

$p^{-1} = 1/3$

$$-\frac{3}{1-3z}$$

$$-\frac{3}{1-3z^{-1}}$$

$$-\frac{3z}{z-3}$$

$$\frac{z}{1-1/3\,z}$$

$p = 3$

$$-\frac{1/3}{1-1/3\,z}$$

$$-\frac{1/3}{1-1/3\,z^{-1}}$$

$$\frac{z}{1-3z}$$

$$\frac{z}{1-3z}$$

$$\frac{3}{8}\,\frac{-1}{(z-1/3)(z-3)} = \left(\frac{1}{z-1/3} - \frac{1}{z-3}\right)$$

$-1, z^{-1}$

$-3^{n+1}\,u(n)$

$-(1')$ $-\dfrac{3}{1-3z}$

$-1, z^{-1}$

$-(\frac{1}{3})^{n+1}\,u(n)$

$-\dfrac{1/3}{1-1/3\,z}$ $-(2')$

$-(7')$ $-\dfrac{z}{1-3z}$

$-3^{n-1}\,u(n-1)$

$-1, z^{-1}$

$-\dfrac{z}{1-1/3\,z}$ $-(8')$

$-(\frac{1}{3})^{n-1}\,u(n-1)$

$$\frac{3}{8}\,\frac{-z^2}{(z-1/3)(z-3)} = \left(\frac{1/3\,z}{z-1/3} - \frac{3z}{z-3}\right)$$

Reciprocal poles in Geometric Series Forms (p=2)

Anti-causal

$p^{-1} = 1/3$

$p = 3$

$p^{-1} = 1/3$

$$\frac{z^{-1}}{1 - 1/3 z^{-1}}$$

$p = 3$

$$\frac{z^{-1}}{1 - 3z^{-1}}$$

$$\frac{z^{+1}}{1 - 1/3 z^{+1}}$$

$$\frac{z^{+1}}{1 - 2z^{+1}}$$

$$\frac{1}{z^{-1} - 1/3}$$

$$\frac{1}{z^{-1} - 2}$$

$$-\frac{3}{1 - 3z^{-1}}$$

$$-\frac{0.5}{1 - 0.5 z^{-1}}$$

$$\frac{3}{8}\frac{-1}{(z - 1/3)(z - 3)} = \left(\frac{1}{z - 1/3} - \frac{1}{z - 3}\right)$$

$3^{n+1}\,u(-n-1)$

(5') $\dfrac{z^{-1}}{1 - 1/3 z^{-1}}$

$\left(\frac{1}{3}\right)^{n+1} u(-n-1)$

$\dfrac{z^{-1}}{1 - 3z^{-1}}$ (6')

$-1, z^{-1}$

$-1, z^{-1}$

$-1, z^{-1}$

(3') $\dfrac{0.5}{1 - 0.5 z^{-1}}$

$3^{n-1}\,u(-n)$

$\dfrac{3}{1 - 3z^{-1}}$ -(4')

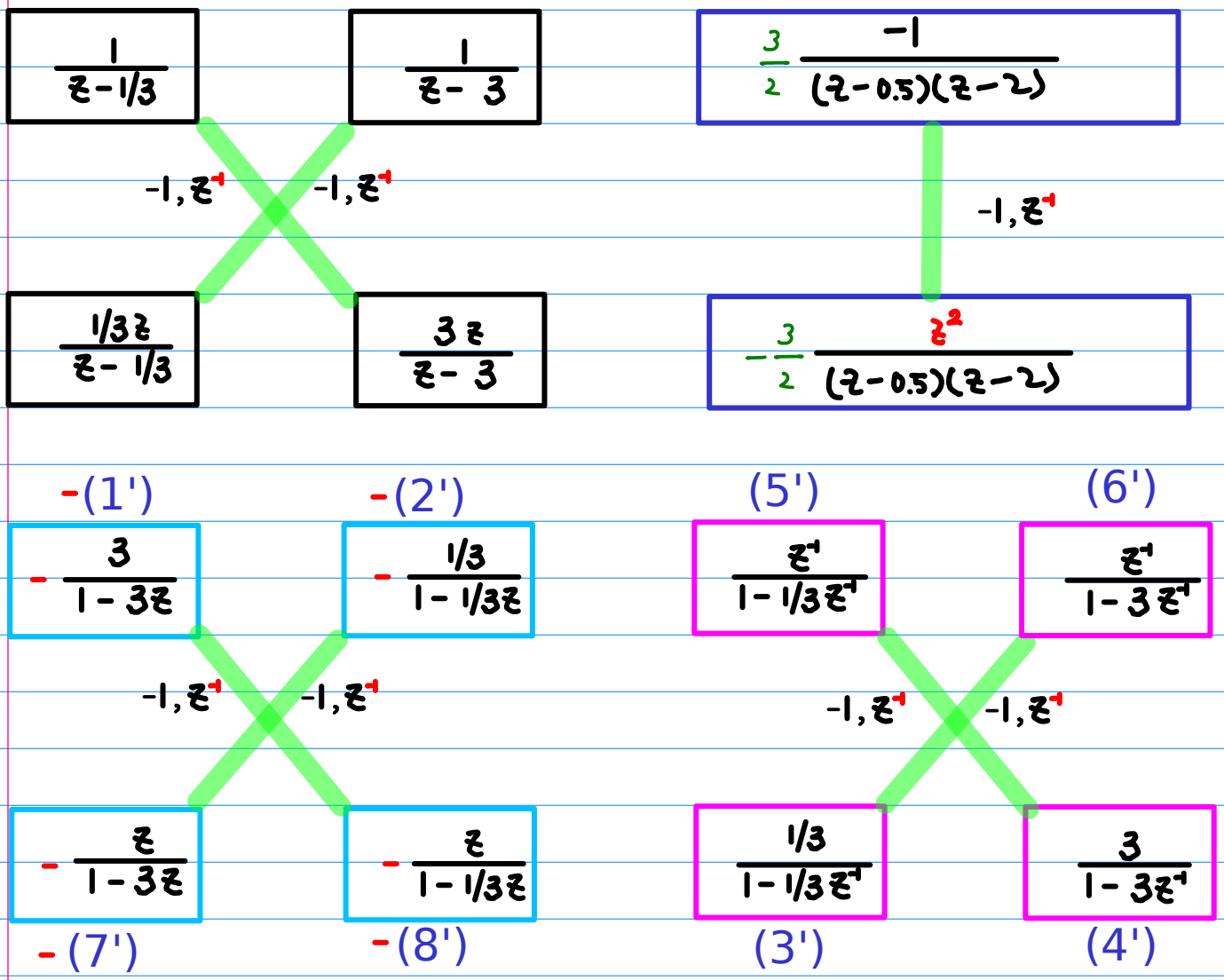
$\left(\frac{1}{3}\right)^{n-1} u(-n)$

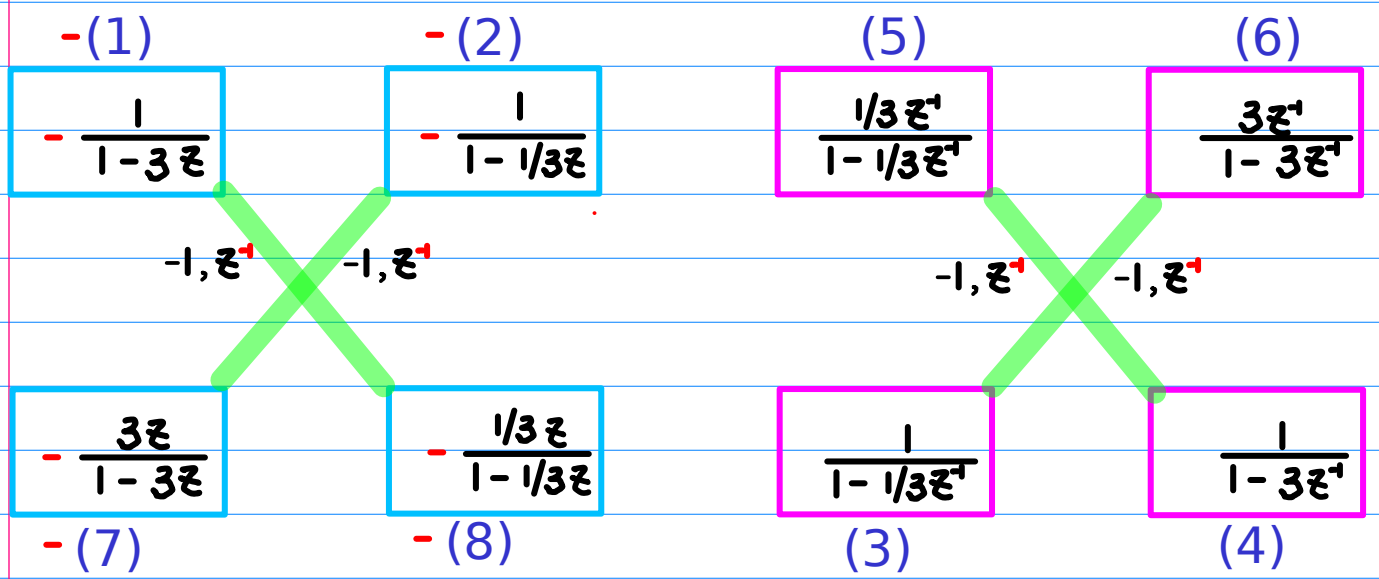
$$\frac{3}{8}\frac{-z^2}{(z - 1/3)(z - 3)} = \left(\frac{1/3\, z}{z - 1/3} - \frac{3z}{z - 3}\right)$$

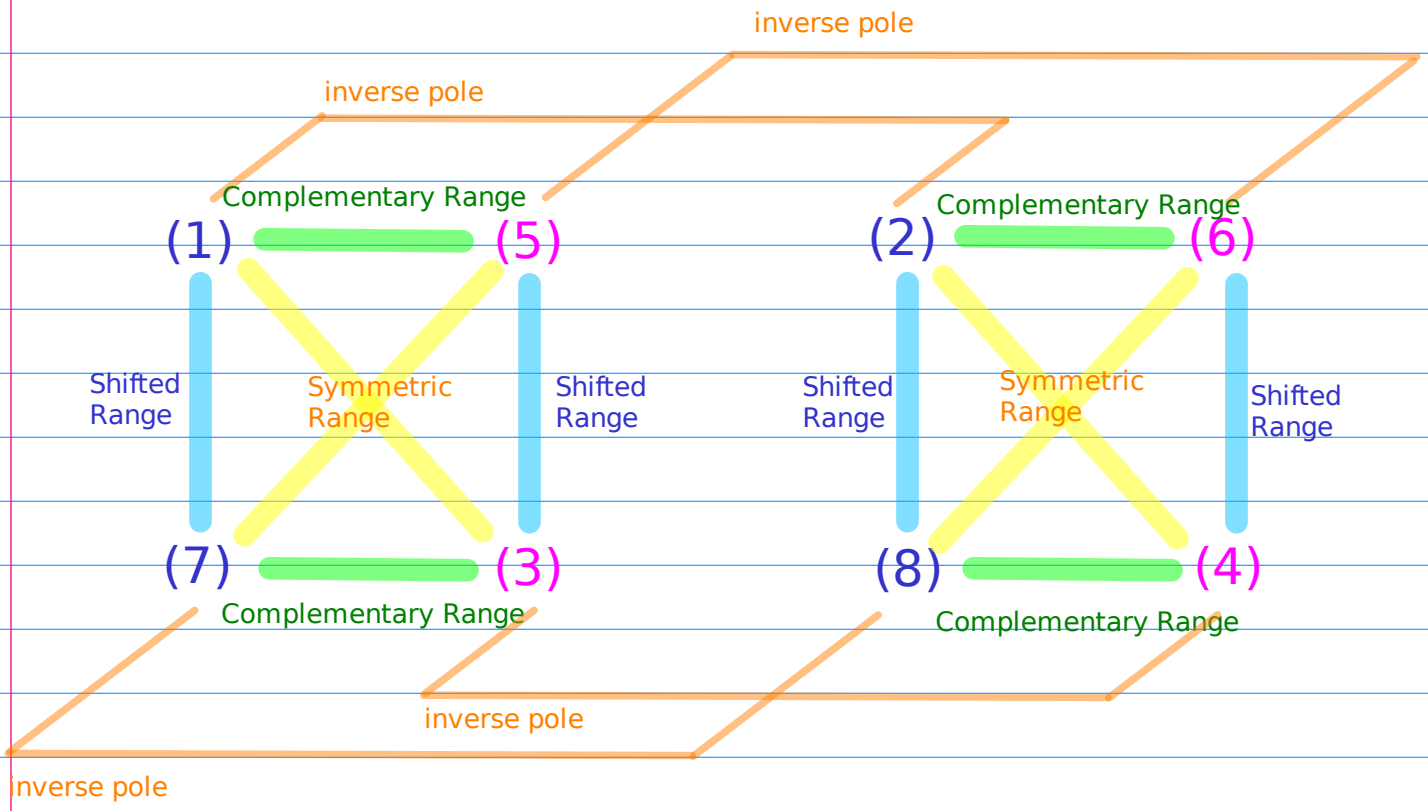
Reciprocal Pole Relation :
Shifted version and
Unshifted version

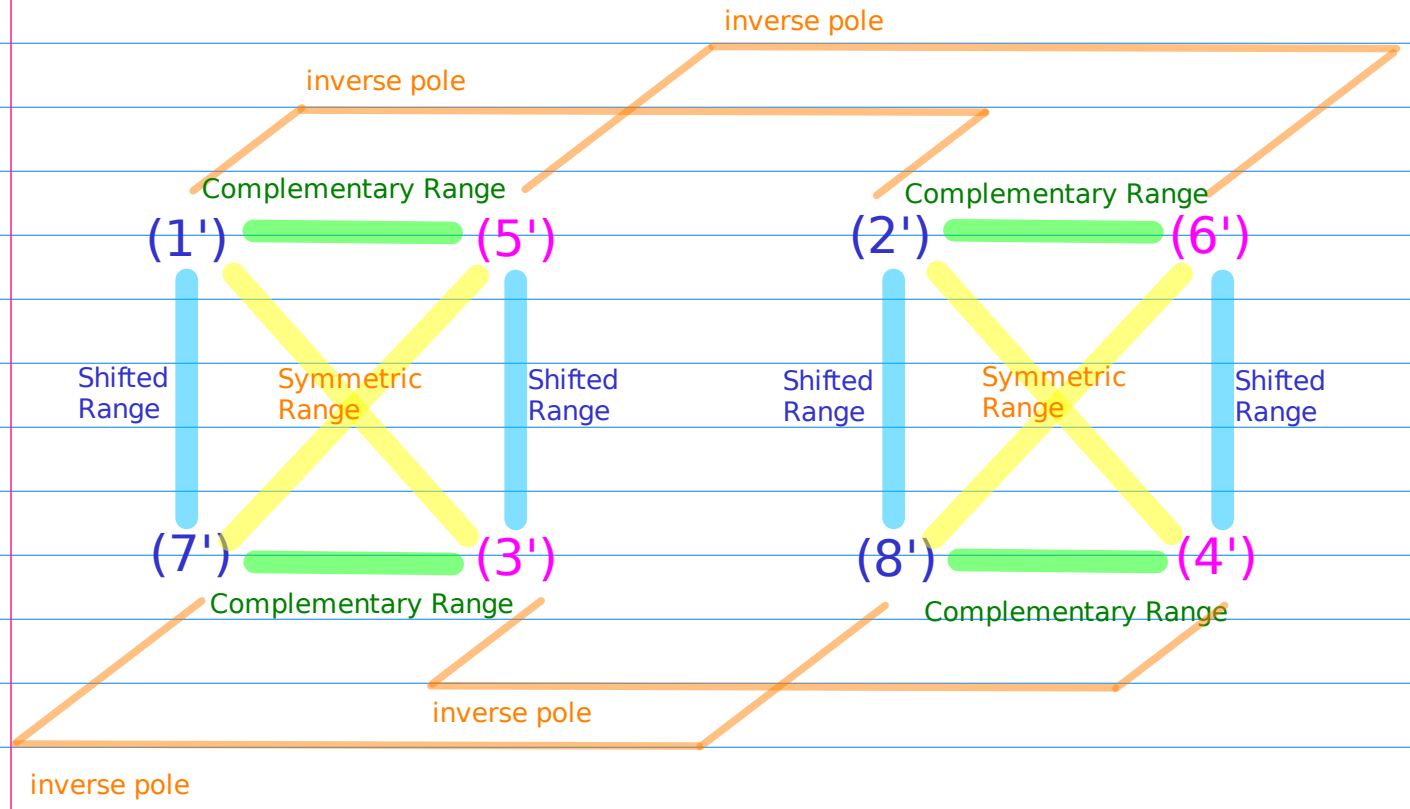
$P^{-1} = 1/3$

$P = 3$



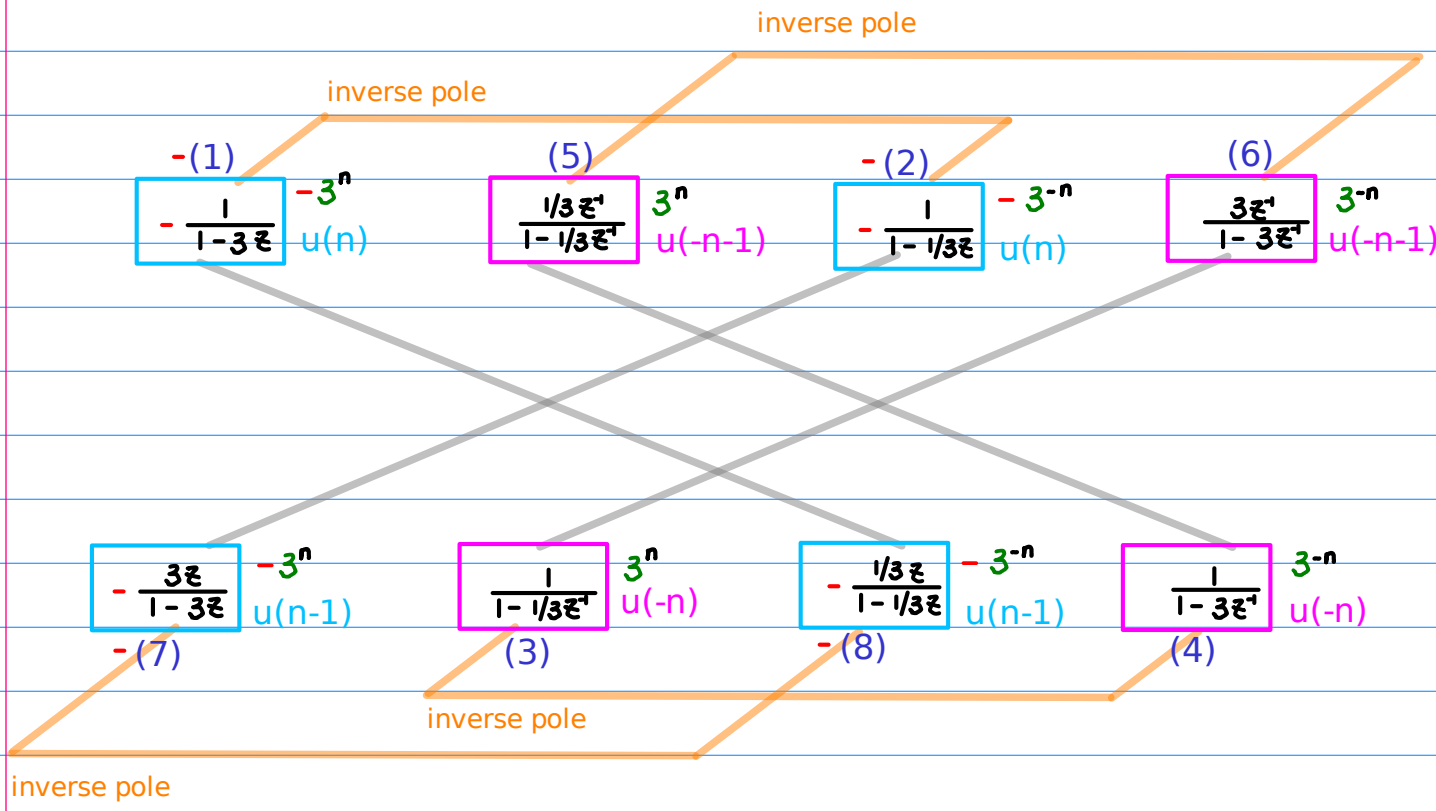


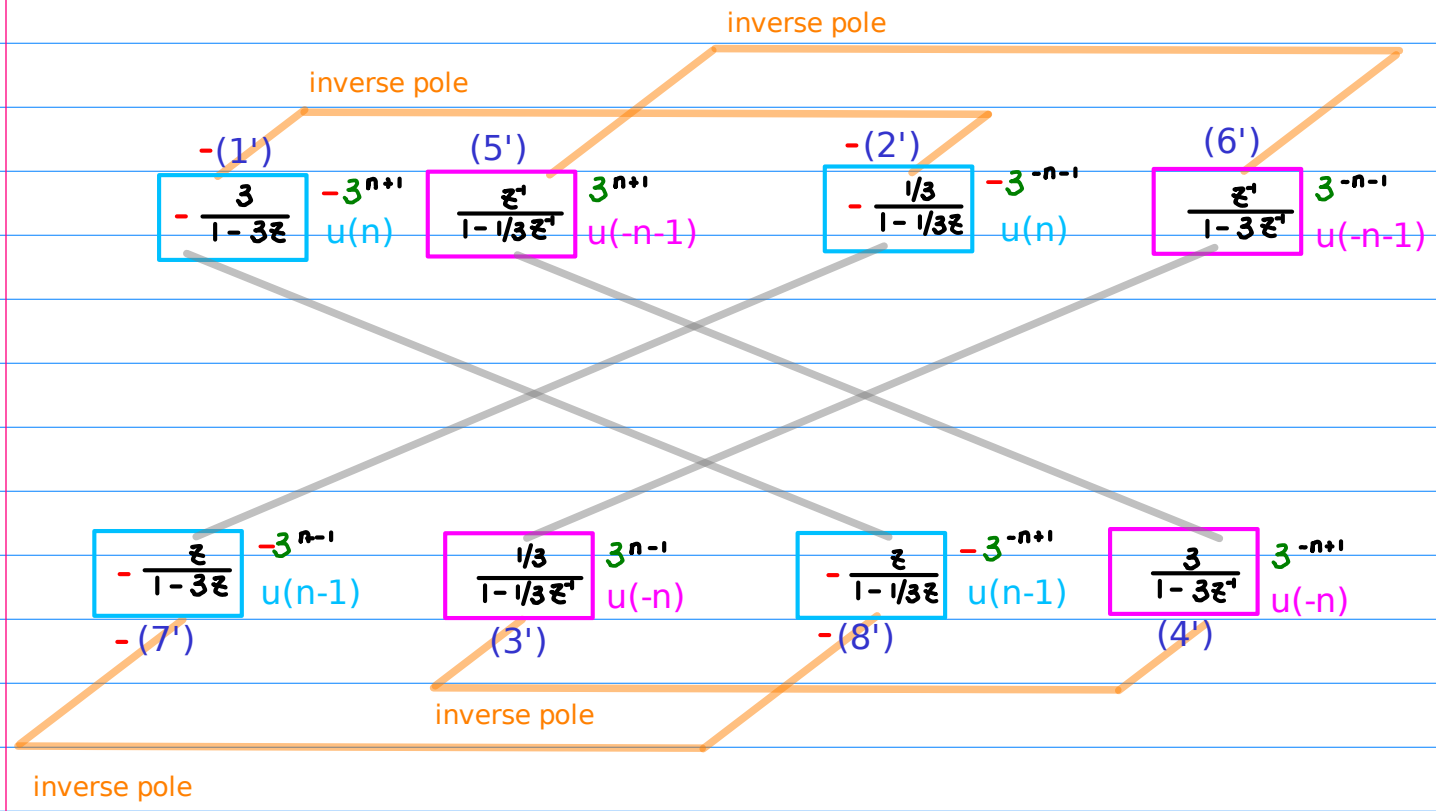

inverse pole
inverse pole
Complementary Range
(1)
(5)
Shifted Range
Symmetric Range
Shifted Range
(7)
(3)
Complementary Range
inverse pole
inverse pole
Complementary Range
(2)
(6)
Shifted Range
Symmetric Range
Shifted Range
(8)
(4)
Complementary Range

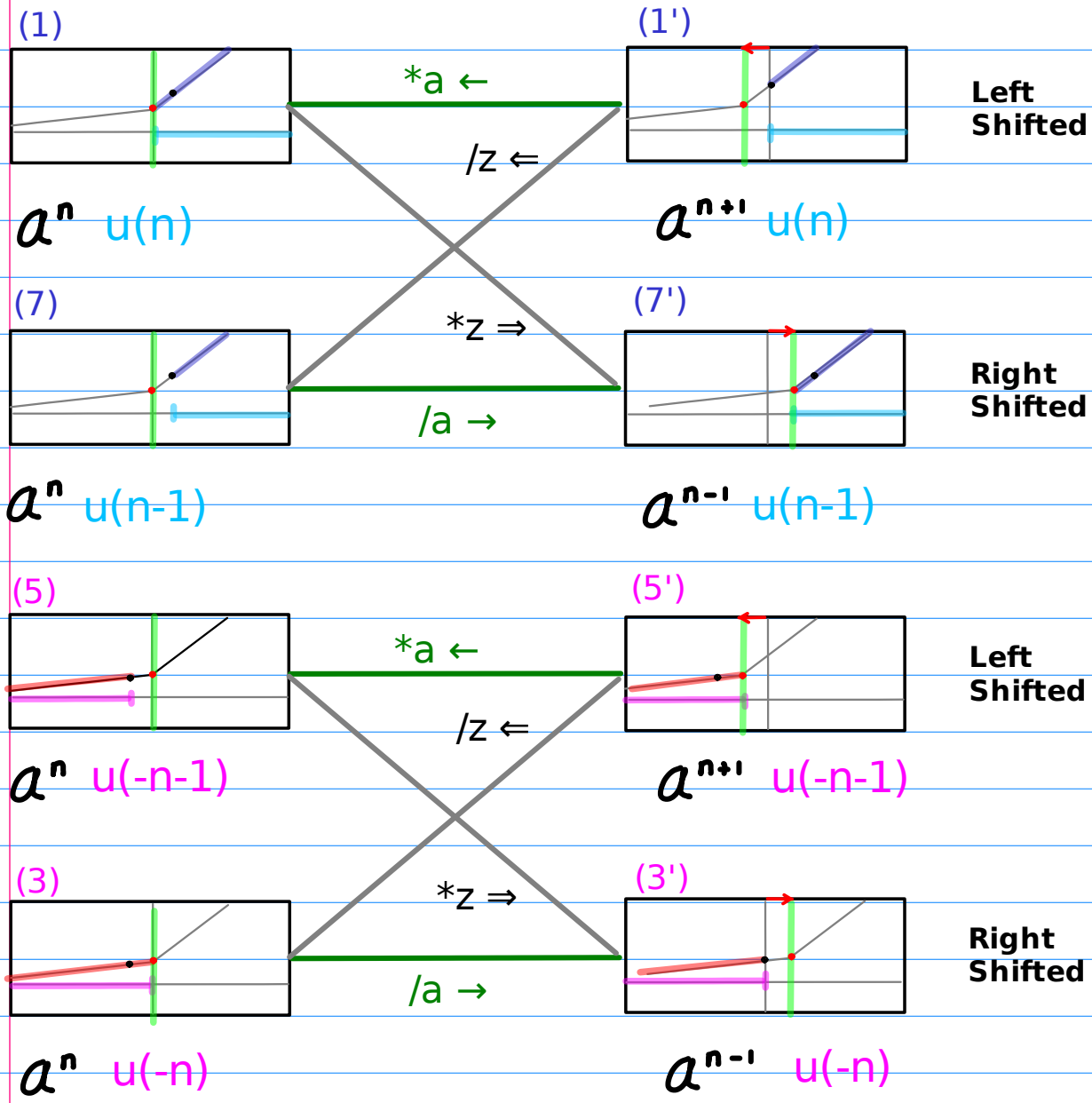

inverse pole
inverse pole
Complementary Range
(1')
(5')
Shifted Range
Symmetric Range
Shifted Range
(7')
(3')
Complementary Range
inverse pole
inverse pole
Complementary Range
(2')
(6')
Shifted Range
Symmetric Range
Shifted Range
(8')
(4')
Complementary Range

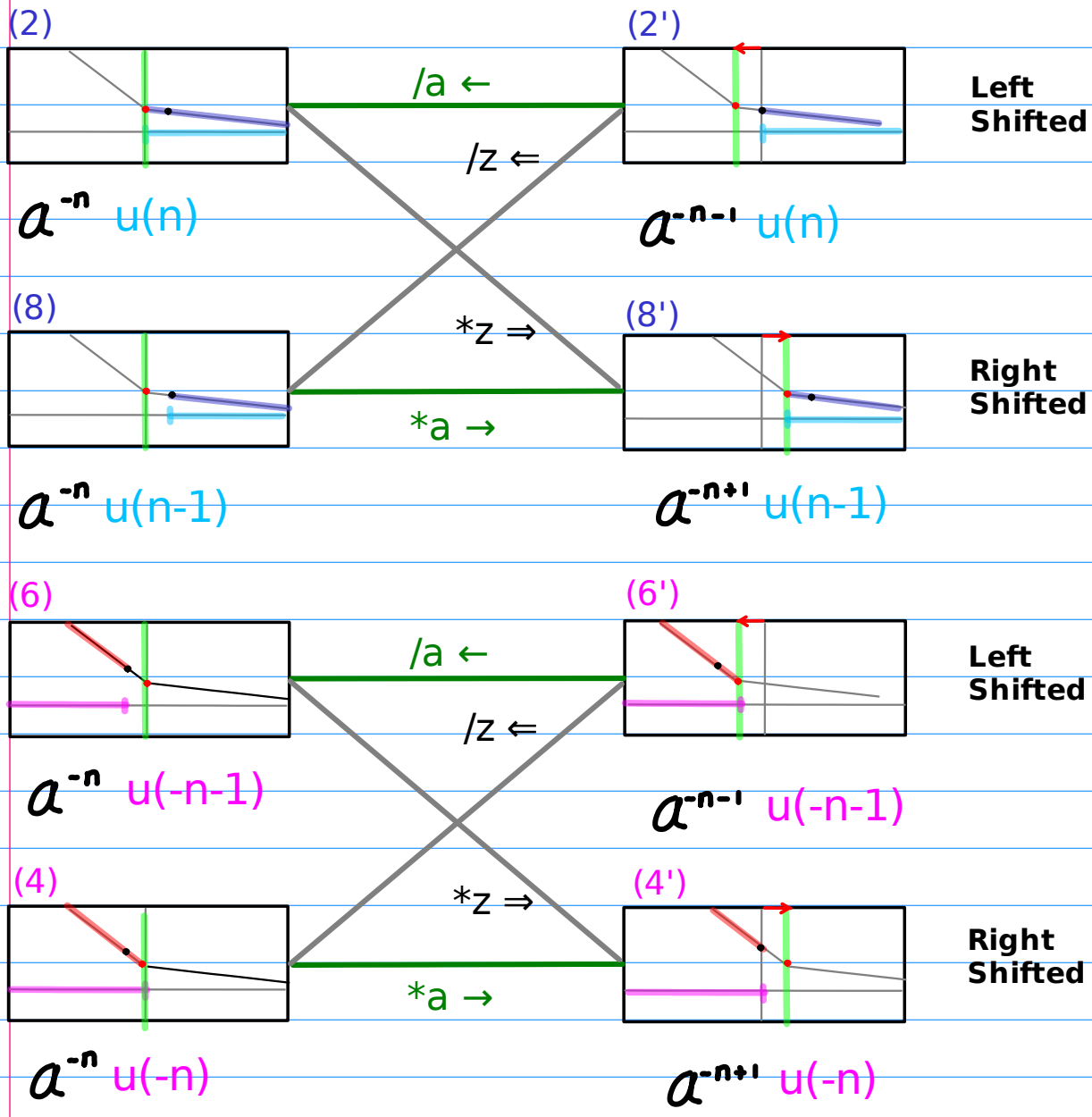
$n \leftarrow -n$

swap{ u(n), u(n-1) }
swap{ u(-n), u(-n-1) }

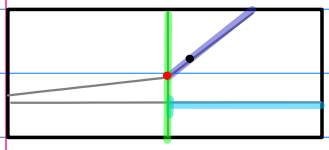





(1)
(1')
*a ←
/z ⇐
Left
Shifted
a^n u(n)
a^{n+1} u(n)
(7)
(7')
*z ⇒
/a →
Right
Shifted
a^n u(n-1)
a^{n-1} u(n-1)
(5)
(5')
*a ←
/z ⇐
Left
Shifted
a^n u(-n-1)
a^{n+1} u(-n-1)
(3)
(3')
*z ⇒
/a →
Right
Shifted
a^n u(-n)
a^{n-1} u(-n)

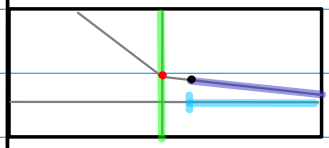

(2)
(2')
/a ←
/z ⇐
Left
Shifted
a^{-n} u(n)
a^{-n-1} u(n)
(8)
(8')
*z ⇒
*a →
Right
Shifted
a^{-n} u(n-1)
a^{-n+1} u(n-1)
(6)
(6')
/a ←
/z ⇐
Left
Shifted
a^{-n} u(-n-1)
a^{-n-1} u(-n-1)
(4)
(4')
*z ⇒
*a →
Right
Shifted
a^{-n} u(-n)
a^{-n+1} u(-n)

(1)



a^n u(n)

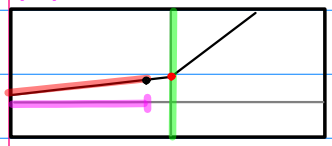
(8)



Left Shifted

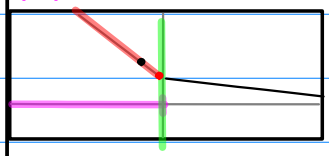
a^{-n} u(n-1)

(5)



a^n u(-n-1)

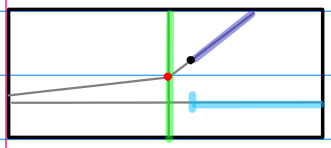
(4)



Left Shifted

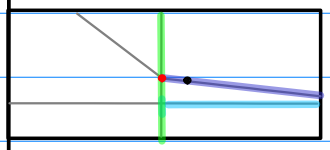
a^{-n} u(-n)

(7)



a^n u(n-1)

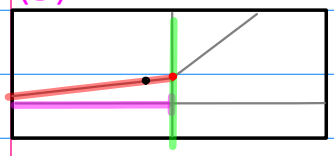
(2)



Right Shifted

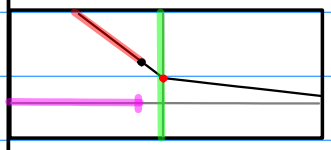
a^{-n} u(n)

(3)



a^n u(-n)

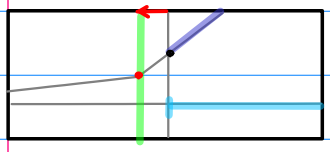
(6)



Right Shifted

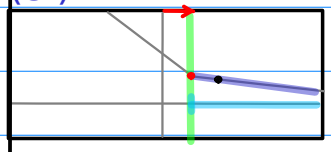
a^{-n} u(-n-1)

(1')



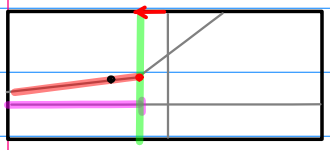
a^{n+1} u(n)

(8')



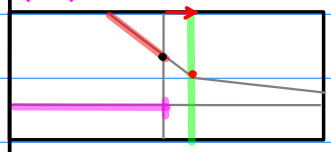
a^{-n+1} u(n-1)

(5')



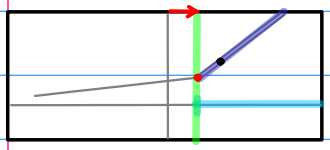
a^{n+1} u(-n-1)

(4')



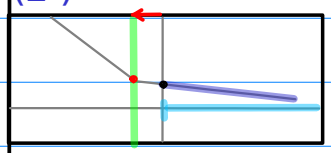
a^{-n+1} u(-n)

(7')



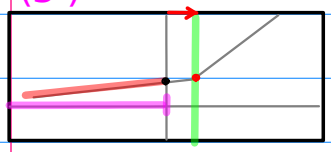
a^{n-1} u(n-1)

(2')



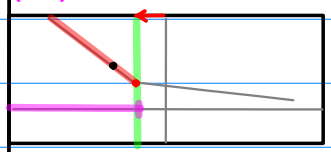
a^{-n-1} u(n)

(3')

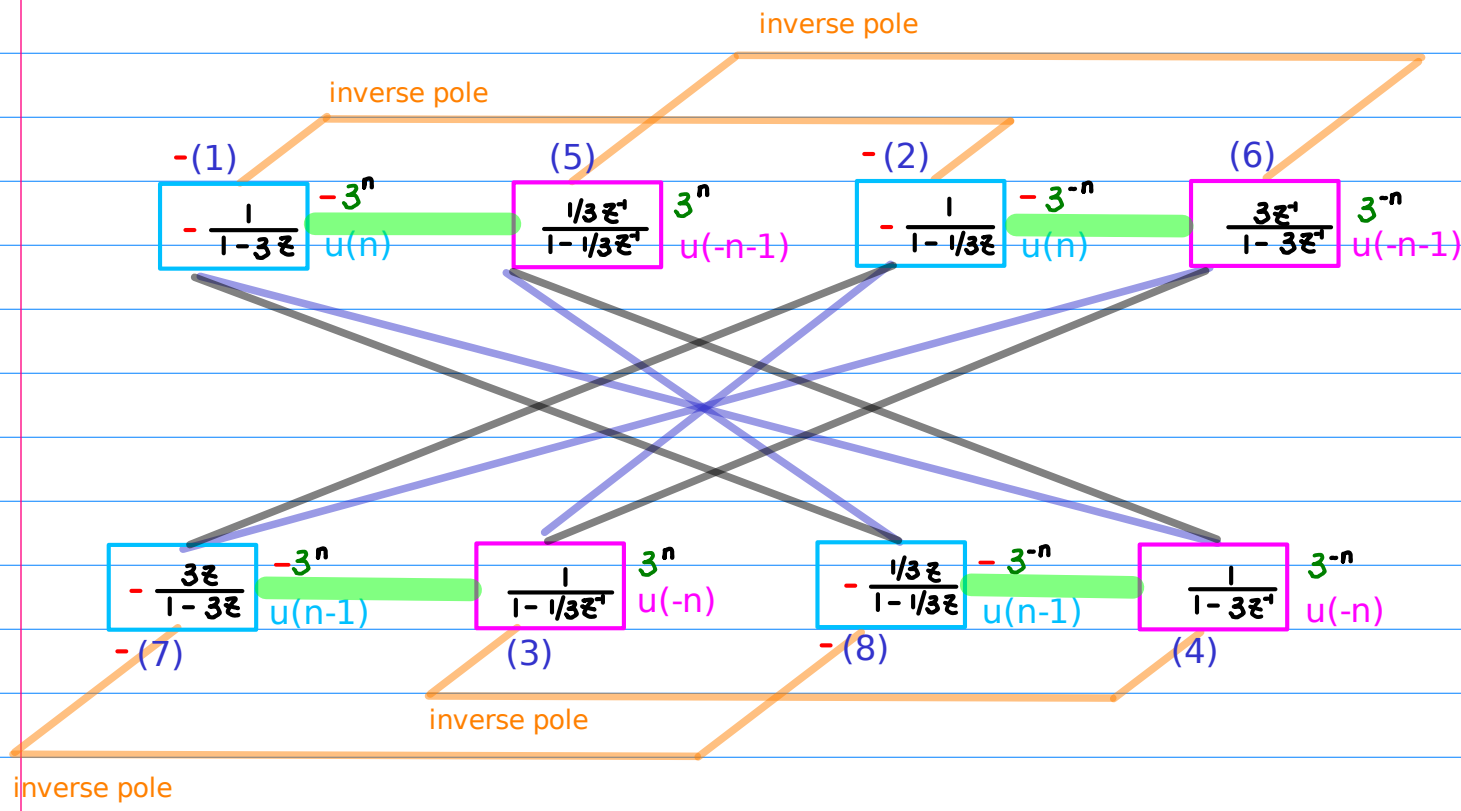


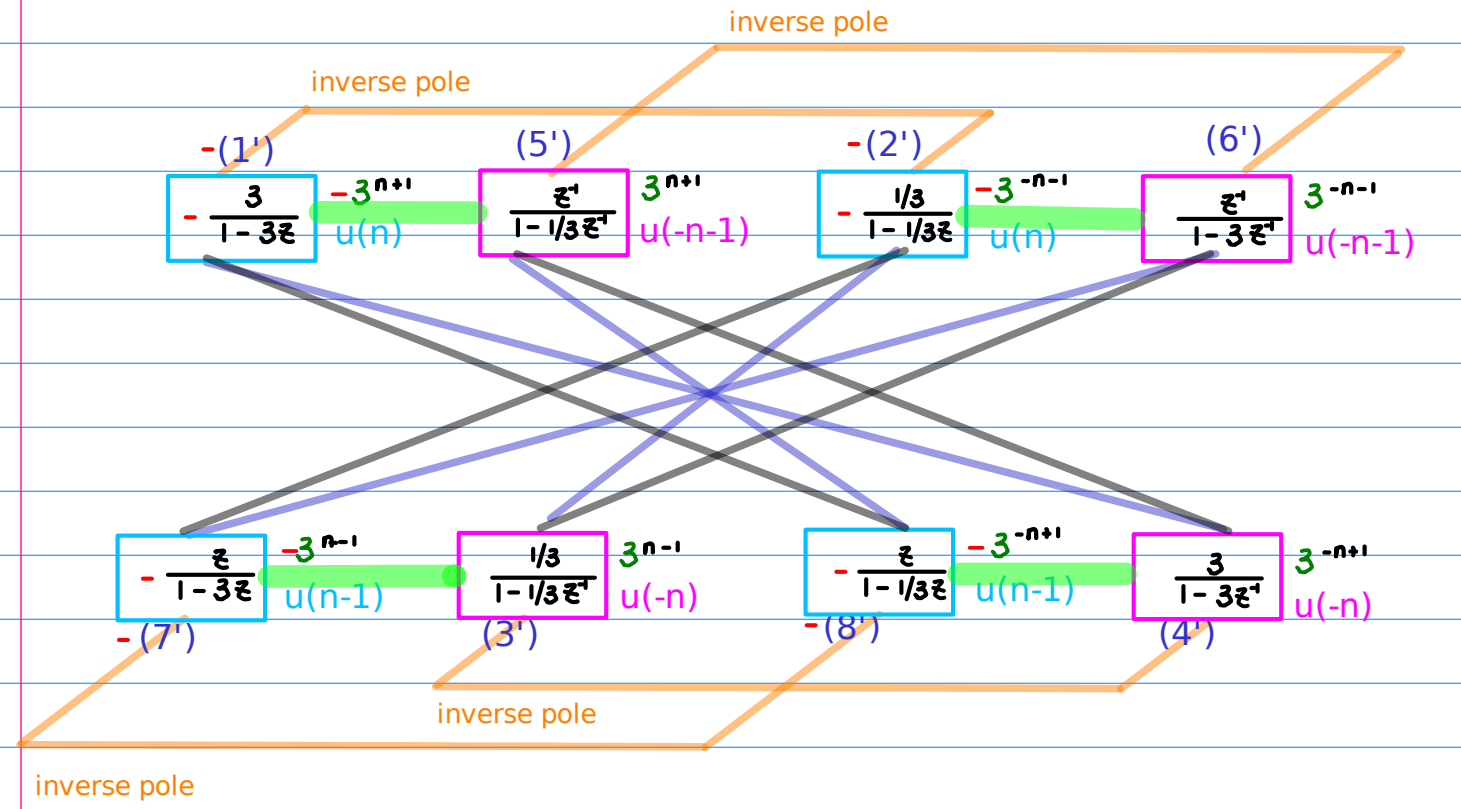
a^{n-1} u(-n)

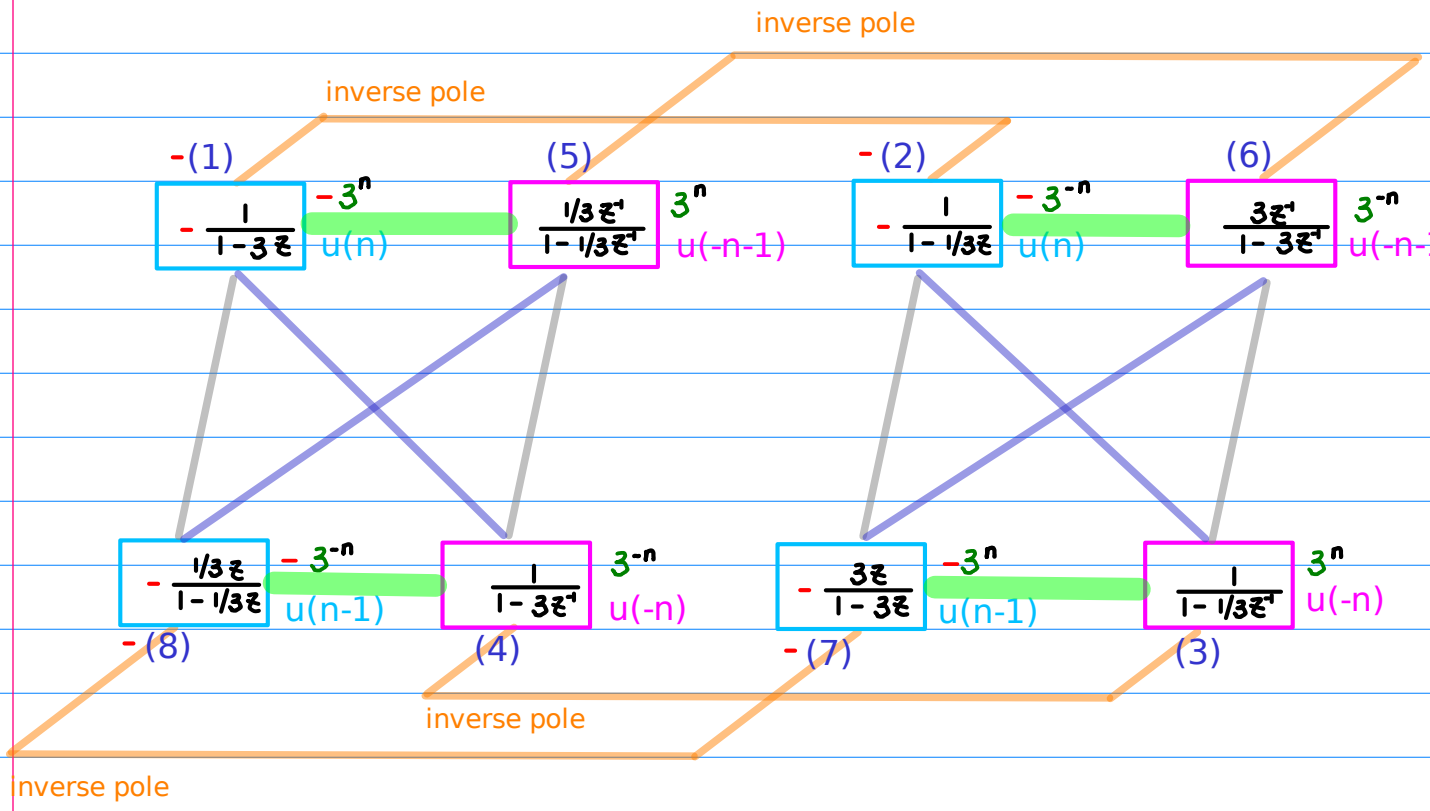
(6')

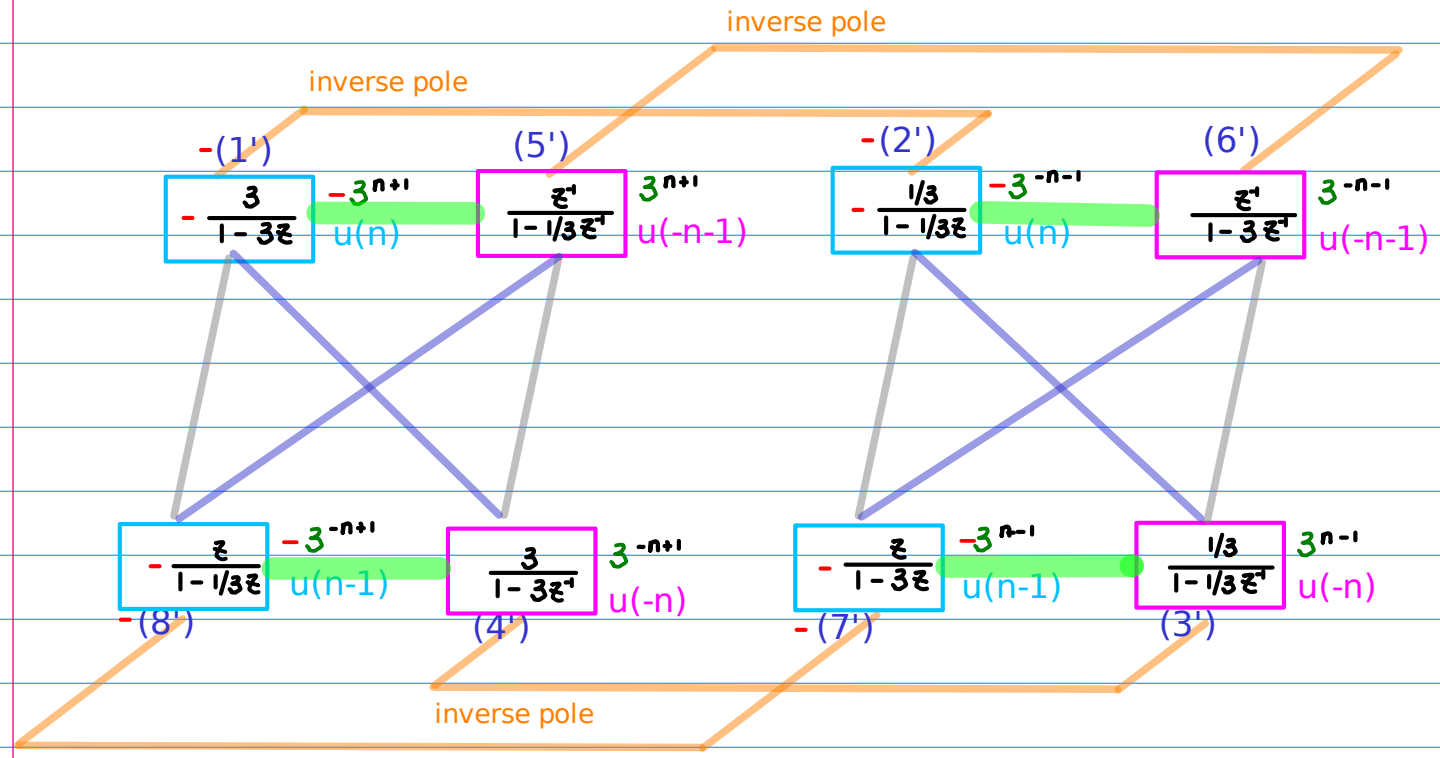


a^{-n-1} u(-n-1)


inverse pole
inverse pole
-(1)
(5)
-(2)
(6)
$-\frac{1}{1-3z}$
-3^{n}
u(n)
$\frac{1/3z^{-1}}{1-1/3z^{-1}}$
3^{n}
u(-n-1)
$-\frac{1}{1-1/3z}$
-3^{-n}
u(n)
$\frac{3z^{-1}}{1-3z^{-1}}$
3^{-n}
u(-n-1)
$-\frac{3z}{1-3z}$
-3^{n}
u(n-1)
$\frac{1}{1-1/3z^{-1}}$
3^{n}
u(-n)
$-\frac{1/3z}{1-1/3z}$
-3^{-n}
u(n-1)
$\frac{1}{1-3z^{-1}}$
3^{-n}
u(-n)
-(7)
(3)
-(8)
(4)
inverse pole
inverse pole


inverse pole
inverse pole
-(1')
(5')
-(2')
(6')
$-\frac{3}{1-3z}$
-3^{n+1}
u(n)
$\frac{z^{-1}}{1-1/3z^{-1}}$
3^{n+1}
u(-n-1)
$-\frac{1/3}{1-1/3z}$
-3^{-n-1}
u(n)
$\frac{z^{-1}}{1-3z^{-1}}$
3^{-n-1}
u(-n-1)
$-\frac{z}{1-3z}$
-3^{n-1}
u(n-1)
$\frac{1/3}{1-1/3z^{-1}}$
3^{n-1}
u(-n)
$-\frac{z}{1-1/3z}$
-3^{-n+1}
u(n-1)
$\frac{3}{1-3z^{-1}}$
3^{-n+1}
u(-n)
-(7')
(3')
-(8')
(4')
inverse pole
inverse pole


inverse pole
inverse pole
-(1)
$-\frac{1}{1-3z}$
-3^{n}
u(n)
(5)
$\frac{1/3z^{-1}}{1-1/3z^{-1}}$
3^{n}
u(-n-1)
-(2)
$-\frac{1}{1-1/3z}$
-3^{-n}
u(n)
(6)
$\frac{3z^{-1}}{1-3z^{-1}}$
3^{-n}
u(-n-1)
$-\frac{1/3z}{1-1/3z}$
-3^{-n}
u(n-1)
-(8)
$\frac{1}{1-3z^{-1}}$
3^{-n}
u(-n)
(4)
$-\frac{3z}{1-3z}$
-3^{n}
u(n-1)
-(7)
$\frac{1}{1-1/3z^{-1}}$
3^{n}
u(-n)
(3)
inverse pole
inverse pole


inverse pole
inverse pole
-(1')
$-\frac{3}{1-3z}$
-3^{n+1}
u(n)
(5')
$\frac{z^{-1}}{1-1/3z^{-1}}$
3^{n+1}
u(-n-1)
-(2')
$-\frac{1/3}{1-1/3z}$
-3^{-n-1}
u(n)
(6')
$\frac{z^{-1}}{1-3z^{-1}}$
3^{-n-1}
u(-n-1)
$-\frac{z}{1-1/3z}$
-3^{-n+1}
u(n-1)
-(8')
$\frac{3}{1-3z^{-1}}$
3^{-n+1}
u(-n)
(4')
$-\frac{z}{1-3z}$
-3^{n-1}
u(n-1)
-(7')
$\frac{1/3}{1-1/3z^{-1}}$
3^{n-1}
u(-n)
(3')
inverse pole
inverse pole

Time reversal x[-n] X(z^-1)

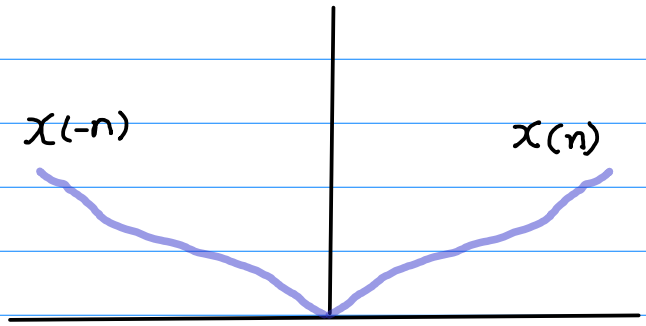
.

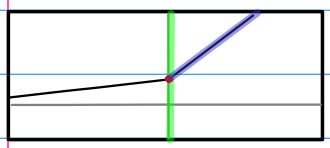
$$\mathcal{Z}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{m}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\left(z^{-1}\right)^{-m}$$

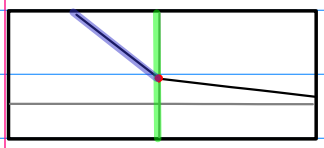
$$= X(z^{-1})$$





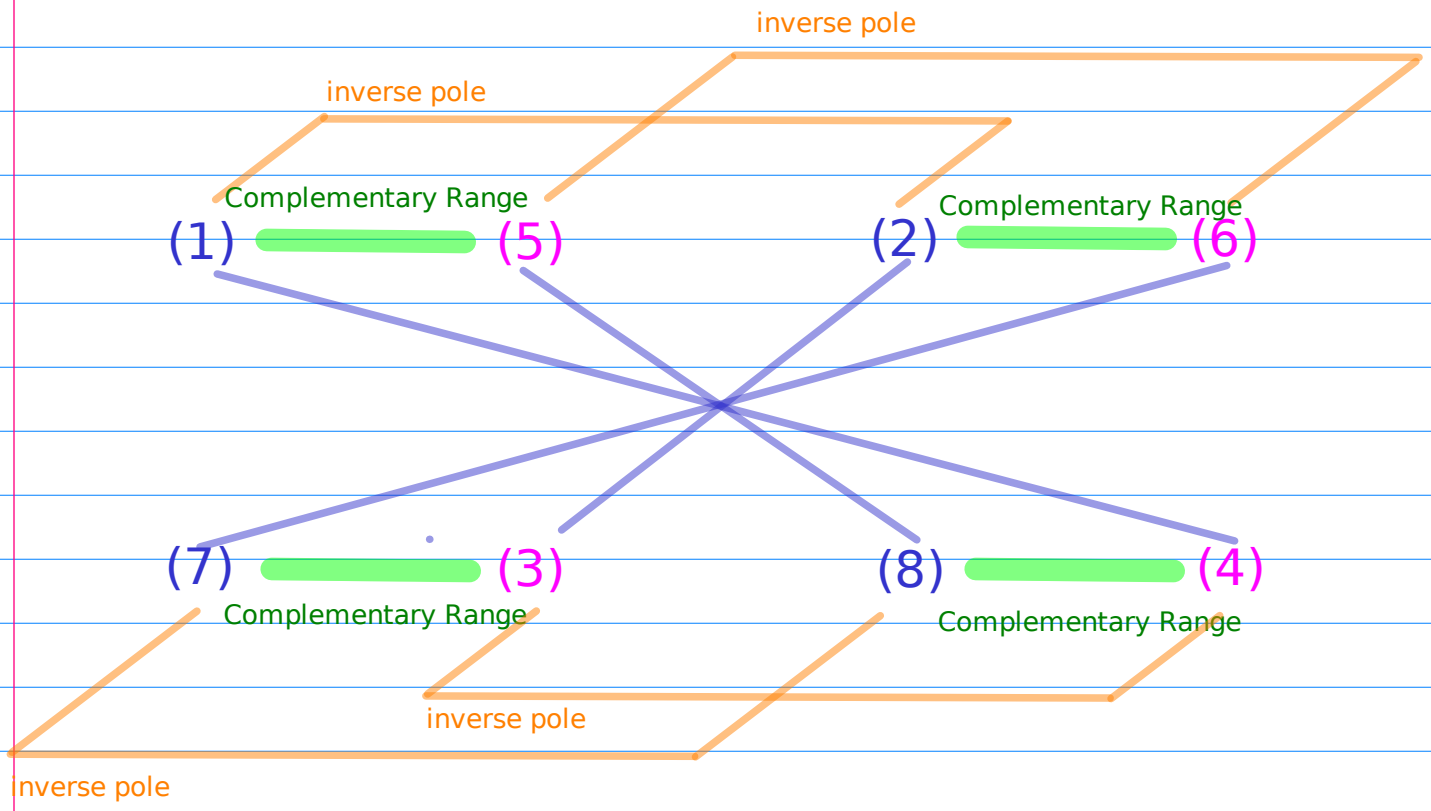
3^n

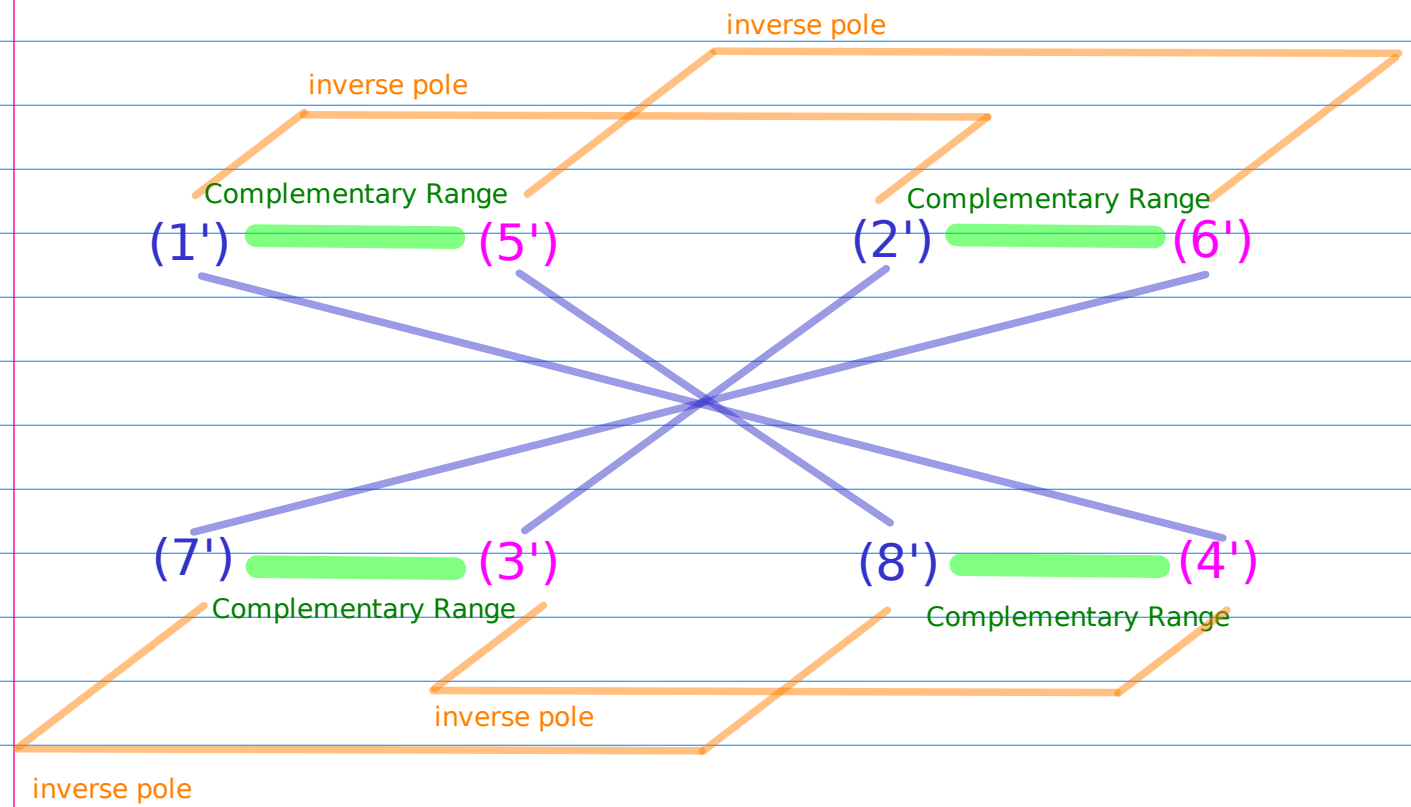
u(n)



3^{-n}

u(-n)

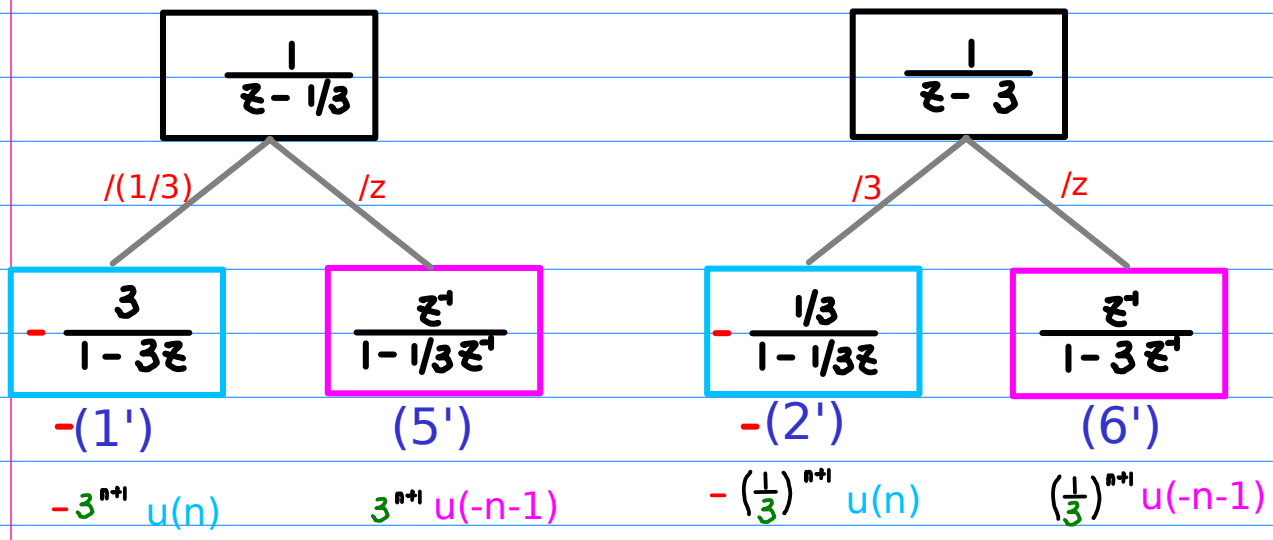

inverse pole
inverse pole
Complementary Range
(1)
(5)
Complementary Range
(2)
(6)
(7)
(3)
(8)
(4)
Complementary Range
Complementary Range
inverse pole
inverse pole

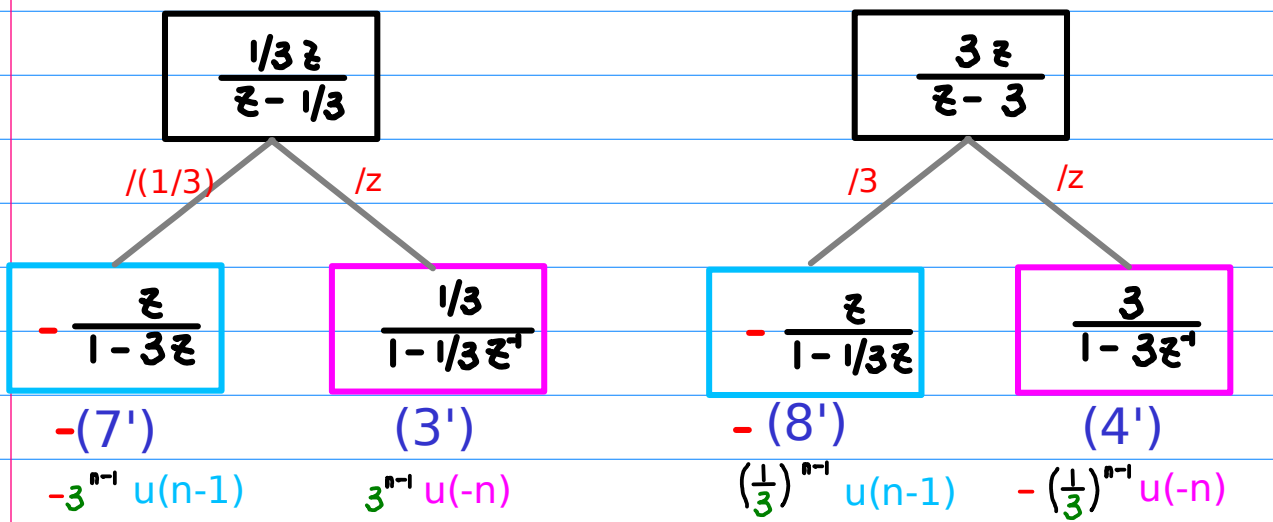

inverse pole
inverse pole
Complementary Range
(1')
(5')
Complementary Range
(2')
(6')
(7')
(3')
(8')
(4')
Complementary Range
Complementary Range
inverse pole
inverse pole

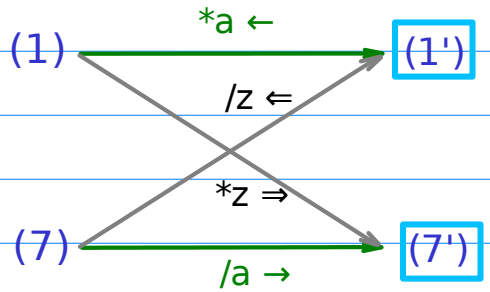
Reciprocal Pole and Shift Relations

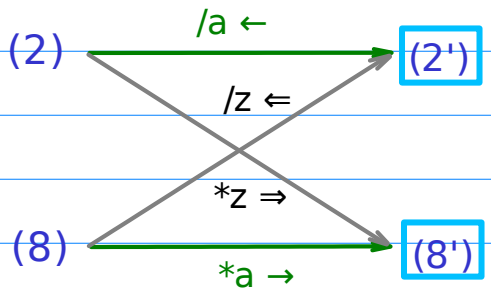
$p^{-1} = 1/3$

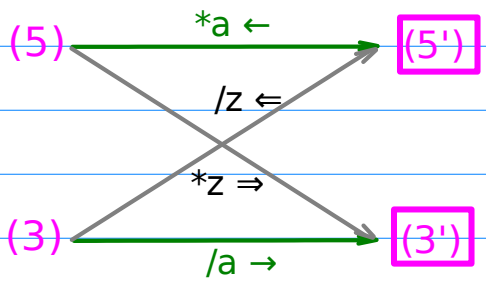
$p = 3$

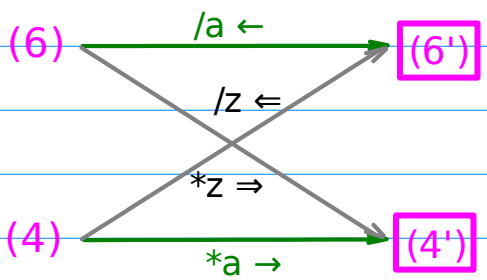












(1) (5)
(7) (3)

(2) (6)
(8) (4)

(1') (5')
(7') (3')

(2')(6')
(8')(4')

Partial fractions and geometric power series

$P^{-1} = 1/3$

$P = 3$

-(1) $-\dfrac{1}{1-3z}$

$-3^n\ u(n)$

(5) $\dfrac{1/3\,z^{-1}}{1-1/3\,z^{-1}}$

$3^n\ u(-n-1)$

-(2) $-\dfrac{1}{1-1/3z}$

$-3^{-n}\ u(n)$

(6) $\dfrac{3z^{-1}}{1-3z^{-1}}$

$3^{-n}\ u(-n-1)$

-(7) $-\dfrac{3z}{1-3z}$

$-3^n\ u(n-1)$

(3) $\dfrac{1}{1-1/3z^{-1}}$

$3^n\ u(-n)$

-(8) $-\dfrac{1/3\,z}{1-1/3z}$

$-3^{-n}\ u(n-1)$

(4) $\dfrac{1}{1-3z^{-1}}$

$3^{-n}\ u(-n)$

-(1') $-\dfrac{3}{1-3z}$

$-3^{n+1}\ u(n)$

(5') $\dfrac{z^{-1}}{1-1/3z^{-1}}$

$3^{n+1}\ u(-n-1)$

-(2') $-\dfrac{1/3}{1-1/3z}$

$-3^{-n-1}\ u(n)$

(6') $\dfrac{z^{-1}}{1-3z^{-1}}$

$3^{-n-1}\ u(-n-1)$

-(7') $-\dfrac{z}{1-3z}$

$-3^{n-1}u(n-1)$

(3') $-\dfrac{1/3}{1-1/3\,z^{-1}}$

$3^{n-1}\ u(-n)$

-(8') $-\dfrac{z}{1-1/3z}$

$-3^{-n+1}u(n-1)$

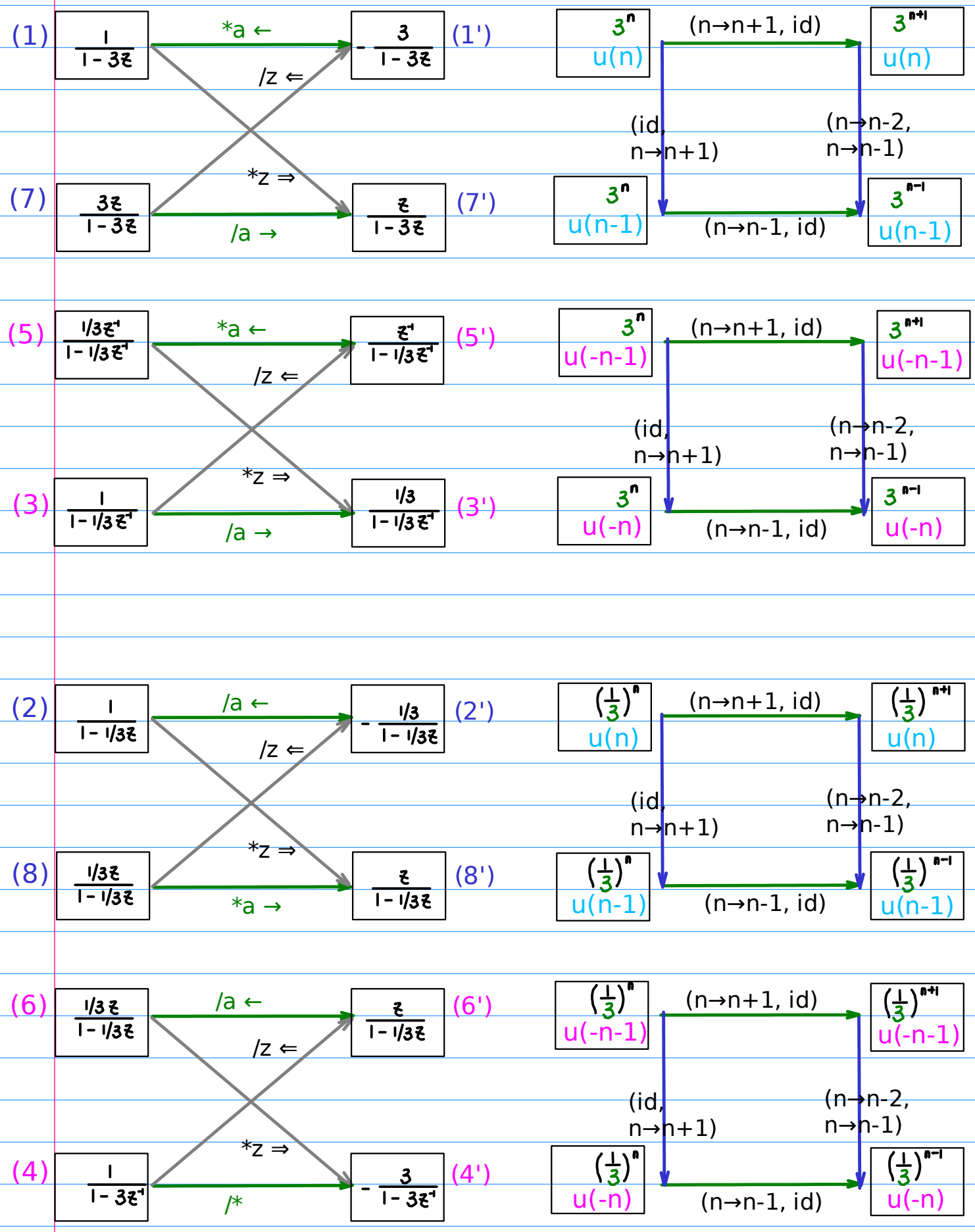
(4') $\dfrac{3}{1-3z^{-1}}$

$3^{-n+1}\ u(-n)$

(Exp, Range)

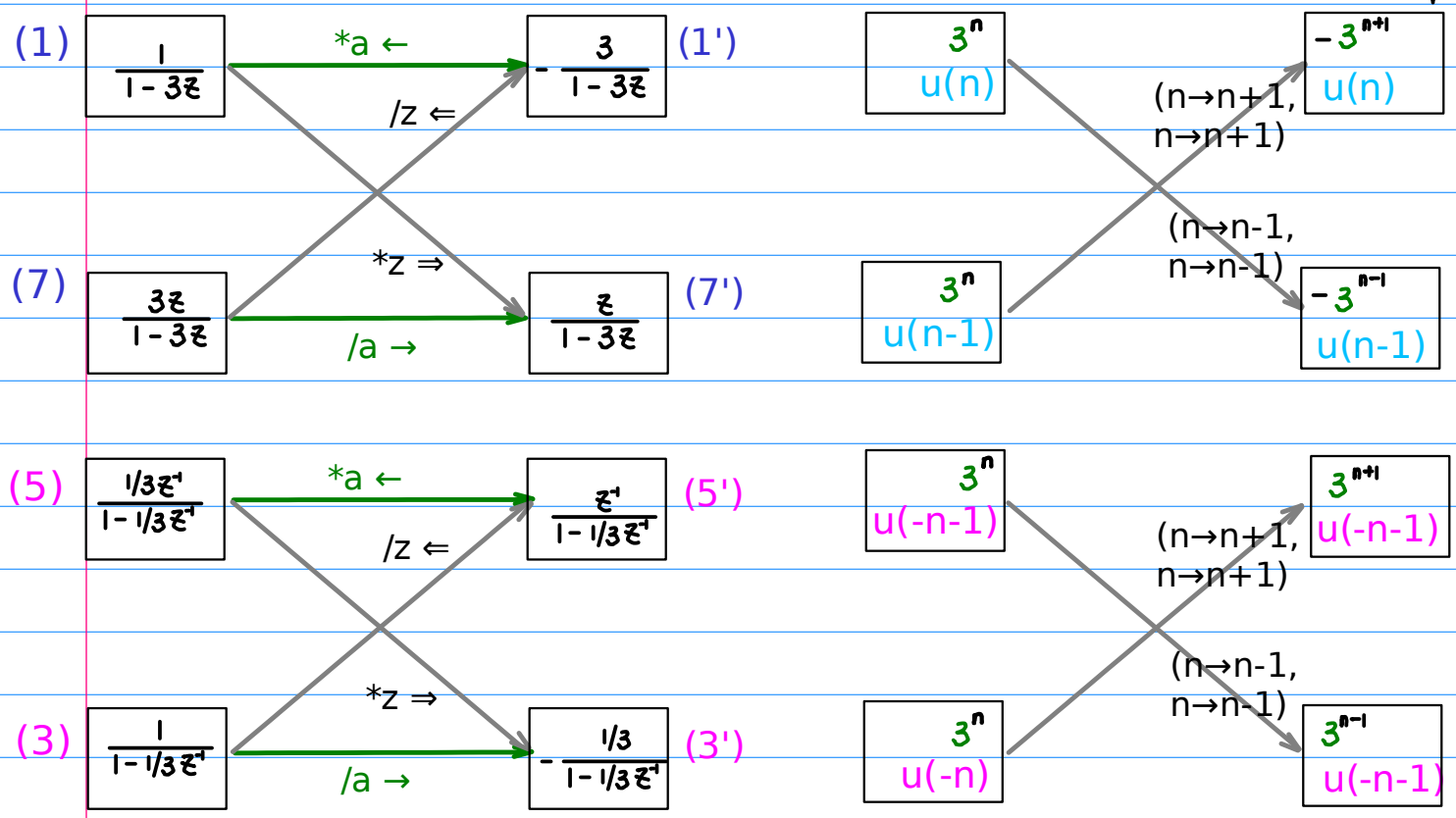
$P^{-1} = 1/3$

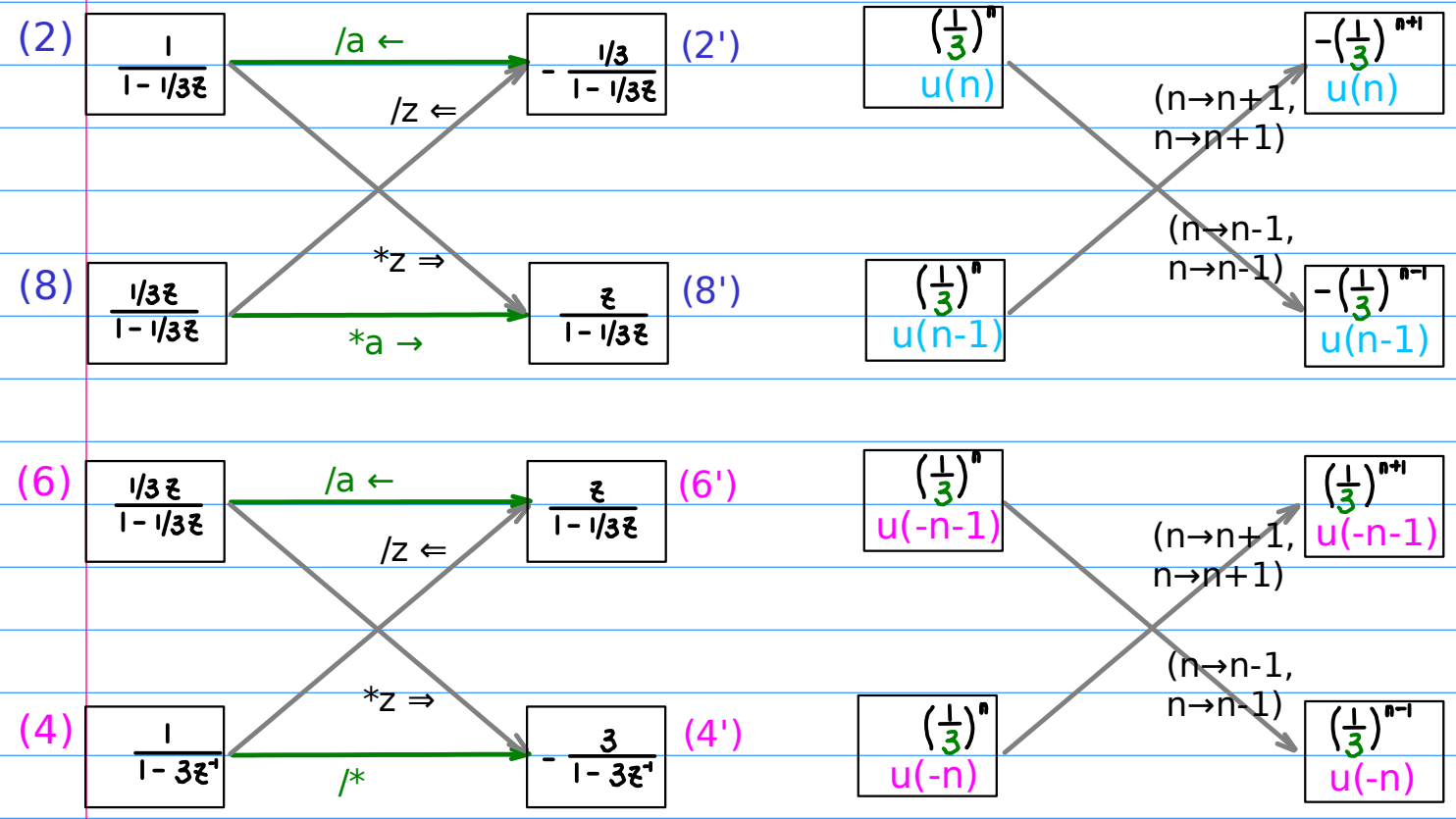
$P = 3$



$P^{-1} = 1/3$

$P = 3$


(1)
$\frac{1}{1-3z}$
*a ←
/z ⇐
$-\frac{3}{1-3z}$
(1')
(7)
$\frac{3z}{1-3z}$
*z ⇒
/a →
$\frac{z}{1-3z}$
(7')
$3^n u(n)$
$-3^{n+1} u(n)$
(n→n+1, n→n+1)
$3^n u(n-1)$
$-3^{n-1} u(n-1)$
(n→n-1, n→n-1)
(5)
$\frac{1/3z^{-1}}{1-1/3z^{-1}}$
*a ←
/z ⇐
$\frac{z^{-1}}{1-1/3z^{-1}}$
(5')
(3)
$\frac{1}{1-1/3z^{-1}}$
*z ⇒
/a →
$-\frac{1/3}{1-1/3z^{-1}}$
(3')
$3^n u(-n-1)$
$3^{n+1} u(-n-1)$
(n→n+1, n→n+1)
$3^n u(-n)$
$3^{n-1} u(-n-1)$
(n→n-1, n→n-1)


(2)
$\frac{1}{1-1/3z}$
/a ←
/z ⇐
$-\frac{1/3}{1-1/3z}$
(2')
(8)
$\frac{1/3z}{1-1/3z}$
*z ⇒
*a →
$\frac{z}{1-1/3z}$
(8')
$(\frac{1}{3})^n u(n)$
$-(\frac{1}{3})^{n+1} u(n)$
(n→n+1, n→n+1)
$(\frac{1}{3})^n u(n-1)$
$-(\frac{1}{3})^{n-1} u(n-1)$
(n→n-1, n→n-1)
(6)
$\frac{1/3z}{1-1/3z}$
/a ←
/z ⇐
$\frac{z}{1-1/3z}$
(6')
(4)
$\frac{1}{1-3z^{-1}}$
*z ⇒
/*
$-\frac{3}{1-3z^{-1}}$
(4')
$(\frac{1}{3})^n u(-n-1)$
$(\frac{1}{3})^{n+1} u(-n-1)$
(n→n+1, n→n+1)
$(\frac{1}{3})^n u(-n)$
$(\frac{1}{3})^{n-1} u(-n)$
(n→n-1, n→n-1)

Partial Fraction Decompositions

$p_1 = 1$
$p_2 = 2$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

$$= \frac{(z-2)-(z-1)}{(z-1)(z-2)}$$

$$= \frac{-1}{(z-1)(z-2)}$$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

$$= \frac{z^2-2z\ -2z^2+2z}{(z-1)(z-2)}$$

$$= \frac{-z^2}{(z-1)(z-2)}$$

$$-\frac{z}{(z-0.5)(z-2)} = \left(\frac{1}{z-1} - \frac{4}{z-2}\right)$$

$$= \frac{(z-2)-2(z-1)}{(z-1)(z-2)}$$

$$= \frac{-z}{(z-1)(z-2)}$$

Partial fractions as geometric power series

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

$P_1 = 1$
$P_2 = 2$

Simple Pole Form

$$\frac{1}{z-1}$$

/1 /z

$$-\frac{1}{1-z} \qquad \frac{z^{-1}}{1-z^{-1}}$$

Geometric Power Series Forms

Simple Pole Form

$$\frac{1}{z-2}$$

/2 /z

$$-\frac{0.5}{1-0.5z} \qquad \frac{z^{-1}}{1-2z^{-1}}$$

Geometric Power Series Forms

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

$P_1 = 1$
$P_2 = 2$

Simple Pole Form

$$\frac{z}{z-1}$$

/1 /z

$$-\frac{z}{1-z} \qquad \frac{1}{1-z^{-1}}$$

Geometric Power Series Forms

Simple Pole Form

$$\frac{2z}{z-2}$$

/2 /z

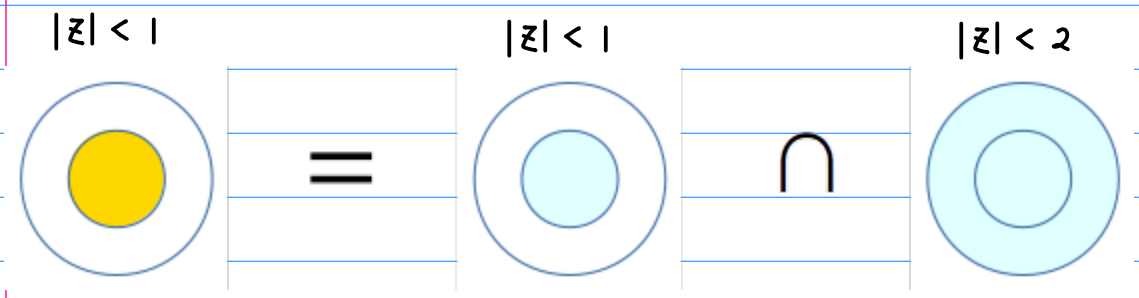
$$-\frac{z}{1-0.5z} \qquad \frac{2}{1-2z^{-1}}$$

Geometric Power Series Forms

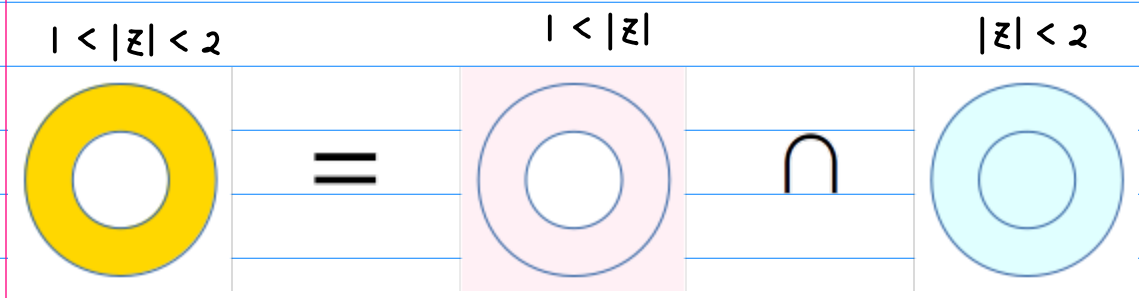
ROC Cases
for irreducible polynomials (z-1), (z-2)

$$\frac{\square}{(z-1)(z-2)}$$

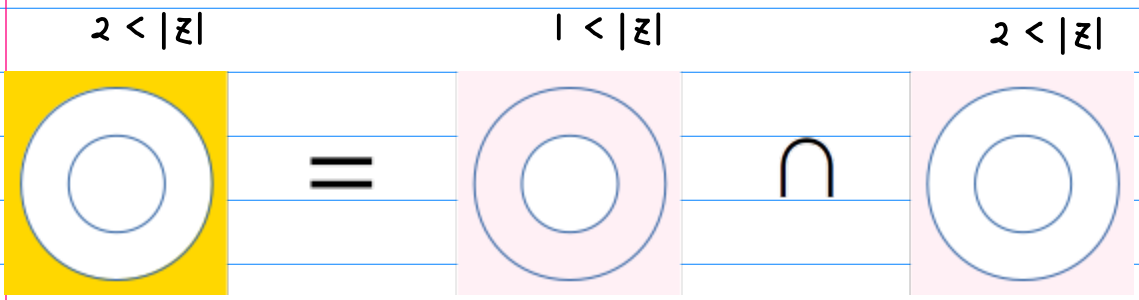
(A)



(B)



(C)



Finding a common ratio

$p_1 = 1$
$p_2 = 2$

$$-\frac{1}{(z-1)(z-2)} \qquad -\frac{z^2}{(z-1)(z-2)} \qquad -\frac{z}{(z-0.5)(z-2)}$$

Irreducible Polynomials

$(z-1)$ $(z-2)$

Geometric Power Series From

$$\frac{1}{(z-1)} \qquad \frac{1}{(z-2)}$$

$$\Downarrow \qquad \Downarrow$$

$$-\frac{1}{1-1z} \qquad -\frac{2^{-1}}{1-2^{-1}z}$$

$$\Downarrow \qquad \Downarrow$$

(1) $\boxed{1z}$ $\qquad$ $\boxed{2^{-1}z}$ (2)

$1^n\,u(n)$ $\qquad$ $2^{-n}\,u(n)$

Finding a geometric series

$P_1 = 1$
$P_2 = 2$

Irreducible Polynomials

Geometric Power Series From

$(z-1)$ $(z-2)$

$\frac{1}{(z-1)}$ $\frac{1}{(z-2)}$

$-\frac{1}{1-1z}$ $-\frac{2^{-1}}{1-2^{-1}z}$

$\Downarrow$ $\Downarrow$

$1 \cdot \frac{1}{1-1z}$ $2^{-1} \cdot \frac{1}{1-2^{-1}z}$

$a=1$ $a^{-1}=2^{-1}$

$1^n\,u(n)$ $2^{-n}\,u(n)$

(1) az $+\frac{1}{1-az}$ $+\frac{1}{1-a^{-1}z}$ $a^{-1}z$ (2)

(3) $a^{-1}z^{-1}$ $-\frac{1}{1-a^{-1}z^{-1}}$ $-\frac{1}{1-az^{-1}}$ az^{-1} (4)

(5) $a^{-1}z^{-1}$ $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $-\frac{az^{-1}}{1-az^{-1}}$ az^{-1} (6)

(7) az $+\frac{az}{1-az}$ $+\frac{a^{-1}z}{1-a^{-1}z}$ $a^{-1}z$ (8)

(1) (1')
(7) (7')

(2) (2')
(8) (8')

(5) (5')
(3) (3')

(6) (6')
(4) (4')

$P_1 = 1$

$P_2 = 2$

Irreducible Polynomials

Geometric Power Series From

$\frac{1}{(z-1)}$ $\frac{1}{(z-2)}$

$\rightarrow -\frac{1}{1-1z}$ $\rightarrow -\frac{2^{-1}}{1-2^{-1}z}$

$\Downarrow$ $\Downarrow$

$\rightarrow 1 \cdot \frac{1}{1-1z}$ $\rightarrow 2^{-1} \cdot \frac{1}{1-2^{-1}z}$

$a = 1$ $a^{-1} = 2^{-1}$

(1) $\frac{1}{1-az}$ *a ← $\frac{a}{1-az}$ — a^n u(n) — a^{n+1} u(n)

(2) $\frac{1}{1-a^{-1}z}$ /a ← $\frac{a^{-1}}{1-a^{-1}z}$ — a^{-n} u(n) — a^{-n-1} u(n)

(7) $\frac{az}{1-az}$ /a → $\frac{z}{1-az}$ — a^n u(n-1) — a^{n-1} u(n-1)

(8) $\frac{a^{-1}z}{1-a^{-1}z}$ *a → $\frac{z}{1-a^{-1}z}$ — a^{-n} u(n-1) — a^{-n+1} u(n-1)

(5) $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ *a ← $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ — a^n u(-n-1) — a^{n+1} u(-n-1)

(6) $-\frac{az^{-1}}{1-az^{-1}}$ /a ← $-\frac{z^{-1}}{1-az^{-1}}$ — a^{-n} u(-n-1) — a^{-n-1} u(-n-1)

(3) $-\frac{1}{1-a^{-1}z^{-1}}$ /a → $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ — a^n u(-n) — a^{n-1} u(-n)

(4) $-\frac{1}{1-az^{-1}}$ *a → $-\frac{a}{1-az^{-1}}$ — a^{-n} u(-n) — a^{-n+1} u(-n)

$a=1$

(1) (2)
(3) (4)
(5) (6)
(7) (8)

(1) $\frac{1}{1-|z} \xrightarrow{*1\leftarrow} \frac{1}{1-|z}$ $|z$ $|^{-1}z$ $\frac{|^{-1}}{1-|^{-1}z} \xleftarrow{/1} \frac{1}{1-|^{-1}z}$ (2)

$|^{n}u(n)$ $|^{n+1}u(n)$ $|^{-n}u(n)$ $|^{-n-1}u(n)$

(3) $\frac{1}{1-|^{-1}z^{-1}} \xrightarrow{/1\rightarrow} \frac{|^{-1}}{1-|^{-1}z^{-1}}$ $|^{-1}z^{-1}$ $|z^{-1}$ $\frac{1}{1-|z^{-1}} \xrightarrow{*1\rightarrow} \frac{1}{1-|z^{-1}}$ (4)

$|^{n}u(-n)$ $|^{n-1}u(-n)$ $|^{-n}u(-n)$ $|^{-n+1}u(-n)$

(5) $\frac{|^{-1}z^{-1}}{1-|^{-1}z^{-1}} \xrightarrow{*1\leftarrow} \frac{z^{-1}}{1-|^{-1}z^{-1}}$ $|^{-1}z^{-1}$ $|z^{-1}$ $\frac{|z^{-1}}{1-|z^{-1}} \xleftarrow{/1} \frac{z^{-1}}{1-|z^{-1}}$ (6)

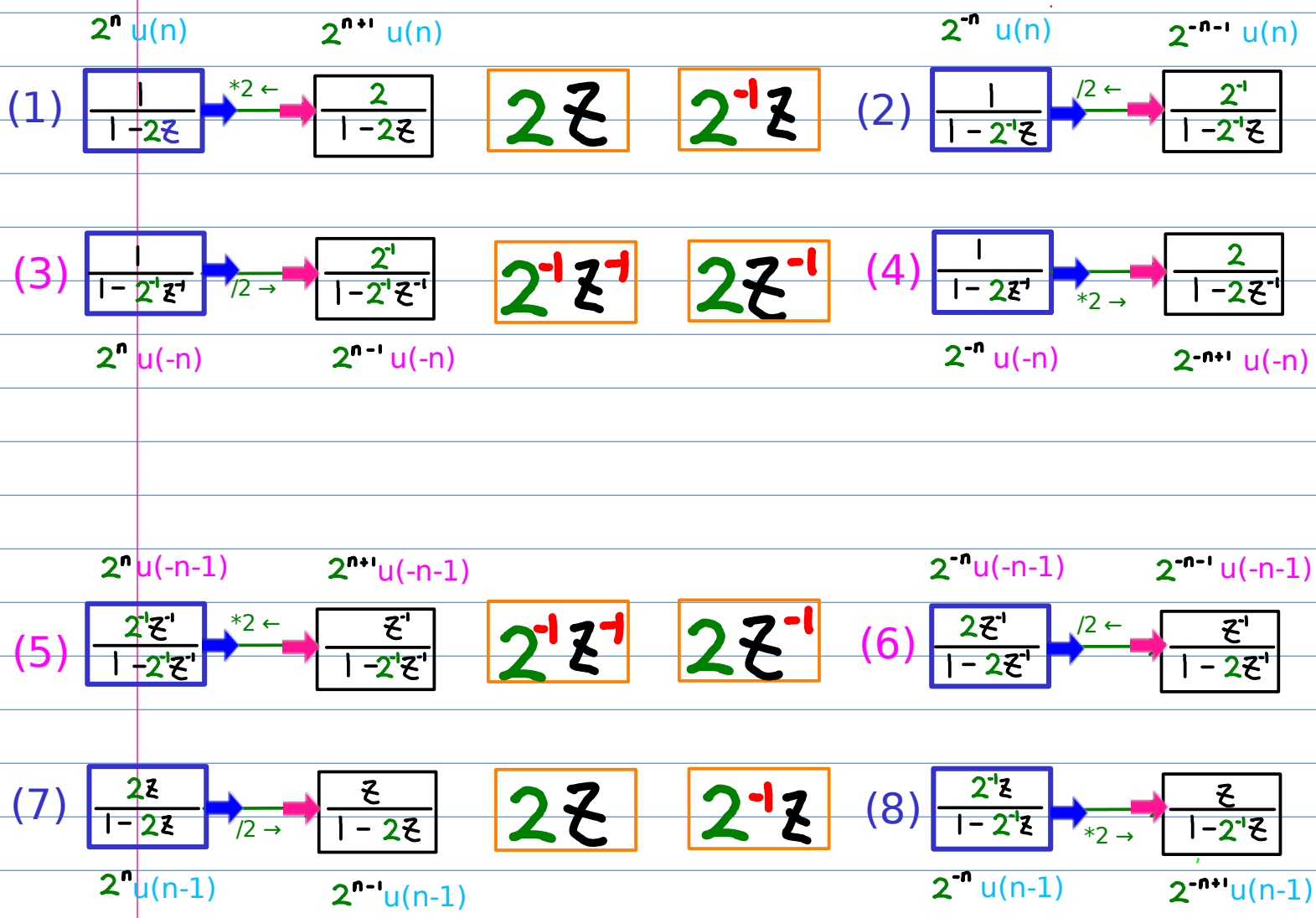
$|^{n}u(-n-1)$ $|^{n+1}u(-n-1)$ $|^{-n}u(-n-1)$ $|^{-n-1}u(-n-1)$

(7) $\frac{|z}{1-|z} \xrightarrow{/1\rightarrow} \frac{z}{1-|z}$ $|z$ $|^{-1}z$ $\frac{|^{-1}z}{1-|^{-1}z} \xrightarrow{*1\rightarrow} \frac{z}{1-|^{-1}z}$ (8)

$|^{n}u(n-1)$ $|^{n-1}u(n-1)$ $|^{-n}u(n-1)$ $|^{-n+1}u(n-1)$

$a=2$

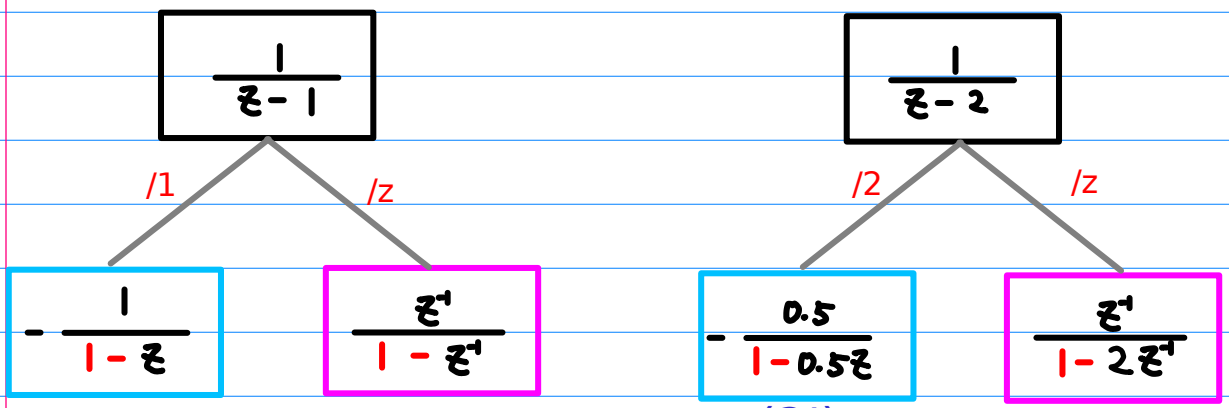
(1) (2)
(3) (4)
(5) (6)
(7) (8)


2^n u(n)
2^{n+1} u(n)
*2 ←
(1)
2z
2^{-1}z
2^{-n} u(n)
2^{-n-1} u(n)
/2 ←
(2)
2^n u(-n)
2^{n-1} u(-n)
/2 →
(3)
2^{-1}z^{-1}
2z^{-1}
2^{-n} u(-n)
2^{-n+1} u(-n)
*2 →
(4)
2^n u(-n-1)
2^{n+1} u(-n-1)
*2 ←
(5)
2^{-n} u(-n-1)
2^{-n-1} u(-n-1)
/2 ←
(6)
2^n u(n-1)
2^{n-1} u(n-1)
/2 →
(7)
2^{-n} u(n-1)
2^{-n+1} u(n-1)
*2 →
(8)

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

$P_1 = 1$

$P_2 = 2$

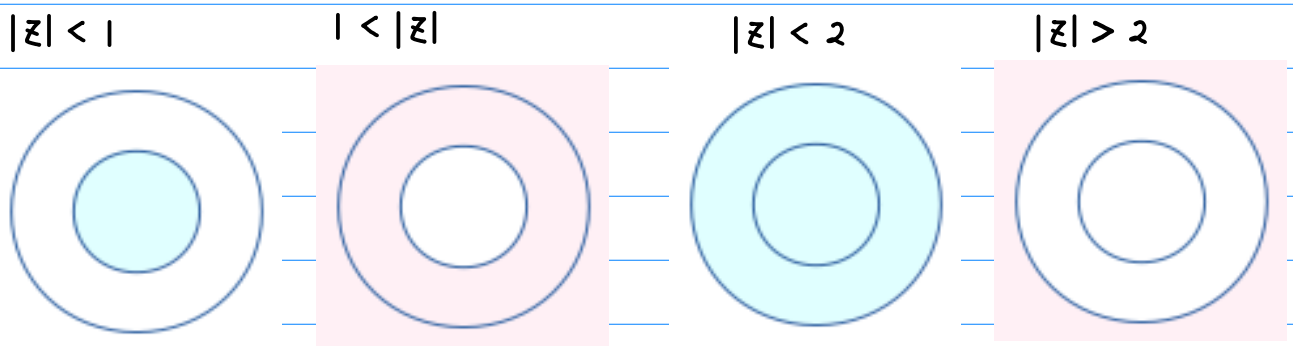


(1') (5') (2') (6')

Complementary Ranges

Complementary Ranges

Case A	$\lvert z\rvert < 1$	(1')	(2')
Case B	$1 < \lvert z\rvert < 2$	(5')	(2')
Case C	$2 < \lvert z\rvert$	(5')	(6')

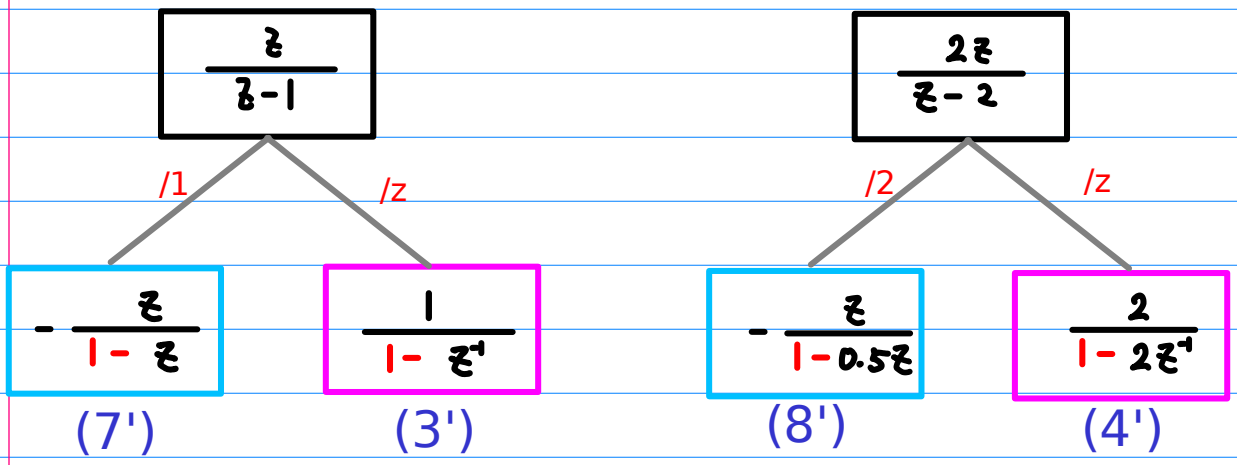


$-\left(\frac{1}{1}\right)^{n+1} u(n)$ $\left(\frac{1}{1}\right)^{n+1} u(-n-1)$ $-\left(\frac{1}{2}\right)^{n+1} u(n)$ $\left(\frac{1}{2}\right)^{n+1} u(-n-1)$

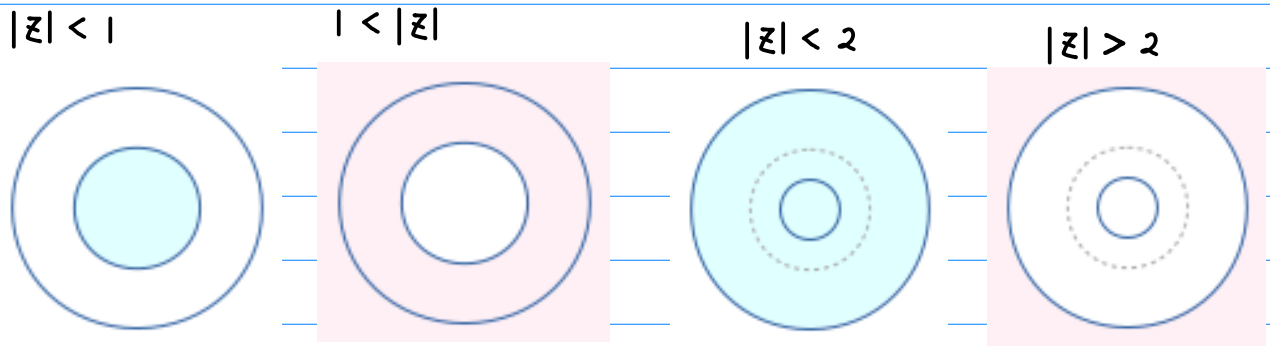
$-1 \cdot \frac{1}{1-1z}$ $1 \cdot \frac{1z^{-1}}{1-1z^{-1}}$ $-0.5 \cdot \frac{1}{1-0.5z}$ $0.5 \cdot \frac{2z^{-1}}{1-2z^{-1}}$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

$P_1 = 1$
$P_2 = 2$



Case A	$\lvert z\rvert < 1$	(7')	(8')
Case B	$1 < \lvert z\rvert < 2$	(3')	(8')
Case C	$2 < \lvert z\rvert$	(3')	(4')



$-\left(\frac{1}{1}\right)^{n+1} u(n)$ $\quad \left(\frac{1}{1}\right)^{n+1} u(-n-1)$ $\quad -\left(\frac{1}{2}\right)^{n-1} u(n-1)$ $\quad \left(\frac{1}{2}\right)^{n-1} u(-n)$

$-1 \cdot \frac{1}{1-1z}$ $\quad 1 \cdot \frac{1z^{-1}}{1-1z^{-1}}$ $\quad -2 \cdot \frac{0.5z}{1-0.5z}$ $\quad 2 \cdot \frac{1}{1-2z^{-1}}$

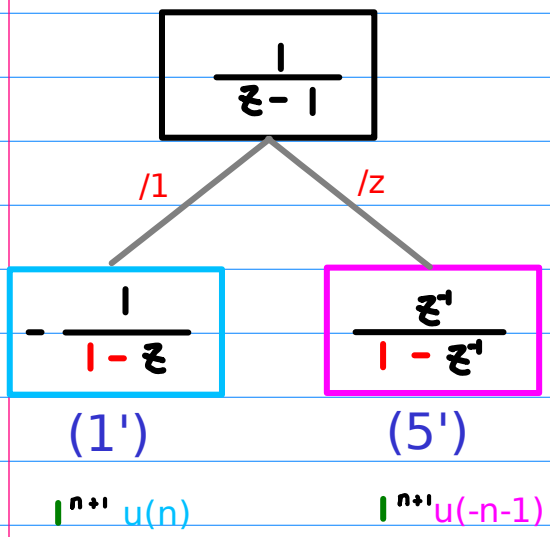
Partial fractions and geometric power series

$P_1 = 1$
$P_2 = 2$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

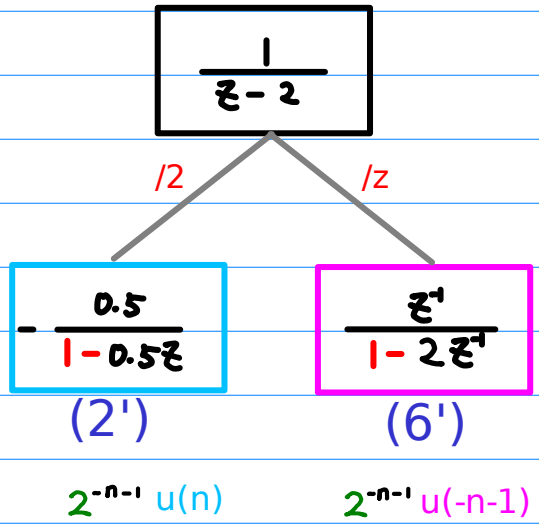
$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

$P_1 = 1$

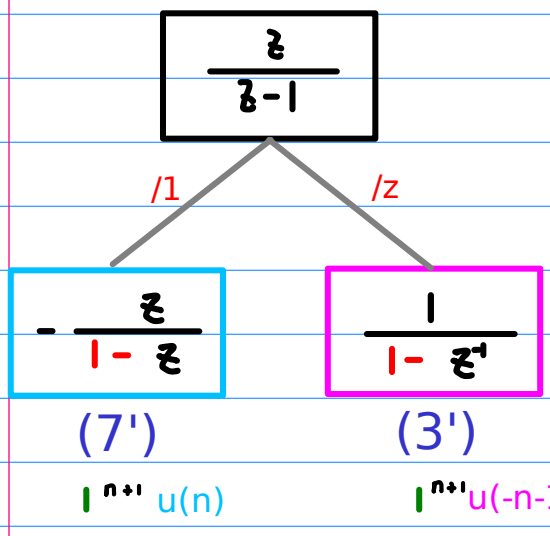


$1^{n+1}\,u(n)$ $\quad 1^{n+1}u(-n-1)$

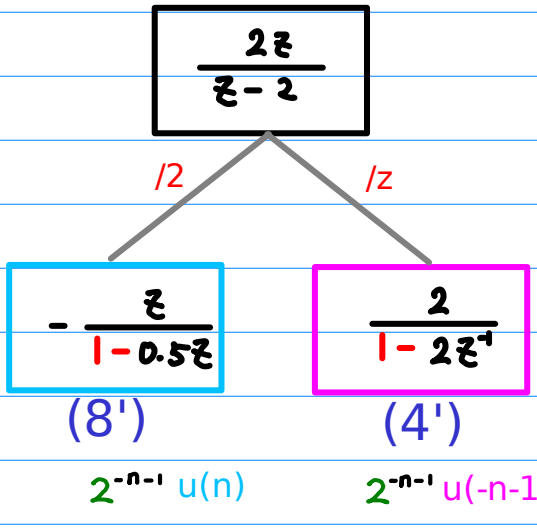
$P_2 = 2$



$2^{-n-1}\,u(n)$ $\quad 2^{-n-1}\,u(-n-1)$



$1^{n+1}\,u(n)$ $\quad 1^{n+1}u(-n-1)$



$2^{-n-1}\,u(n)$ $\quad 2^{-n-1}\,u(-n-1)$

Reciprocal Pole Relation (p1=1, p2=2)

$$\frac{-1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

z^{-1}

$$\frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \left(\frac{1}{z^{-1}-1} - \frac{1}{z^{-1}-2}\right)$$

$$\frac{-z^2}{(1-1\cdot z)(1-2z)} = \left(\frac{z}{(1-1\cdot z)} - \frac{z}{(1-2z)}\right)$$

$$\frac{-z^2 \cdot 1 \cdot 0.5}{(1-z)(0.5-z)} = \left(\frac{z}{(1-z)} - \frac{0.5z}{(0.5-z)}\right)$$

$$\frac{1}{2}\,\frac{-z^2}{(z-1)(z-0.5)} = \left(-\frac{z}{z-1} + \frac{0.5z}{z-0.5}\right)$$

$z-1 \qquad -z+1$

$$\frac{1}{2}\frac{-1}{(z-1)(z-1)} = \left(\frac{1}{z-1} - \frac{1}{z-1}\right)$$

$$\frac{1}{2}\frac{-1}{(z^{-1}-1)(z^{-1}-1)} = \left(\frac{1}{z^{-1}-1} - \frac{1}{z^{-1}-1}\right)$$

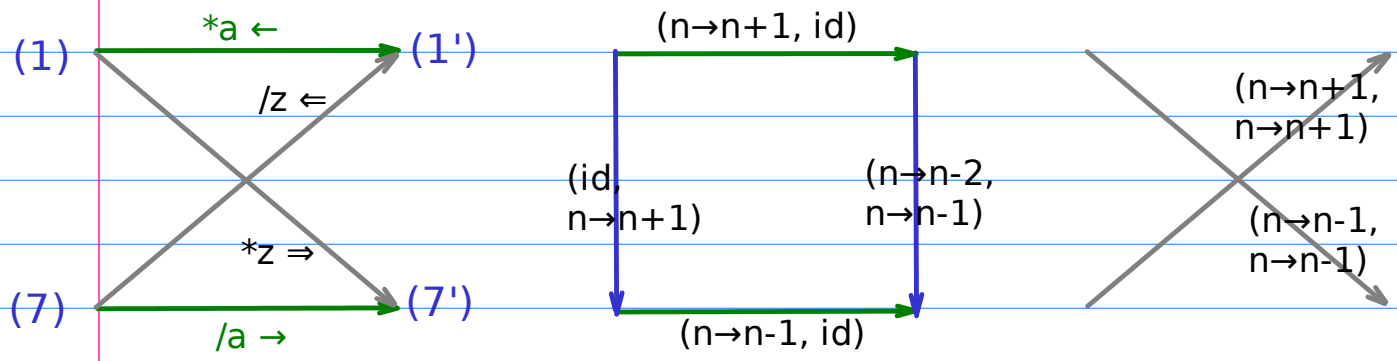
z^{-1}

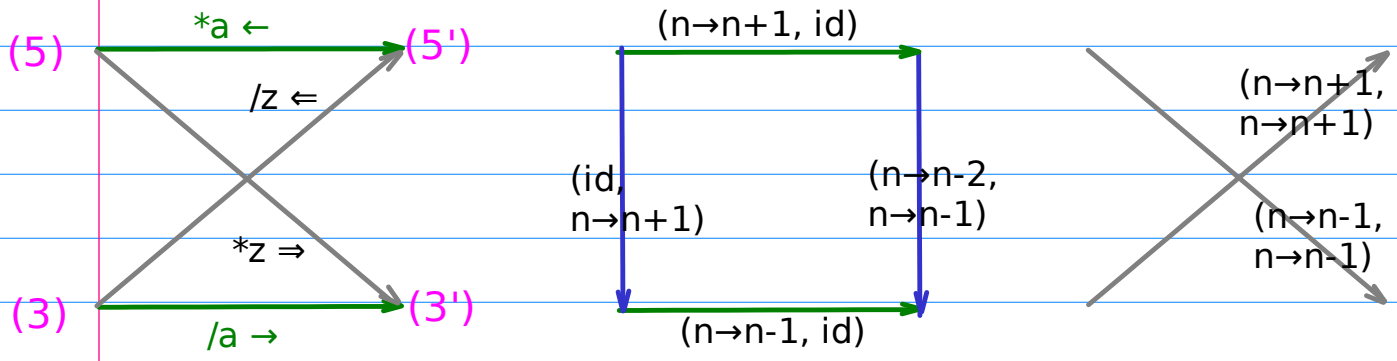
$$\frac{1}{2}\frac{-z^2}{(1-1\cdot z)(1-2z)} = \left(\frac{z}{(1-1\cdot z)} - \frac{z}{(1-2z)}\right)$$

$$\frac{1}{2}\frac{-z^2\cdot 1\cdot 0.5}{(1-z)(0.5-z)} = \left(\frac{z}{(1-z)} - \frac{0.5z}{(0.5-z)}\right)$$

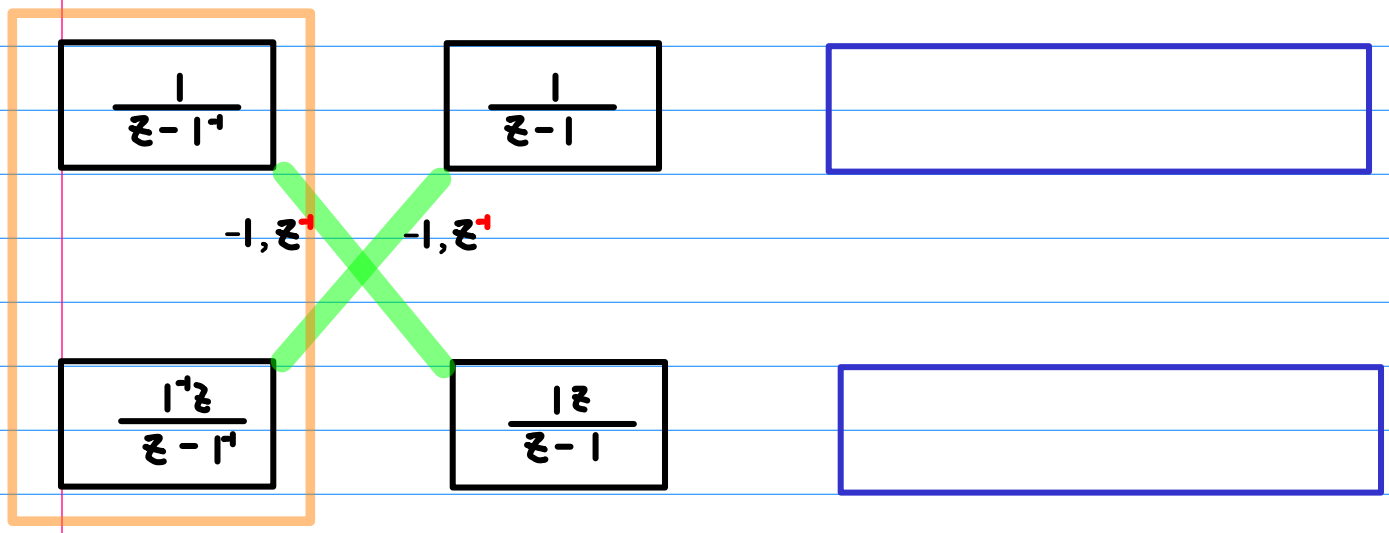
$$\frac{1}{2}\frac{-z^2}{(z-1)(z-0.5)} = \left(-\frac{z}{z-1} + \frac{0.5z}{z-0.5}\right)$$

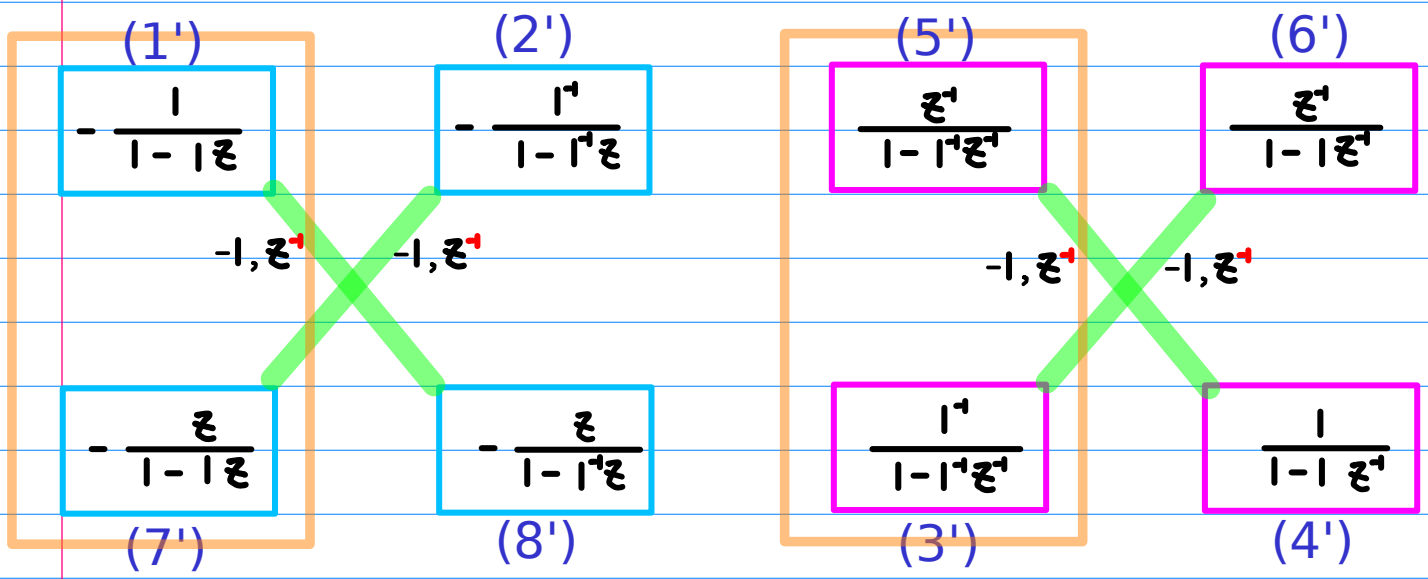
(Exp, Range)

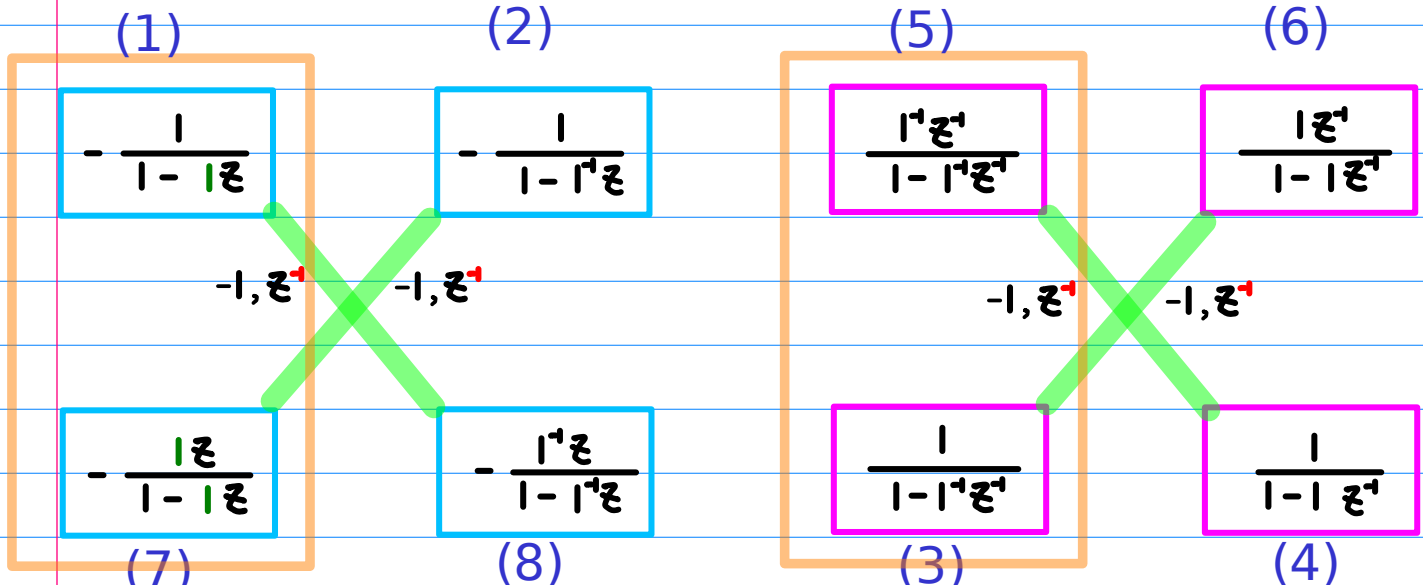

(1)
*a ←
(1')
/z ⇐
*z ⇒
(7)
(7')
/a →
(n→n+1, id)
(id,
n→n+1)
(n→n-2,
n→n-1)
(n→n-1, id)
(n→n+1,
n→n+1)
(n→n-1,
n→n-1)


(5)
*a ←
(5')
/z ⇐
*z ⇒
(3)
(3')
/a →
(n→n+1, id)
(id,
n→n+1)
(n→n-2,
n→n-1)
(n→n-1, id)
(n→n+1,
n→n+1)
(n→n-1,
n→n-1)

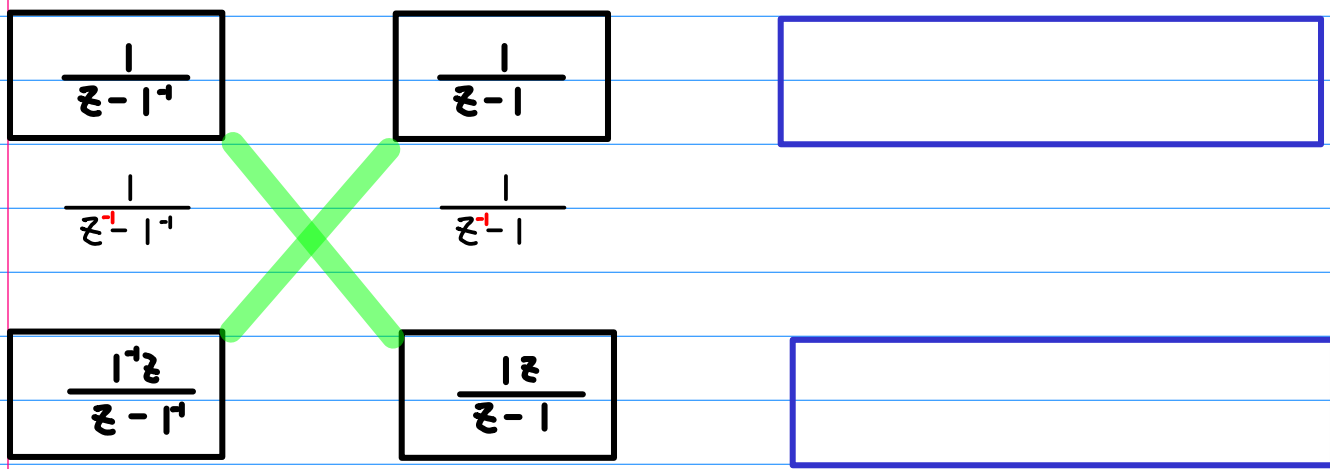
$P_1 = 1$ $P_2 = 1$

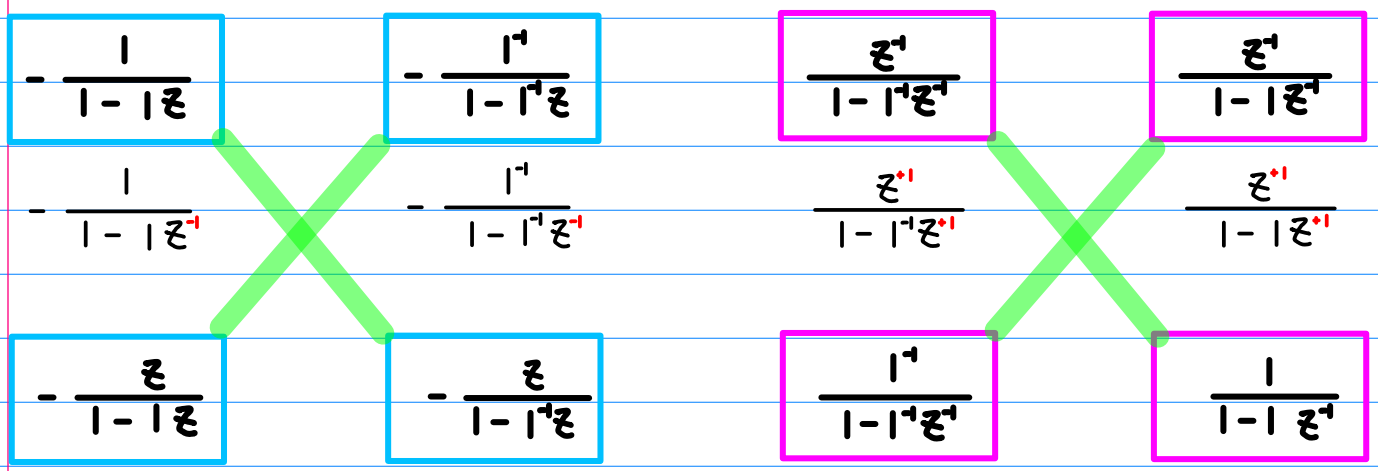

$\frac{1}{z-1^{-1}}$
$\frac{1}{z-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$\frac{1^{-1}z}{z-1^{-1}}$
$\frac{1z}{z-1}$

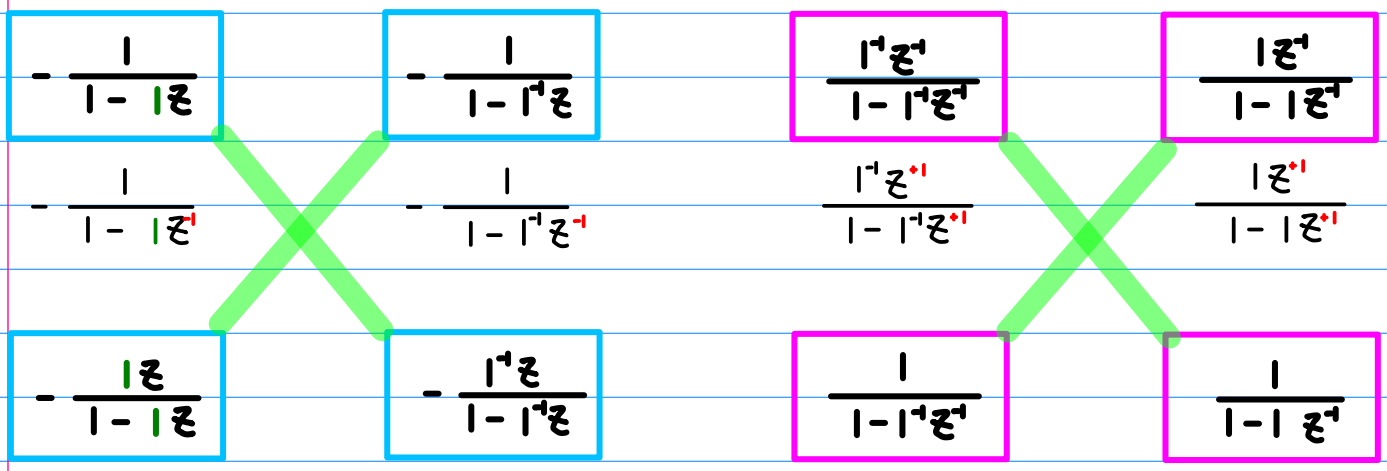

(1')
(2')
(5')
(6')
$-\frac{1}{1-1z}$
$-\frac{1^{-1}}{1-1^{-1}z}$
$\frac{z^{-1}}{1-1^{-1}z^{-1}}$
$\frac{z^{-1}}{1-1z^{-1}}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-\frac{z}{1-1z}$
$-\frac{z}{1-1^{-1}z}$
$\frac{1^{-1}}{1-1^{-1}z^{-1}}$
$\frac{1}{1-1z^{-1}}$
(7')
(8')
(3')
(4')


(1)
(2)
(5)
(6)
$-\frac{1}{1-1z}$
$-\frac{1}{1-1^{-1}z}$
$\frac{1^{-1}z^{-1}}{1-1^{-1}z^{-1}}$
$\frac{1z^{-1}}{1-1z^{-1}}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-\frac{1z}{1-1z}$
$-\frac{1^{-1}z}{1-1^{-1}z}$
$\frac{1}{1-1^{-1}z^{-1}}$
$\frac{1}{1-1z^{-1}}$
(7)
(8)
(3)
(4)

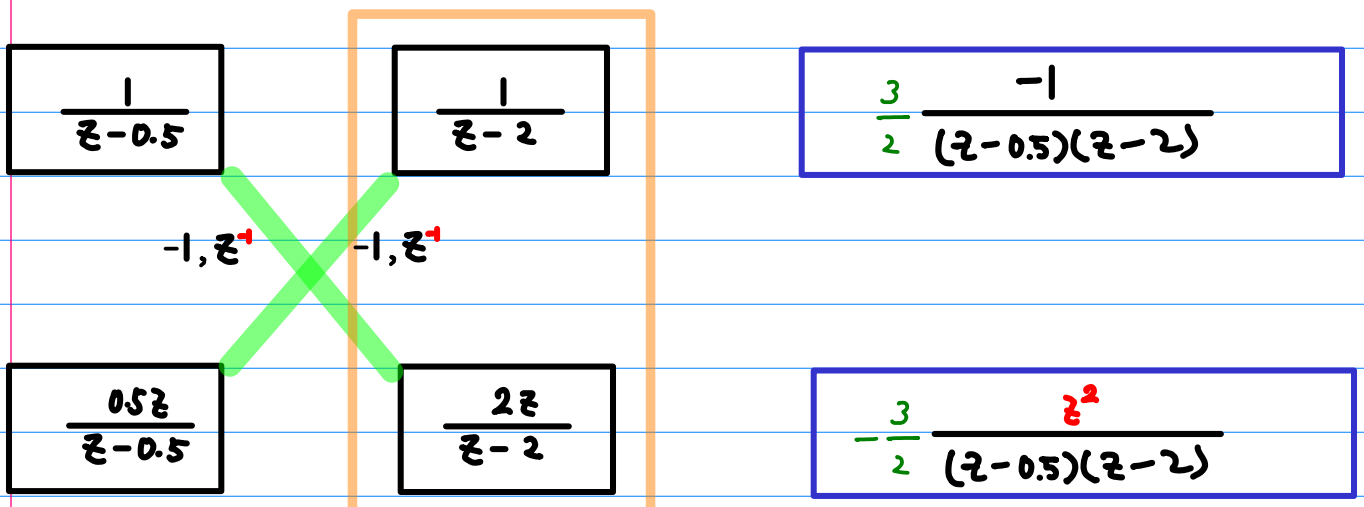
$P_1 = 1$ $P_2 = 1$

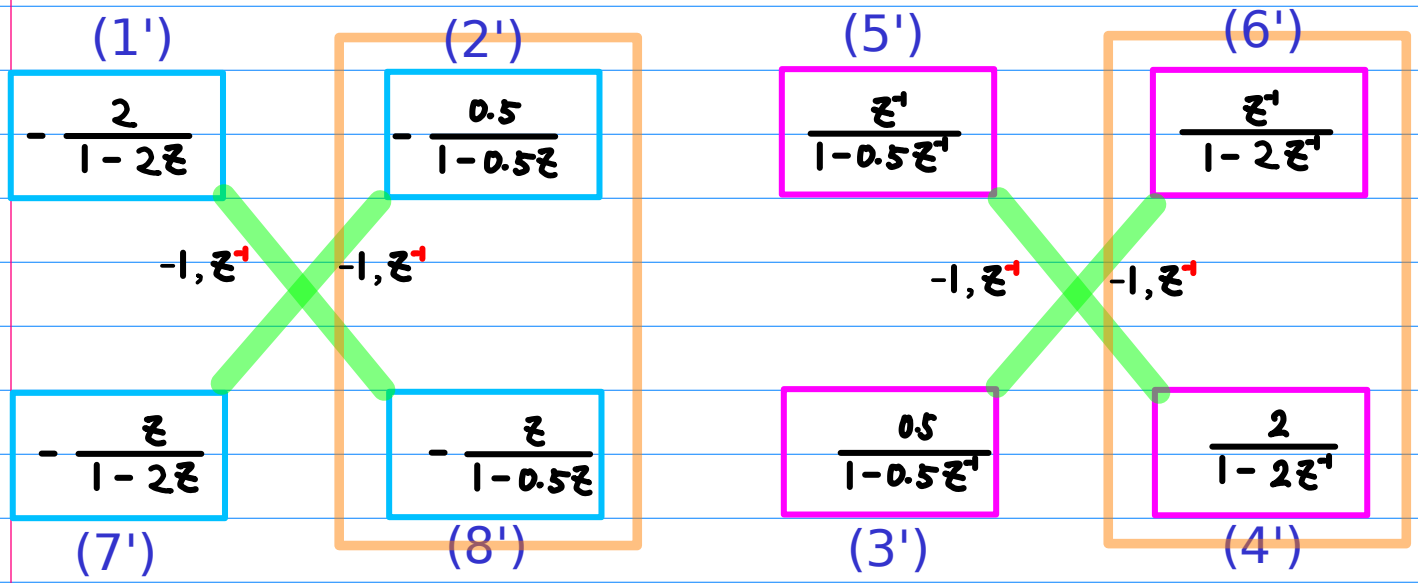

$\frac{1}{z - 1^{-1}}$
$\frac{1}{z - 1}$
$\frac{1}{z^{-1} - 1^{-1}}$
$\frac{1}{z^{-1} - 1}$
$\frac{1^{-1}z}{z - 1^{-1}}$
$\frac{1z}{z - 1}$

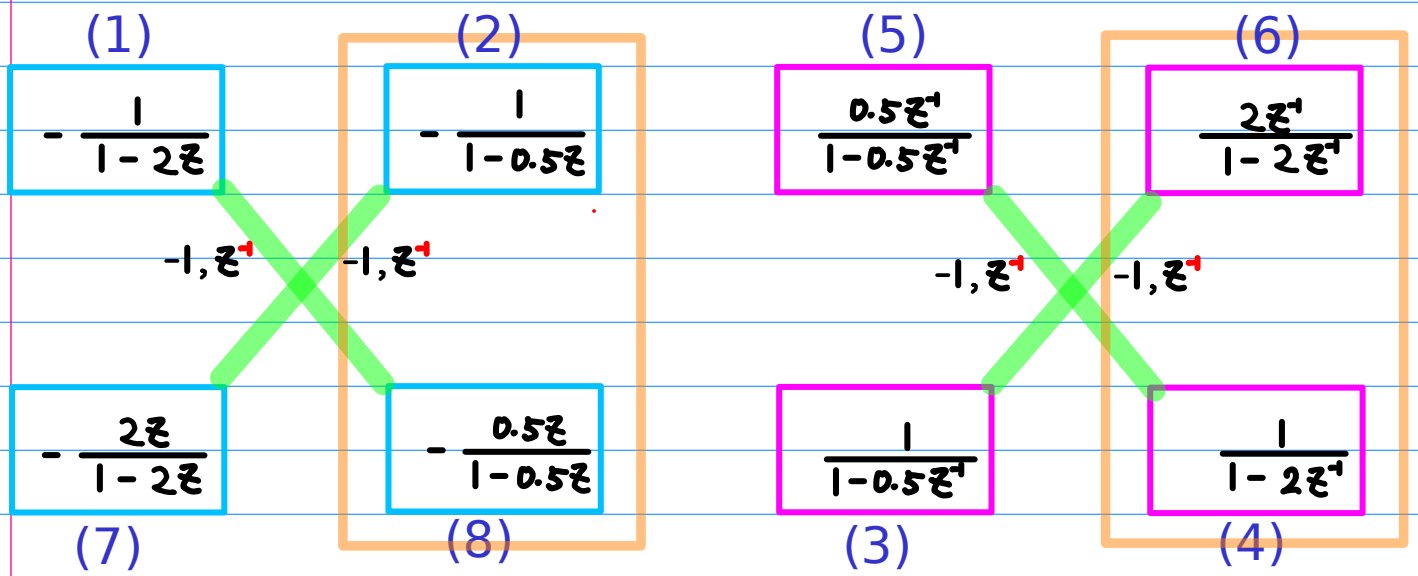

$-\frac{1}{1 - 1z}$
$-\frac{1^{-1}}{1 - 1^{-1}z}$
$\frac{z^{-1}}{1 - 1^{-1}z^{-1}}$
$\frac{z^{-1}}{1 - 1z^{-1}}$
$-\frac{1}{1 - 1z^{-1}}$
$-\frac{1^{-1}}{1 - 1^{-1}z^{-1}}$
$\frac{z^{+1}}{1 - 1^{-1}z^{+1}}$
$\frac{z^{+1}}{1 - 1z^{+1}}$
$-\frac{z}{1 - 1z}$
$-\frac{z}{1 - 1^{-1}z}$
$\frac{1^{-1}}{1 - 1^{-1}z^{-1}}$
$\frac{1}{1 - 1z^{-1}}$


$-\frac{1}{1 - 1z}$
$-\frac{1}{1 - 1^{-1}z}$
$\frac{1^{-1}z^{-1}}{1 - 1^{-1}z^{-1}}$
$\frac{1z^{-1}}{1 - 1z^{-1}}$
$-\frac{1}{1 - 1z^{-1}}$
$-\frac{1}{1 - 1^{-1}z^{-1}}$
$\frac{1^{-1}z^{+1}}{1 - 1^{-1}z^{+1}}$
$\frac{1z^{+1}}{1 - 1z^{+1}}$
$-\frac{1z}{1 - 1z}$
$-\frac{1^{-1}z}{1 - 1^{-1}z}$
$\frac{1}{1 - 1^{-1}z^{-1}}$
$\frac{1}{1 - 1z^{-1}}$

$P_1 = 0.5$ $P_2 = 2$


$\frac{1}{z-0.5}$
$\frac{1}{z-2}$
$-1, z^{-1}$
$-1, z^{-1}$
$\frac{0.5z}{z-0.5}$
$\frac{2z}{z-2}$
$\frac{3}{2}\frac{-1}{(z-0.5)(z-2)}$
$-\frac{3}{2}\frac{z^2}{(z-0.5)(z-2)}$


(1')
(2')
(5')
(6')
$-\frac{2}{1-2z}$
$-\frac{0.5}{1-0.5z}$
$\frac{z^{-1}}{1-0.5z^{-1}}$
$\frac{z^{-1}}{1-2z^{-1}}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-\frac{z}{1-2z}$
$-\frac{z}{1-0.5z}$
$\frac{0.5}{1-0.5z^{-1}}$
$\frac{2}{1-2z^{-1}}$
(7')
(8')
(3')
(4')


(1)
(2)
(5)
(6)
$-\frac{1}{1-2z}$
$-\frac{1}{1-0.5z}$
$\frac{0.5z^{-1}}{1-0.5z^{-1}}$
$\frac{2z^{-1}}{1-2z^{-1}}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-1, z^{-1}$
$-\frac{2z}{1-2z}$
$-\frac{0.5z}{1-0.5z}$
$\frac{1}{1-0.5z^{-1}}$
$\frac{1}{1-2z^{-1}}$
(7)
(8)
(3)
(4)

$P_1 = 0.5 \qquad P_2 = 2$

$\frac{1}{z-0.5}$ $\qquad$ $\frac{1}{z-2}$ $\qquad$ $\frac{3}{2}\frac{-1}{(z-0.5)(z-2)}$

$\frac{1}{z^{-1}-0.5}$ $\qquad$ $\frac{1}{z^{-1}-2}$

$\frac{0.5z}{z-0.5}$ $\qquad$ $\frac{2z}{z-2}$ $\qquad$ $-\frac{3}{2}\frac{z^2}{(z-0.5)(z-2)}$

$-\frac{2}{1-2z}$ $\qquad$ $-\frac{0.5}{1-0.5z}$ $\qquad$ $\frac{z^{-1}}{1-0.5z^{-1}}$ $\qquad$ $\frac{z^{-1}}{1-2z^{-1}}$

$-\frac{2}{1-2z^{-1}}$ $\qquad$ $-\frac{0.5}{1-0.5z^{-1}}$ $\qquad$ $\frac{z^{+1}}{1-0.5z^{+1}}$ $\qquad$ $\frac{z^{+1}}{1-2z^{+1}}$

$-\frac{z}{1-2z}$ $\qquad$ $-\frac{z}{1-0.5z}$ $\qquad$ $\frac{0.5}{1-0.5z^{-1}}$ $\qquad$ $\frac{2}{1-2z^{-1}}$

$-\frac{1}{1-2z}$ $\qquad$ $-\frac{1}{1-0.5z}$ $\qquad$ $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $\qquad$ $\frac{2z^{-1}}{1-2z^{-1}}$

$-\frac{1}{1-2z^{-1}}$ $\qquad$ $-\frac{1}{1-0.5z^{-1}}$ $\qquad$ $\frac{0.5z^{+1}}{1-0.5z^{+1}}$ $\qquad$ $\frac{2z^{+1}}{1-2z^{+1}}$

$-\frac{2z}{1-2z}$ $\qquad$ $-\frac{0.5z}{1-0.5z}$ $\qquad$ $\frac{1}{1-0.5z^{-1}}$ $\qquad$ $\frac{1}{1-2z^{-1}}$

(1) (5)
(7) (3)

(2) (6)
(8) (4)

(1') (5')
(7') (3')

(2')(6')
(8')(4')

Partial fractions and geometric power series

$P_1 = 1$

$P_2 = 2$

(1) $-\frac{1}{1-1z}$ $1^n\ u(n)$

(5) $\frac{1z^{-1}}{1-1z^{-1}}$ $1^n\ u(-n-1)$

(2) $-\frac{1}{1-0.5z}$ $2^{-n}\ u(n)$

(6) $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n}\ u(-n-1)$

(7) $-\frac{1z}{1-1z}$ $1^n\ u(n-1)$

(3) $\frac{1}{1-1z^{-1}}$ $1^n\ u(-n)$

(8) $-\frac{0.5z}{1-0.5z}$ $2^{-n}\ u(n-1)$

(4) $\frac{1}{1-2z^{-1}}$ $2^{-n}\ u(-n)$

(1') $-\frac{1}{1-1z}$ $1^{n+1}\ u(n)$

(5') $\frac{z^{-1}}{1-1z^{-1}}$ $1^{n+1}u(-n-1)$

(2') $-\frac{0.5}{1-0.5z}$ $2^{-n-1}\ u(n)$

(6') $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1}\ u(-n-1)$

(7') $-\frac{z}{1-1z}$ $1^{n-1}u(n-1)$

(3') $\frac{1}{1-1z^{-1}}$ $1^{n-1}\ u(-n)$

(8') $-\frac{z}{1-0.5z}$ $2^{-n+1}u(n-1)$

(4') $\frac{2}{1-2z^{-1}}$ $2^{-n+1}\ u(-n)$

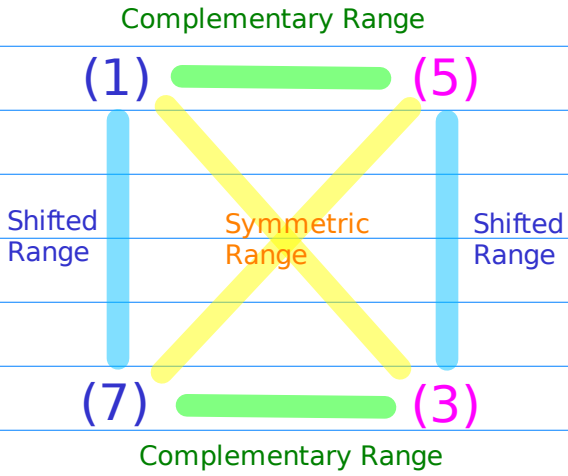
(1) (5)
(7) (3)

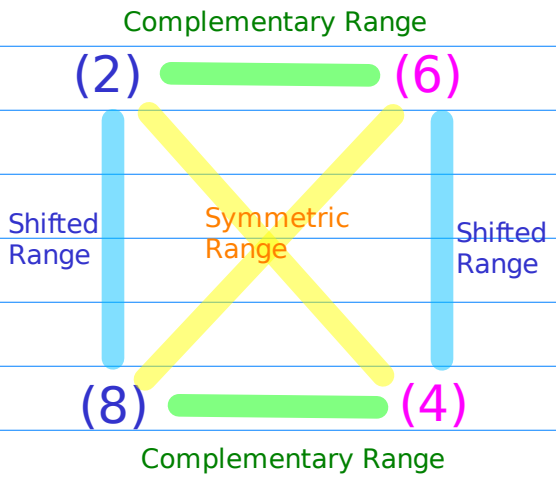
(2) (6)
(8) (4)

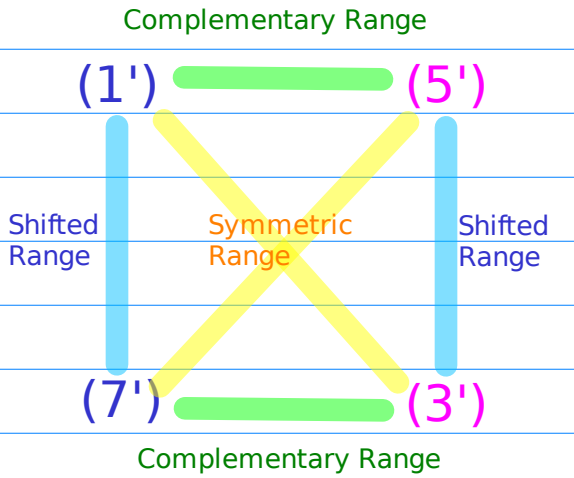
(1') (5')
(7') (3')

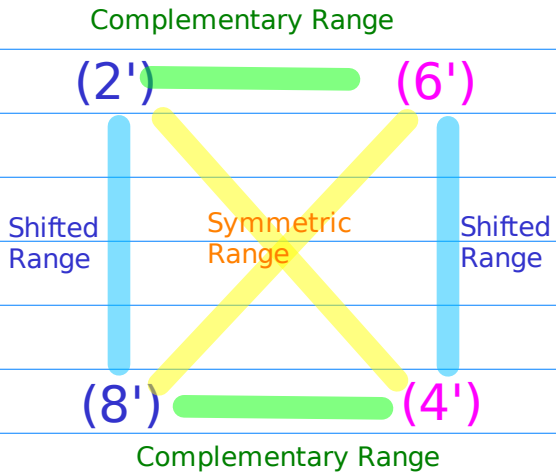
(2') (6')
(8') (4')

Complementary, Shifted, Symmetric Ranges



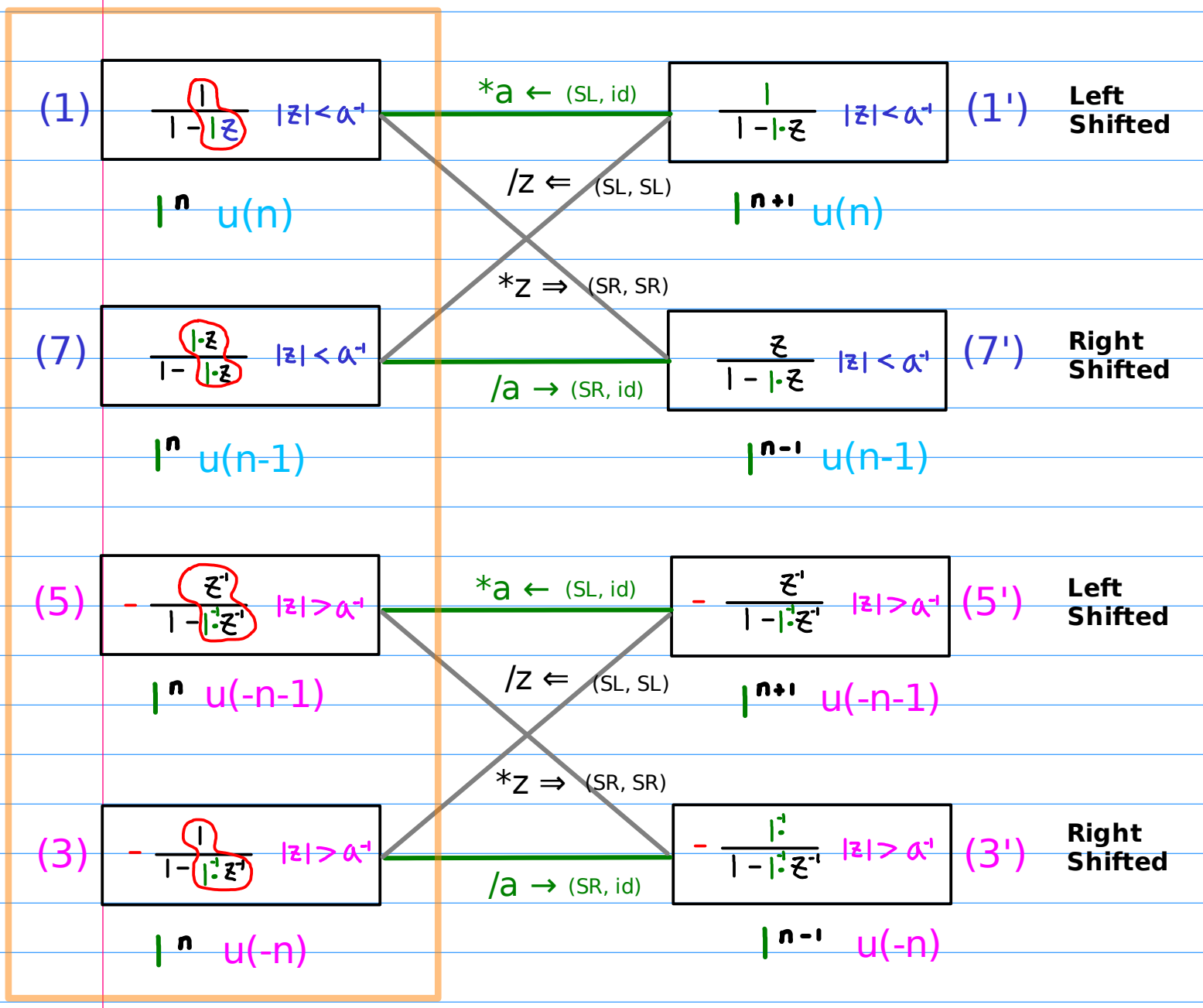






Shifting Geometric Series (1) positive exponent

Positive Exponent $a=1$ /z $n \leftarrow n+1$ *z $n \leftarrow n-1$

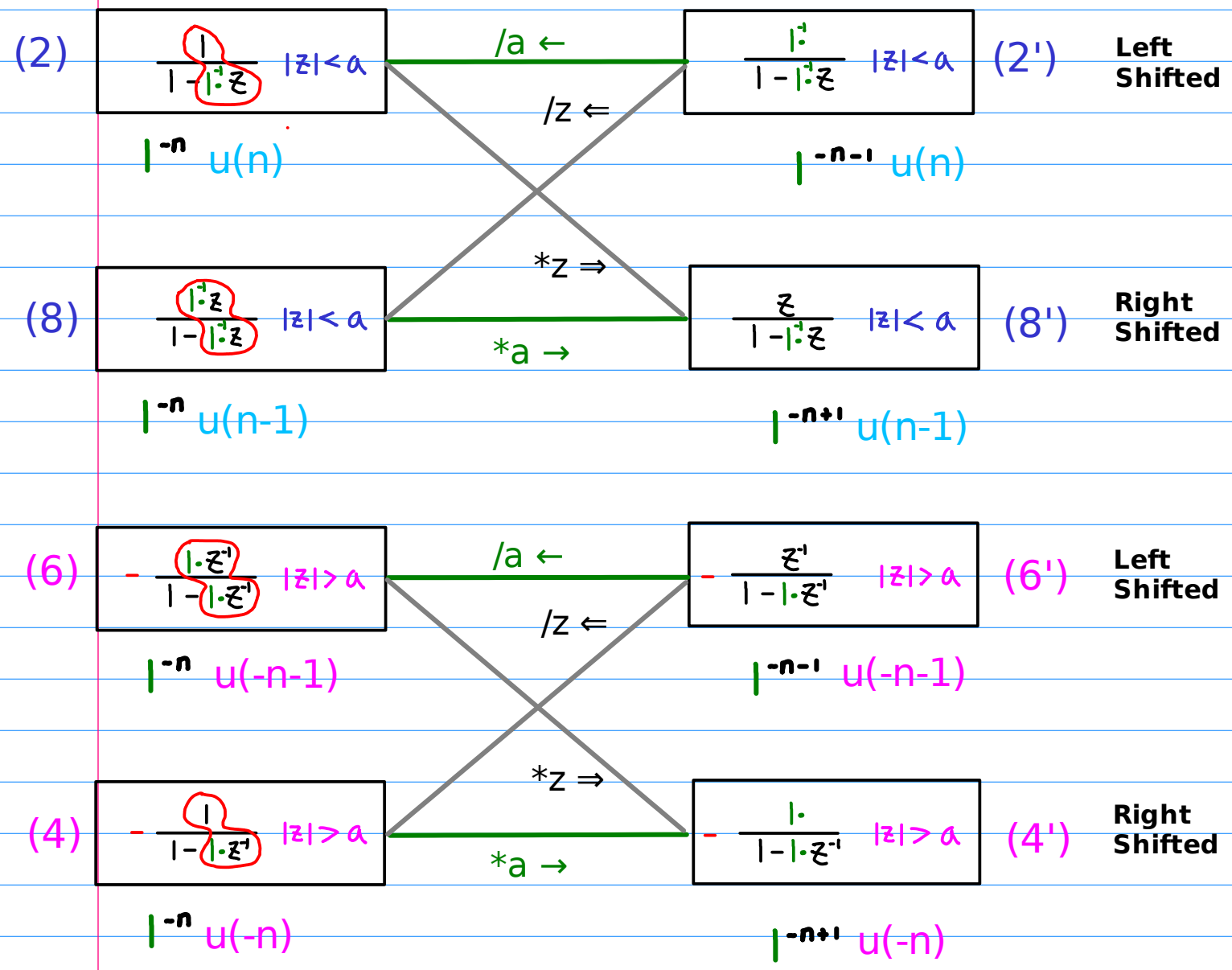


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series (2) negative exponent

Negative Exponent $a^{-1}=1^{-1}$ /z n ← n+1 *z n ← n-1

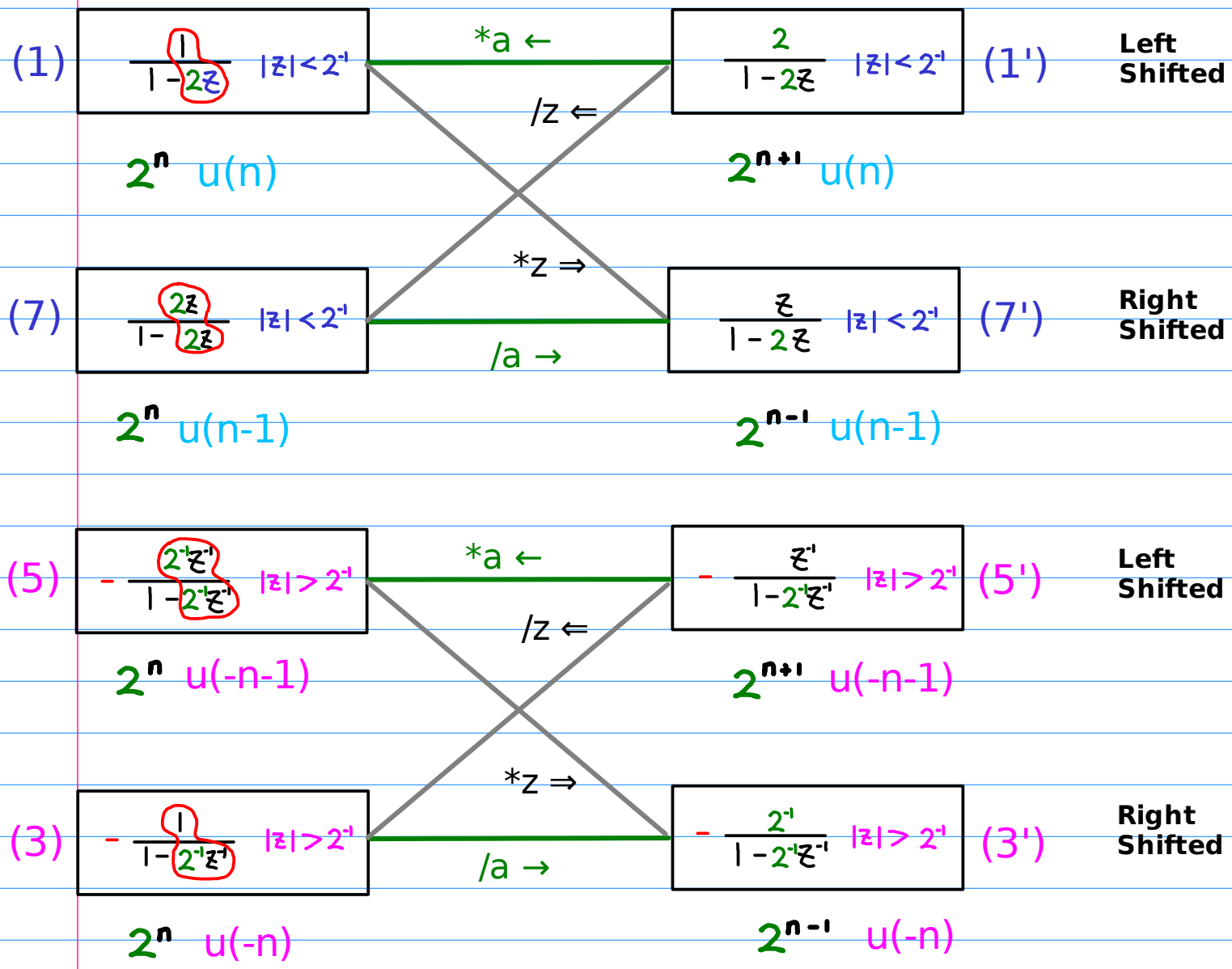


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series (1) positive exponent

Positive Exponent $a=2$ /z n ← n+1 *z n ← n-1

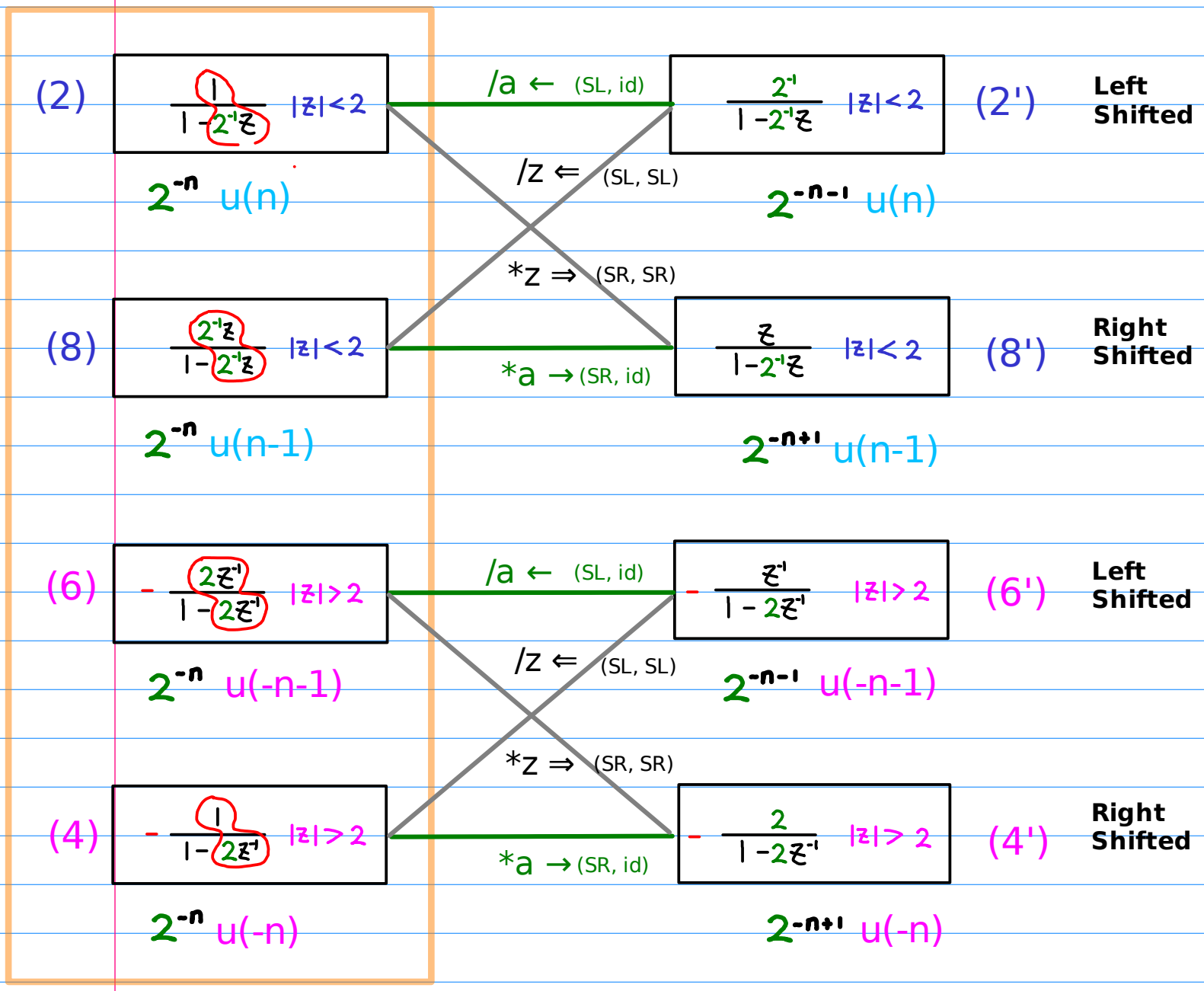


Causal	u(n)	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	u(-n-1)	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series (2) negative exponent

Negative Exponent $a^{-1}=2^{-1}$ /z $n \leftarrow n+1$ *z $n \leftarrow n-1$



Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Decoding Geometric Series

Positive Exponent

$$\square z,\ \square^{-1} z^{-1} \rightarrow \square^{n}$$

$$2z,\ 2^{-1} z^{-1} \rightarrow 2^{n}$$

Negative Exponent

$$\square^{-1} z,\ \square z^{-1} \rightarrow \square^{-n}$$

$$2^{-1} z,\ 2 z^{-1} \rightarrow 2^{-n}$$

Causal

$$\square z \rightarrow u(n),\ u(n-1)$$

$$\frac{1}{1-\square z} \rightarrow u(n)$$

$$\frac{\square z}{1-\square z} \rightarrow u(n-1)$$

Anti-causal

$$\square z^{-1} \rightarrow u(-n),\ u(-n-1)$$

$$\frac{1}{1-\square z^{-1}} \rightarrow u(-n)$$

$$\frac{\square z^{-1}}{1-\square z^{-1}} \rightarrow u(-n-1)$$

$a=1$ (1) (2) (3) (4)

$a^{-1}=2^{-1}$ (5) (6) (7) (8)

Unshifted combinations

including the origin

$P_1 = 1$

$P_2 = 2$

$a=1$

(1)

$1\,z$

$$f(z) = \frac{1}{1-1z} \qquad |z|<1^{-1}$$

$1^n\ u(n) \qquad (n \geq 0)$

(2)

$2^{-1}z$

$$g(z) = \frac{1}{1-2^{-1}z} \qquad |z|<2$$

$\left(\frac{1}{2}\right)^n u(n) \qquad (n \geq 0)$

(3)

$1^{-1}z^{-1}$

$$\bar{f}_1(z) = \frac{1}{1-1^{-1}z^{-1}} \qquad |z|>1^{-1}$$

$1^n\ u(-n) \qquad (n<1)$

(4)

$2z^{-1}$

$$\bar{g}_1(z) = \frac{1}{1-2z^{-1}} \qquad |z|>2$$

$\left(\frac{1}{2}\right)^n u(-n) \qquad (n<1)$

(5)

$1^{-1}z^{-1}$

$$\bar{f}(z) = \frac{1^{-1}z^{-1}}{1-1^{-1}z^{-1}} \qquad |z|>1^{-1}$$

$1^n\ u(-n-1) \qquad (n<0)$

(6)

$2z^{-1}$

$$\bar{g}(z) = \frac{2z^{-1}}{1-2z^{-1}} \qquad |z|>2$$

$\left(\frac{1}{2}\right)^n u(-n-1) \qquad (n<0)$

(7)

$1\,z$

$$f_1(z) = \frac{1z}{1-1z} \qquad |z|<1^{-1}$$

$1^n\ u(n-1) \qquad (n \geq 1)$

$a=2$

(8)

$2^{-1}z$

$$g_1(z) = \frac{2^{-1}z}{1-2^{-1}z} \qquad |z|<2$$

$\left(\frac{1}{2}\right)^n u(n-1) \qquad (n \geq 1)$

$a=1$ (1') (2') (3') (4')

$a^{-1}=2^{-1}$ (5') (6') (7') (8')

Shifted combinations

excluding the origin

$P_1 = 1$

$P_2 = 2$

$a=1$

(1')

$1\,z$

$f_2(z) = \dfrac{1}{1-z}$ $\quad |z|<1^{-1}$

$1^{n+1}\,u(n)$ $\quad (n \geq 0)$

(2')

$2^{-1}z$

$g_2(z) = \dfrac{2^{-1}}{1-2^{-1}z}$ $\quad |z|<2$

$\left(\frac{1}{2}\right)^{n+1} u(n)$ $\quad (n \geq 0)$

(3')

$1^{-1}z^{-1}$

$\bar{f}_3(z) = \dfrac{1^{-1}}{1-1^{-1}z^{-1}}$ $\quad |z|>1^{-1}$

$1^{n-1}\,u(-n)$ $\quad (n<1)$

(4')

$2z^{-1}$

$\bar{g}_3(z) = \dfrac{2}{1-2z^{-1}}$ $\quad |z|>2$

$\left(\frac{1}{2}\right)^{n-1} u(-n)$ $\quad (n<1)$

(5')

$1^{-1}z^{-1}$

$\bar{f}_2(z) = \dfrac{z^{-1}}{1-1^{-1}z^{-1}}$ $\quad |z|>1^{-1}$

$1^{n+1}\,u(-n-1)$ $\quad (n<0)$

(6')

$2z^{-1}$

$\bar{g}_2(z) = \dfrac{z^{-1}}{1-2z^{-1}}$ $\quad |z|>2$

$\left(\frac{1}{2}\right)^{n+1} u(-n-1)$ $\quad (n<0)$

(7')

$1\,z$

$f_3(z) = \dfrac{z}{1-1z}$ $\quad |z|<1^{-1}$

$1^{n-1}u(n-1)$ $\quad (n \geq 1)$

(8')

$2^{-1}z$

$g_3(z) = \dfrac{z}{1-2^{-1}z}$ $\quad |z|<2$

$\left(\frac{1}{2}\right)^{n-1} u(n-1)$ $\quad (n \geq 1)$

$a=2$

(1') (2')
(1) (2)

(5') (6')
(5) (6)

Geometric series for unshifted combinations

$$-\frac{1}{(z-1)(z-2)}$$

$a=1$

(1')

$f_2(z) = \frac{1}{1-z}$	$\lvert z\rvert < 1^{-1}$
$1^{n+1}\,u(n)$	$(n \geq 0)$

(1)

$f(z) = \frac{1}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n}\,u(n)$	$(n \geq 0)$

(5')

$\bar{f}_2(z) = \frac{z^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n+1}\,u(-n-1)$	$(n < 0)$

(5)

$\bar{f}(z) = \frac{1^{-1}z^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n}\,u(-n-1)$	$(n < 0)$

$a=2$

(2')

$g_2(z) = \frac{2^{-1}}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n+1}\,u(n)$	$(n \geq 0)$

(2)

$g(z) = \frac{1}{1-2^{-1}z}$	$\lvert z\rvert < 2^{1}$
$\left(\frac{1}{2}\right)^{n}\,u(n)$	$(n \geq 0)$

(6')

$\bar{g}_2(z) = \frac{z^{-1}}{1-2z^{-1}}$	$\lvert z\rvert > 2$
$\left(\frac{1}{2}\right)^{n+1}\,u(-n-1)$	$(n < 0)$

(6)

$\bar{g}(z) = \frac{2z^{-1}}{1-2z^{-1}}$	$\lvert z\rvert > 2^{1}$
$\left(\frac{1}{2}\right)^{n}\,u(-n-1)$	$(n < 0)$

(7') (8')
(7) (8)

(3') (4')
(3) (4)

Geometric series for unshifted combinations

$$-\frac{z^2}{(z-1)(z-2)}$$

$a=1$

(7')

$f_3(z) = \frac{z}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n-1}u(n-1)$	$(n \geq 1)$

(7)

$f_1(z) = \frac{1z}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n}\,u(n-1)$	$(n \geq 1)$

(3')

$\bar{f}_3(z) = \frac{1^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n-1}u(-n)$	$(n < 1)$

(3)

$\bar{f}_1(z) = \frac{1}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n}\,u(-n)$	$(n < 1)$

$a=2$

(8')

$g_3(z) = \frac{z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$(\frac{1}{2})^{n-1}u(n-1)$	$(n \geq 1)$

(8)

$g_1(z) = \frac{2^{-1}z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$(\frac{1}{2})^{n}\,u(n-1)$	$(n \geq 1)$

(4')

$\bar{g}_3(z) = \frac{2}{1-2z^{-1}}$	$\lvert z\rvert > 2$
$(\frac{1}{2})^{n-1}u(-n)$	$(n < 1)$

(4)

$\bar{g}_1(z) = \frac{1}{1-2z^{-1}}$	$\lvert z\rvert > 2$
$(\frac{1}{2})^{n}\,u(-n)$	$(n < 1)$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

Case A $|z| < 0.5$

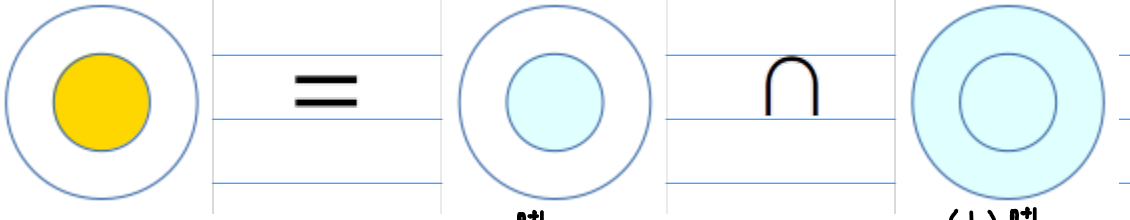
$$-\frac{1}{(z-1)(z-2)}$$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

Simple Pole Form

Geometric Power Series Forms

$\frac{1}{z-1}$	$\frac{1}{z-2}$
$-\frac{1}{1-z}$	$-\frac{0.5}{1-0.5z}$
$\lvert z\rvert < 1$	$\lvert z\rvert < 1$	$\lvert z\rvert < 2$



$-1^{n+1}\,u(n)$ $\qquad$ $-\left(\frac{1}{2}\right)^{n+1}u(n)$

$|z| < 1$ $\qquad f(z) = -\frac{1}{1-z} + \frac{0.5}{1-0.5z}$ $\qquad -1^{n+1}\,u(n) + \left(\frac{1}{2}\right)^{n+1}u(n)$

(1')

$f_2(z) = \frac{1}{1-z}$	$\lvert z\rvert < 1^{-1}$
$1^{n+1}\,u(n)$	$(n \geq 0)$

(2')

$g_2(z) = \frac{2^{-1}}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n+1}u(n)$	$(n \geq 0)$

(1)

$f(z) = \frac{1}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n}\,u(n)$	$(n \geq 0)$

$g(z) = \frac{1}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n}u(n)$	$(n \geq 0)$

Case B $0.5 < |z| < 2$

$$-\frac{1}{(z-1)(z-2)}$$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

Simple Pole Form

Geometric Power Series Forms

	$\frac{1}{z-1}$	$\frac{1}{z-2}$
	$\frac{z^{-1}}{1-z^{-1}}$	$-\frac{0.5}{1-0.5z}$
	$1 < \lvert z \rvert$	$\lvert z \rvert < 2$

$1 < |z| < 2$ = $1 < |z|$ $\cap$ $|z| < 2$

$1^{n+1}\,u(-n-1)$ $\quad$ $-\left(\frac{1}{2}\right)^{n+1}u(n)$

$$1 < |z| < 2 \qquad f(z) = \frac{z^{-1}}{1-z^{-1}} + \frac{0.5}{1-0.5z} \qquad 1^{n+1}\,u(-n-1) + \left(\frac{1}{2}\right)^{n+1}u(n)$$

(5')

$\bar{f}_2(z) = \frac{z^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z \rvert > 1^{-1}$
$1^{n+1}\,u(-n-1)$	$(n < 0)$

(5)

$\bar{f}(z) = \frac{1^{-1}z^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z \rvert > 1^{-1}$
$1^{n}\,u(-n-1)$	$(n < 0)$

(2')

$g_2(z) = \frac{2^{-1}}{1-2^{-1}z}$	$\lvert z \rvert < 2$
$\left(\frac{1}{2}\right)^{n+1}u(n)$	$(n \geq 0)$

(2)

$g(z) = \frac{1}{1-2^{-1}z}$	$\lvert z \rvert < 2$
$\left(\frac{1}{2}\right)^{n}u(n)$	$(n \geq 0)$

Case C $2 < |z|$

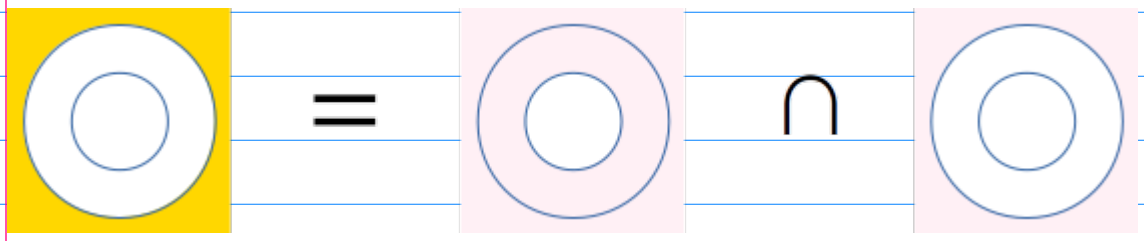
$$-\frac{1}{(z-1)(z-2)}$$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

Simple Pole Form

Geometric Power Series Forms

	$\frac{1}{z-1}$	$\frac{1}{z-2}$
	$\frac{z^{-1}}{1-z^{-1}}$	$\frac{z^{-1}}{1-2z^{-1}}$
	$1 < \|z\|$	$2 < \|z\|$
	$1^{n+1}\,u(-n-1)$	$\left(\frac{1}{2}\right)^{n+1}u(-n-1)$



$2 < |z|$ $\quad f(z) = \frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{1-2z^{-1}}$ $\quad 1^{n+1}\,u(-n-1) - \left(\frac{1}{2}\right)^{n+1}u(-n-1)$

(5')

$\bar{f}_2(z) = \frac{z^{-1}}{1-1^{-1}z^{-1}}$	$\|z\| > 1^{-1}$
$1^{n+1}\,u(-n-1)$	$(n<0)$

(6')

$\bar{g}_2(z) = \frac{z^{-1}}{1-2z^{-1}}$	$\|z\| > 2$
$\left(\frac{1}{2}\right)^{n+1}u(-n-1)$	$(n<0)$

(5)

$\bar{f}(z) = \frac{1^{-1}z^{-1}}{1-1^{-1}z^{-1}}$	$\|z\| > 1^{-1}$
$1^{n}\,u(-n-1)$	$(n<0)$

(6)

$\bar{g}(z) = \frac{2z^{-1}}{1-2z^{-1}}$	$\|z\| > 2$
$\left(\frac{1}{2}\right)^{n}u(-n-1)$	$(n<0)$

Laurent Series and z Transform

$$-\frac{1}{(z-1)(z-2)}$$

$$-\frac{1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

$|z| < 1$ $\quad f(z) = -\frac{1}{1-z} + \frac{0.5}{1-0.5z}$

Laurent Series $\quad -1^{n+1}\,u(n) + \left(\frac{1}{2}\right)^{n+1} u(n)$

z Transform $\quad -1^{-n+1}u(-n) + \left(\frac{1}{2}\right)^{-n+1}u(-n)$

$-\left(\frac{1}{1}\right)^{n-1}u(-n) + 2^{n-1}\,u(-n)$

$1 < |z| < 2$ $\quad f(z) = \frac{z^{-1}}{1-z^{-1}} + \frac{0.5}{1-0.5z}$

Laurent Series $\quad 1^{n+1}\,u(-n-1) + \left(\frac{1}{2}\right)^{n+1}u(n)$

z Transform $\quad 1^{-n+1}\,u(n-1) + \left(\frac{1}{2}\right)^{-n+1}u(-n)$

$\left(\frac{1}{1}\right)^{n-1}u(n-1) + 2^{n-1}\,u(-n)$

$2 < |z|$ $\quad f(z) = \frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{1-2z^{-1}}$

Laurent Series $\quad 1^{n+1}\,u(-n-1) - \left(\frac{1}{2}\right)^{n+1}u(-n-1)$

z Transform $\quad 1^{-n+1}u(n-1) - \left(\frac{1}{2}\right)^{-n+1}u(n-1)$

$\left(\frac{1}{1}\right)^{n-1}u(n-1) + 2^{n-1}\,u(n-1)$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

Case A $|z| < 0.5$

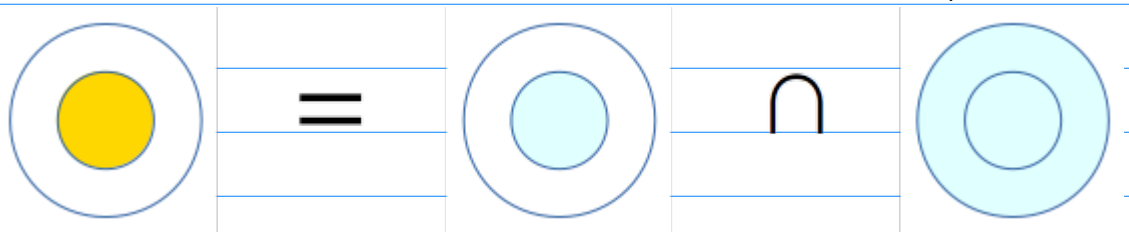
$$-\frac{z^2}{(z-1)(z-2)}$$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

Simple Pole Form

Geometric Power Series Forms

$\frac{z}{z-1}$	$\frac{2z}{z-2}$
$-\frac{z}{1-z}$	$-\frac{z}{1-0.5z}$
$\lvert z\rvert < 1$	$\lvert z\rvert < 1$	$\lvert z\rvert < 2$



$-1^{n-1}\,u(n)$ $\qquad$ $-\left(\frac{1}{2}\right)^{n-1}u(n)$

$|z| < 1$ $\qquad f(z) = -\frac{z}{1-2z} + \frac{z}{1-0.5z}$ $\qquad$ $-1^{n-1}u(n) + \left(\frac{1}{2}\right)^{n-1}u(n)$

(7')

$f_3(z) = \frac{z}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n-1}u(n-1)$	$(n \geq 1)$

(8')

$g_3(z) = \frac{z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n-1}u(n-1)$	$(n \geq 1)$

(7)

$f_1(z) = \frac{1z}{1-1z}$	$\lvert z\rvert < 1^{-1}$
$1^{n}\,u(n-1)$	$(n \geq 1)$

(8)

$g_1(z) = \frac{2^{-1}z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n}u(n-1)$	$(n \geq 1)$

Case B $0.5 < |z| < 2$

$$-\frac{z^2}{(z-1)(z-2)}$$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

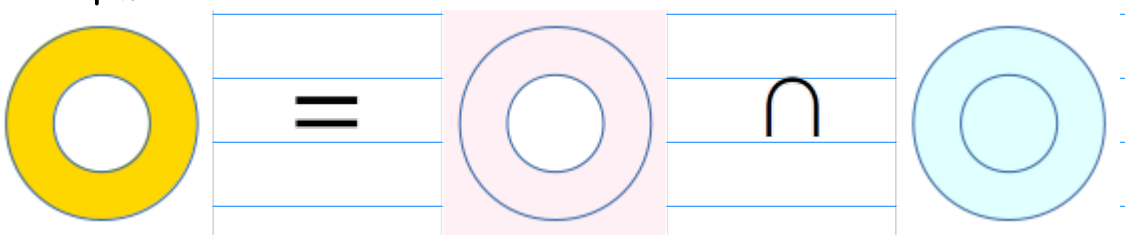
Simple Pole Form

$\frac{z}{z-1}$ $\qquad$ $\frac{2z}{z-2}$

Geometric Power Series Forms

$\frac{1}{1-z^{-1}}$ $\qquad$ $-\frac{z}{1-0.5z}$

$1 < |z| < 2$ $\qquad$ $1 < |z|$ $\qquad$ $|z| < 2$



$1^{n-1}\,u(-n-1)$ $\qquad$ $-\left(\frac{1}{2}\right)^{n-1}u(n)$

$$1 < |z| < 2 \qquad f(z) = \frac{1}{1-z^{-1}} + \frac{z}{1-0.5z} \qquad 1^{n-1}\,u(-n-1) + \left(\frac{1}{2}\right)^{n-1}u(n)$$

(3')

$\bar{f}_3(z) = \frac{1^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n-1}\,u(-n)$	$(n < 1)$

(3)

$\bar{f}_1(z) = \frac{1}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n}\,u(-n)$	$(n < 1)$

(8')

$g_3(z) = \frac{z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n-1}u(n-1)$	$(n \geq 1)$

(8)

$g_1(z) = \frac{2^{-1}z}{1-2^{-1}z}$	$\lvert z\rvert < 2$
$\left(\frac{1}{2}\right)^{n}u(n-1)$	$(n \geq 1)$

Case C $2 < |z|$

$$-\frac{z^2}{(z-1)(z-2)}$$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

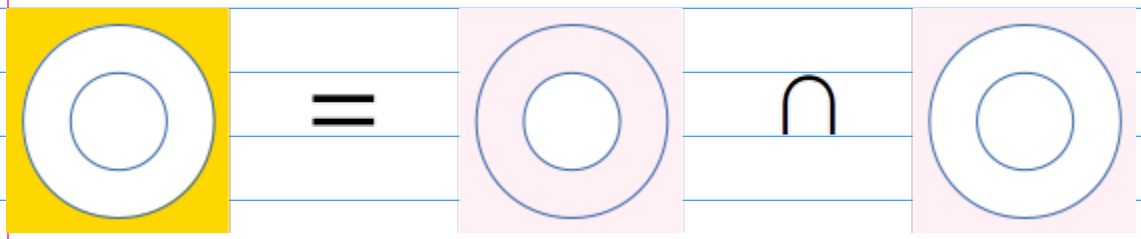
Simple Pole Form

$\frac{z}{z-1}$ $\qquad$ $\frac{2z}{z-2}$

Geometric Power Series Forms

$\frac{1}{1-z^{-1}}$ $\qquad$ $\frac{2}{1-2z^{-1}}$

$2 < |z|$ $\qquad$ $1 < |z|$ $\qquad$ $2 < |z|$



$1^{n-1}\,u(-n-1)$ $\qquad$ $\left(\frac{1}{2}\right)^{n-1}u(-n-1)$

$2 < |z|$ $\qquad f(z) = \frac{1}{1-z^{-1}} - \frac{2}{1-2z^{-1}}$ $\qquad 1^{n-1}\,u(-n-1) - \left(\frac{1}{2}\right)^{n-1}u(-n-1)$

(3')

$\bar{f}_3(z) = \frac{1^{-1}}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n-1}\,u(-n)$	$(n < 1)$

(4')

$\bar{g}_3(z) = \frac{2}{1-2z^{-1}}$	$\lvert z\rvert > 2$
$\left(\frac{1}{2}\right)^{n-1}u(-n)$	$(n < 1)$

(3)

$\bar{f}_1(z) = \frac{1}{1-1^{-1}z^{-1}}$	$\lvert z\rvert > 1^{-1}$
$1^{n}\,u(-n)$	$(n < 1)$

(4)

$\bar{g}_1(z) = \frac{1}{1-2z^{-1}}$	$\lvert z\rvert > 2$
$\left(\frac{1}{2}\right)^{n}u(-n)$	$(n < 1)$

Laurent Series and z Transform

$$-\frac{z^2}{(z-1)(z-2)}$$

$$-\frac{z^2}{(z-1)(z-2)} = \left(\frac{z}{z-1} - \frac{2z}{z-2}\right)$$

$|z| < 1$ $\quad f(z) = -\frac{z}{1-2z} + \frac{z}{1-0.5z}$

Laurent Series $\quad -1^{n-1}u(n) + \left(\frac{1}{2}\right)^{n-1}u(n)$

z Transform $\quad -1^{-n-1}u(-n) + \left(\frac{1}{2}\right)^{-n-1}u(-n)$

$-\left(\frac{1}{1}\right)^{n+1}u(-n) + 2^{n+1}u(-n)$

$1 < |z| < 2$ $\quad f(z) = \frac{1}{1-z^{-1}} + \frac{z}{1-0.5z}$

Laurent Series $\quad 1^{n-1}u(-n-1) + \left(\frac{1}{2}\right)^{n-1}u(n)$

z Transform $\quad 1^{-n-1}u(n-1) + \left(\frac{1}{2}\right)^{-n-1}u(-n)$

$\left(\frac{1}{1}\right)^{n+1}u(n-1) + 2^{n+1}u(-n)$

$2 < |z|$ $\quad f(z) = \frac{1}{1-z^{-1}} - \frac{2}{1-2z^{-1}}$

Laurent Series $\quad 1^{n-1}u(-n-1) - \left(\frac{1}{2}\right)^{n-1}u(-n-1)$

z Transform $\quad 1^{-n-1}u(n-1) - \left(\frac{1}{2}\right)^{-n-1}u(n-1)$

$\left(\frac{1}{1}\right)^{n+1}u(n-1) + 2^{n+1}u(n-1)$

$$f(z) = g(z^{-1})$$
$$g(z) = f(z^{-1})$$

$\{1, 2\} \rightarrow \{1, 0.5\}$

$$\frac{-1}{(z-1)(z-2)} \quad \overset{z^{-1}}{\longleftrightarrow} \quad \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$\frac{-1}{(z-1)(z-2)} = \left(\frac{1}{z-1} - \frac{1}{z-2}\right)$$

z^{-1}

$$\frac{-1}{(z^{-1}-1)(z^{-1}-2)} = \left(\frac{1}{z^{-1}-1} - \frac{1}{z^{-1}-2}\right)$$

$$= \left(\frac{z}{1-z} - \frac{z}{1-2z}\right)$$

$$= \left(\frac{-z}{z-1} + \frac{0.5z}{z-0.5}\right)$$

$$= z\left(\frac{-1}{z-1} + \frac{0.5}{z-0.5}\right)$$

$$= z\left(\frac{-0.5z}{(z-1)(z-0.5)}\right)$$

$$= \frac{-0.5z^2}{(z-1)(z-0.5)}$$

$$\frac{-0.5z^2}{(z-1)(z-0.5)} = \left(-\frac{z}{(z-1)} + \frac{0.5z}{(z-0.5)}\right)$$