Draft:A card game for Bell's theorem and its loopholes

WikiJournal of Science
Open access • Publication charge free • Public peer review

This pre-print is undergoing public peer review

First submitted: 22 February 2018

Last updated: 25 February 2018

Reviewer comments
Last reviewed version

Contents

Footnotes

This is an unpublished pre-print. It is undergoing peer review.

Authors: Guy Vandegrift, Joshua Stomel

Wright State University Lake Campus, Celina, Ohio, 45822

Author correspondence: by online form

Abstract

Instead of teaching the proof of Bell's inequality, let students see for themselves that this card game is impossible to win. The solitaire version of the game so simple it can be used to teach elementary statistics without mentioning physics or Bell's theorem. Things get interesting in the partners' version because Alice and Bob can win, but only if they cheat. We have identified three cheats, and each corresponds to a Bell's theorem "loophole". This gives us an excuse to discuss detector error, causality, and why there is a maximum speed at which information can travel.

A simple Bell's theorem experiment

The 1964 ~~discovery~~ by Bell[1][2][3] reinforces a view that the laws of physics are not constrained to obey what might be called intuitive or common notions. It is customary to name the particles in a Bell's theorem experiment "Alice" and "Bob", an anthropomorphism that serves to emphasize the fact that a pair of humans cannot win the card game ... unless they cheat. To some experts, a "loophole" is a constraint on any "hidden variable" theory that might replace quantum mechanics.[4] It is also possible to view a loophole as a physical mechanism by which the outcome of a Bell's theorem experiment might seem less "spooky". In this paper, we associate loopholes with ways to cheat at the partners' version of the card game. It should be noted that the three loophole mechanisms introduced in this paper raise questions that are even spookier than quantum mechanics: *Are the photons "communicating" with each other? Do they "know" the future? Do they "persuade" the measuring devices to fail when the cards are unfavorable?*

Since entanglement is so successfully modelled by quantum mechanics, one can argue that there is no need for a mechanism that "explains" it. Nevertheless, there are reasons for investigating loopholes. At the most fundamental level, history shows that a successful physical theory can be later shown to be an approximation to a deeper theory, and the need for this new theory is typically associated with a failure of the old paradigm. It is plausible that a breakdown of quantum mechanics might be discovered using a Bell's theorem experiment designed to investigate a loophole. But the vast majority of us (including most working physicists) need other reasons to care about loopholes: Many find it interesting that we seem to live in a universe governed by fundamental laws, and Bell's theorem yields insights into the bizarre nature of those laws. Also, those who teach can use these card games to motive introductory discussions about statistical inference, polarization, and modern physics.

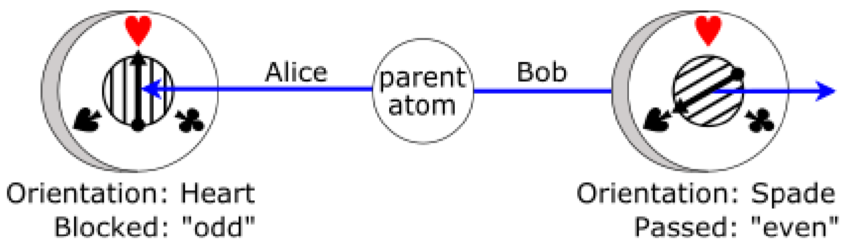


Figure 1 | The outside casing of each device remains stationary while the circle with parallel lines rotates with the center arrow pointing in one of three directions (♥, ♣, ♠.) If Jacks are used to represent these directions, Alice will see J♥ as her question card. She will respond with an "odd"-numbered answer card (3♥) to indicate that she is blocked by the filter. If Bob passes through a filter with the "spade" orientation, he sees J♠ as the question card, and answers with the "even" numbered 2♠. This wins one point for the team because they gave different answers to different questions.

Figure 1 shows an idealized experiment involving two entangled photons simultaneously emitted by a single (parent) atom. After the photons have been separated by some distance, each is exposed to a measurement that determines whether the photon would pass or be blocked by the polarizing filter.[5] To ensure that the results seem "spooky" it should be possible to rotate the filter while the photons are en route so that the filter's angle of orientation is not "known" to either photon until the it encounters the filter. If the filters are rotated between only three polarization angles, we may use card suits (hearts ♥, clubs ♣, spades ♠) to represent these angles. These three polarization angles are associated with "question" cards, because the ~~the~~ measurement essentially asks the photon a question:

"Will you pass through a filter oriented at this angle?"

For simplicity we restrict our discussion to symmetric angles (0°, 120°, 240°.) The filter's axis of polarization is shown in the figure as parallel lines, with the center line pointing to the heart, club, or spade. Any face card can be used to "ask" the question, and the four face cards (jack, queen, king, ace) are equivalent. If the detectors are flawless, each measurement is binary: The photon either passes or is blocked by the filter (subsequent measurements on a photon would yield nothing interesting.) The measurement's outcome is represented by an even or odd numbered "answer" card (of the same suit). The numerical value of an answer card is not important: all odd numbers (3,5,7,9) are equivalent and represent a photon passing through the filter, while the even cards (2,4,6,8) represent a photon being blocked.

Although Bell's inequality is easy to prove[6], we avoid it here because the card game reverses roles regarding probability: Instead of the investigators attempting to ascertain the photons' so-called hidden variables, the players are acting as particles attempting to win the game by guessing the measurement angles. Another complication is that the original form of Bell's inequality does not adequately model the partners' version of the game because humans have the freedom to exhibit a behavior not observed by entangled particles (under ideal experimental conditions). In the partners' version of the card game, a penalty must be deducted from the partners' score whenever they are caught using a forbidden strategy (which we shall later call the β-strategy). The minimum required penalty is calculated in the appendix, but fortunately students need not master this calculation because the actual penalty should often be whatever it takes to encourage a strategy that mimics this aspect of entanglement (which we shall call the

"humans have the freedom to exhibit a behavior not observed in entangled particles"

which is this behavior?

how is it observed in humans and not observed in particles?

α-strategy.)

The solitaire card game

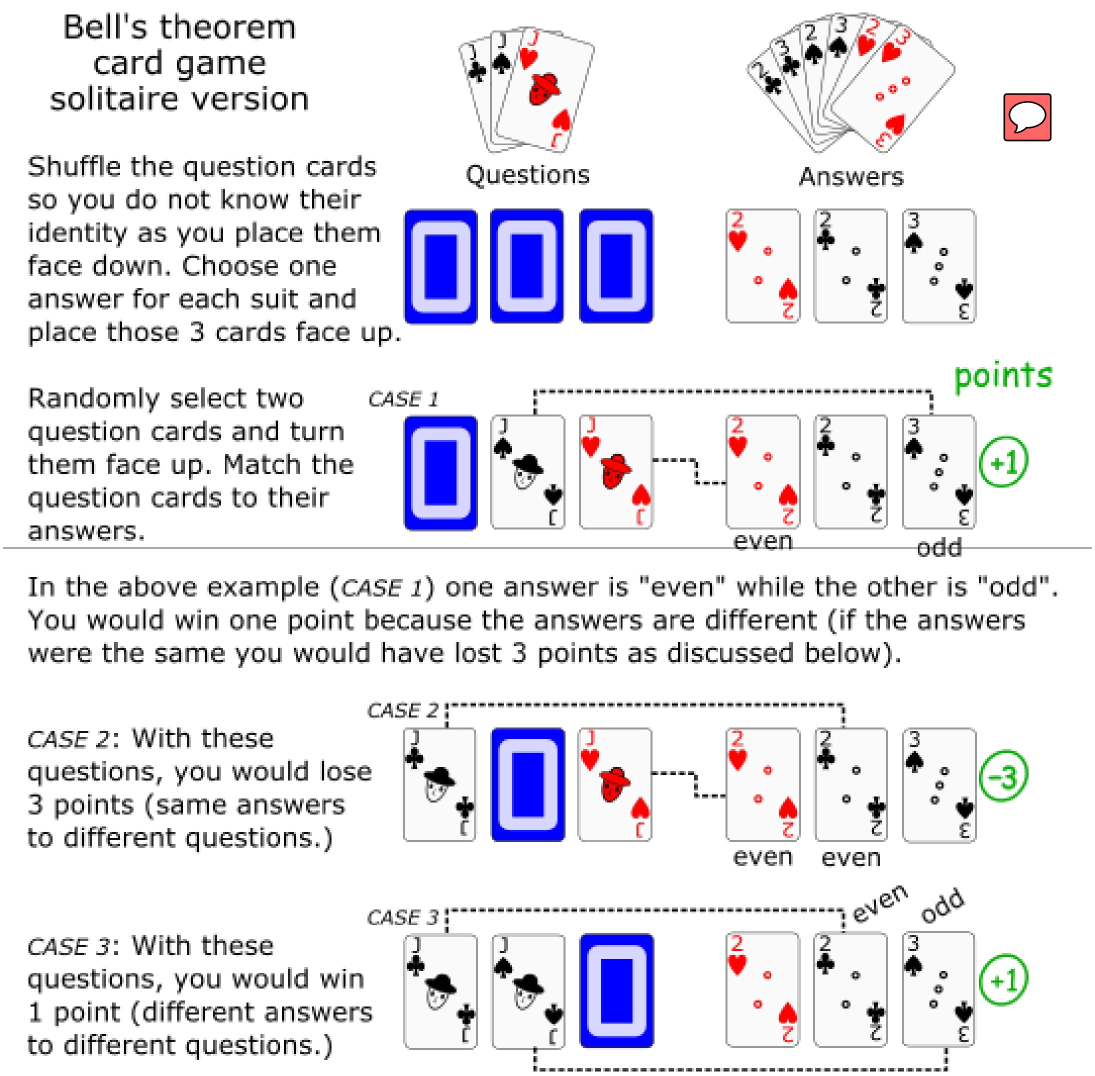


Figure 2 | solitaire version of game. Cases 1, 2, and 3 represent three possible outcomes if the player chooses the α-strategy, which is for one answer ("odd" for ♠) to differ from that given for the other two questions (i.e., "even" for♥ & ♣.)

Figure 2 shows the three possible outcomes associated with one hand of the solitaire version of the game. The solitaire version requires nine cards. The figure uses a set with three "jacks" (♥, ♣ ♠) for the questions, and (2,3) for the six (even/odd) answer cards. To play one round of the game, the player first shuffles the three question cards and places them face down so their identity is not known. Next, for each of the three suits, the player selects an even or odd answer card. The figure shows the player choosing the heart and club to be even, while the spade is odd: 2♥, 2♣ 3♠. This strategy in which one suit has a different answer (here the spade) shall be called the α-strategy. In the solitaire version, this is the only viable strategy, because the forbidden β-strategy is not possible in the solitaire version.

After three answer cards are selected and turned face up, two of the three question cards are randomly selected and also turned face up. Figure 2 depicts all three equally probable outcomes, or ways to select two out of three cards *(3 choose 2.)* The round is scored by adding or subtracting points, as shown in Table 1: First the suit of each of the two upturned question of subtracting points, as shown in Table 1. First the suit of each of the two upturned question cards is matched to the corresponding answer card. In case 1 (shown in the figure), the player wins one point because answers are different: ♥ is an even number, while ♠ is odd. The player loses three points case 2 because the♥ and ♣ are the same (even). Case 3 wins one point for the player because the answers are different. It is evident that the player has a 2/3 probability winning a round. The conundrum of Bell's theorem is that, entangled particles in an actual experiment manage to win with a probability of 3/4. Table 1 shows that this scoring system causes humans to average a loss of at least 1/3 of a point per round,[7] while entangled particles maintain an average score of zero. How do particles succeed where humans fail?

Table 1: Solitaire Scoring

Points	Answers are:	Example
+1	different	2♥ and 3♠
−3	same	2♣ and 2♥

The game for entangled partners

In the partners' version of the game, Alice and Bob each play one (even/odd) answer card in response to the suit of a question card. Each round is played in two distinctly different phases. Alice and Bob are allowed to discuss strategy during phase 1 because it simulates the fact that the particles are (effectively) "inside" the parent atom before it emits photons. Then, all communication between the partners must cease during phase 2, which simulates the arrival of the photons at the detectors for measurement under conditions where communication is impossible. In this phase each player silently plays an (even/odd) answer that matches the question's suit. The player cannot know the other's question or answer during phase 2.

The partners' version differs from the solitaire version because it is now possible for Alice and Bob to be given question cards of the same suit. Whenever asked the same question, Alice and Bob would always give the same answer, if they were entangled particles.[5] It should be noted that actual Bell's theorem experiments can register such events due to detection error flaws. To encourage Alice and Bob to behave like entangled particles, it is necessary to deduct Q points whenever they give different answers to the same question (no points are awarded for giving the same answer to the same question.) The minimum penalty that should be imposed depends on how often the partners are given question cards of the same suit, and is derived in the appendix:

$$Q \geq \frac{4}{3}\left(\frac{1-P_S}{P_S}\right) \,, \qquad (1)$$

where P_S is the probability that Alice and Bob are asked the same question. The equality holds if $P_S = 1/4$ and $Q = 4$, which can be accomplished by randomly selecting two question cards from nine (K♠, K♥, K♣, Q♠, Q♥, Q♣,J♠, J♥, J♣), as shown in Fig. 3. If the equality in (1) holds, the partners are "neutral" with respect to the selection of two different strategies, one of which risks incurring the 4 point penalty associated with giving different answers to the same question. Both strategies lose, but the loss rate is reduced to −1/4 points per round, because the referee must dilute the number of times different questions are asked.

A sample round begins in the top part of Fig. 3 as phase 1, where the pipe smoking referee has selected different questions (hearts and spades). In a classroom setting, consider allowing Alice and Bob to side-by-side, facing slightly away from each other during phase 2. Arrange for the audience to sit close enough to listen and watch for evidence of surreptitious communication between Alice and Bob. The prospect of cheating not only makes the game more fun, but also allows us to introduce "loopholes". The "thought-bubbles" above the partners show a *tentative* agreement by the partners to play the same α-strategy introduced in the solitaire version (both say "even" to ♥♣, and "odd" to ♠.) ***It is important to allow both players to hold all the answer cards in phase 2 so that each can change his or her mind upon seeing the actual question.*** The figure shows them following their original plan and winning because the referee selected a heart for Alice and a spade for Bob.

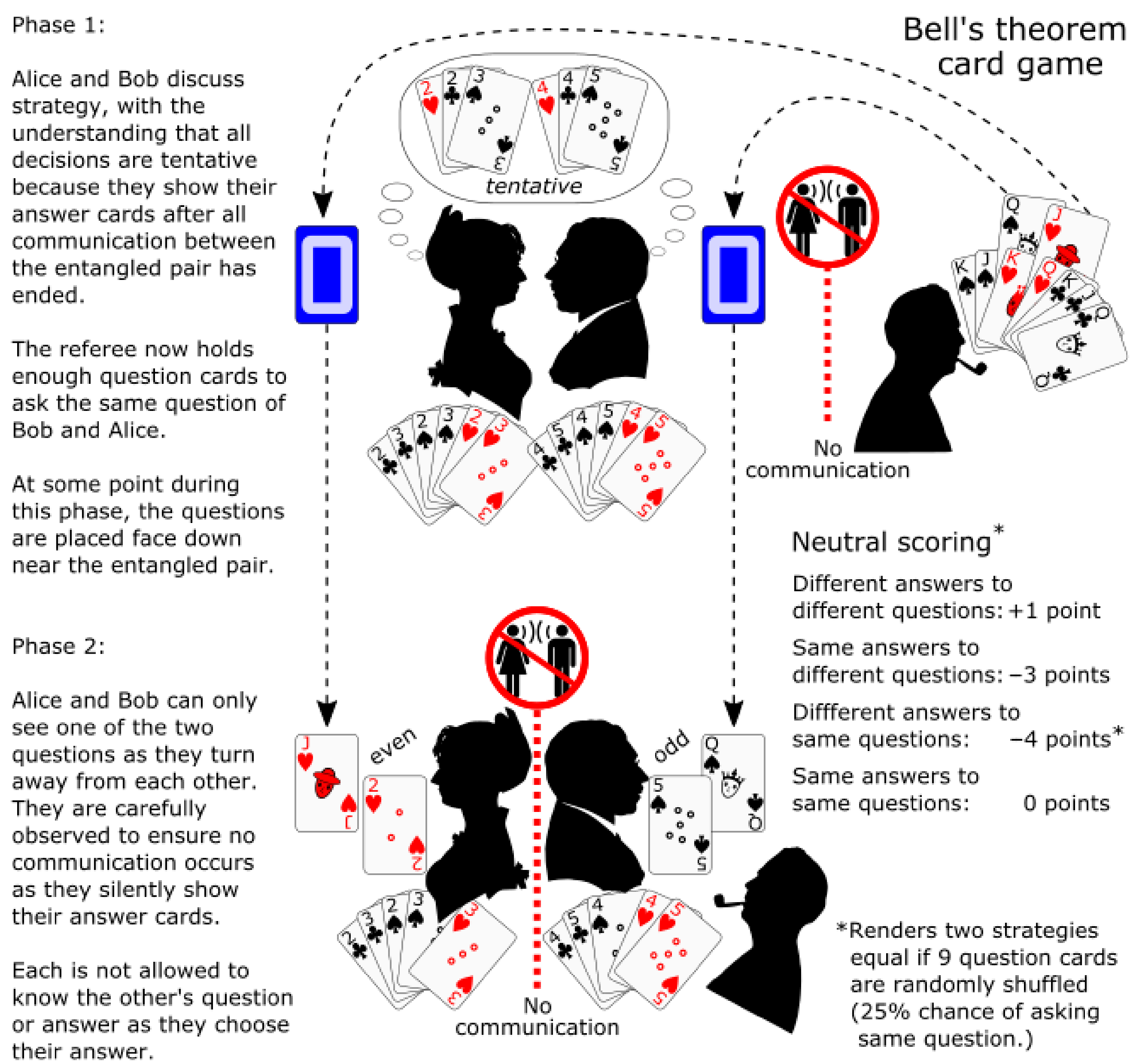


Figure 3 | One round of the partners' version with Alice and Bob employing the α-strategy. The scoring is "neutral" if the referee randomly selects from the nine question cards. The penalty of 4 points for giving different answers to the same question ensures that the α and β strategies yield the same expected score.

But the partners have another strategy that might win: Suppose Alice agrees to answer "even" to any question, while Bob answer is always "odd". This wins ($+1$) if different questions are asked, and loses ($-Q$) if the same question is asked. This is called the β-strategy, and the appendix establishes that no other strategy is superior to the α and/or β strategies:

α-strategy: Alice and Bob select their answers in advance, in such a way that both give the same answer if asked the same question. For example, they might both agree that ♥♣ are even, while ♠ is odd. This strategy was ensured in the solitaire version because only three cards are played: If the heart is chosen to be "even", the solitaire version models a situation where both Alice and Bob would answer "even" to "heart". This α-strategy requires that one answer differs from the other two (i.e., all "even" or all "odd" is never a good strategy). The expected loss is 1/3 for each round whenever different questions are asked.

β-strategy: One partner always answers "even" while the other always answers "odd". This strategy gains one point if different questions are asked, and loses Q points if the same question is asked.

Table 2: Examples of neutral and biased scoring

Neutral scoring Shuffle 9 face cards to ask the same question exactly 25% of the time.		**Biased scoring** Shuffle 6 face cards or ask the same question more than 2/11 = 18$\overline{18}$% of the time.	
Points	Alice and Bob give...	Example	Points
+1	different answers to diferent questions	"even" to hearts and "odd" to spades	+1
−3	the same answer to diferent questions	"even" to clubs and "even" to hearts	−3
−4	different answers to the same question	"even" to clubs and "odd" to clubs	−6
0	the same answer to the same question	"even" to clubs (for both players)	0

For pedagogical reasons, the instructor may wish to discourage the β-strategy. If Alice and Bob are not asked the same question often, they might choose to risk large losses for the possibility winning just a few rounds using the β-strategy, perhaps terminating the game prematurely with a claim that they lost "quantum entanglement". To counter this, the referee can raise the penalty to six points and randomly shuffle only six question cards that result from the merging of two solitaire decks. We refer to any scoring that favors the players' use of the α-strategy as "biased scoring". To further inhibit use of the β-strategy, the referee should routinely override the shuffle and deliberately select question cards of the same suit. Examples of both scoring systems are shown in Table 2. The distinction between biased and neutral scoring lies in whether the equality or the inequality holds in (1).

Cheating at cards and Bell's theorem "loopholes"

In the card game, Alice and Bob could either win by surreptitiously communicating after they see their question cards, or by colluding with the referee to learn the questions in advance. Which seems more plausible, information travelling faster than light, or atoms acting as if they "know" the future? A small poll of undergraduate math and science college students suggests that they are inclined to favor superluminal (faster-than-light) communication as the more plausible "loophole". We shall use a space-time diagram to illustrate how superluminal (faster-than-light) communication violates causality by allowing people to send signals to their own past. And, without taking sides in the debate, we shall argue that decisions made today by humans regarding how and where to perform a Bell's theorem experiment next week, might be mysteriously connected to the behavior of an obscure atom in a distant galaxy billions of years ago.

The third loophole was a surprise for us. In an early trial of the partners' game, a student[8] stopped playing and attempted to construct a modified version of the α-strategy that uses the new information a player gains upon seeing his or her question card. After convincing ourselves that no superior strategy exists, we realized that a player could cheat by terminating the game after seeing his or own question card, but before playing the answer card. This is related to an important detector error loophole.[9] The student's discovery also alerted us to the fact that our original calculation of (1) was just a lucky guess based on flawed logic.

Magic phones: Communications loophole

Alice and Bob could win every round of the partners' version if they cheat by communicating with each other after seeing their question cards in phase 2. In an actual experiment, this loophole is closed by making the measurements far apart in space and nearly simultaneous, which in effect requires that these communications travel faster than the speed of light.[10] While any superluminal (faster-than-light) communication is inconsistent with special relativity, we shall limit our discussion to information that travels at nearly infinite speed.[11]

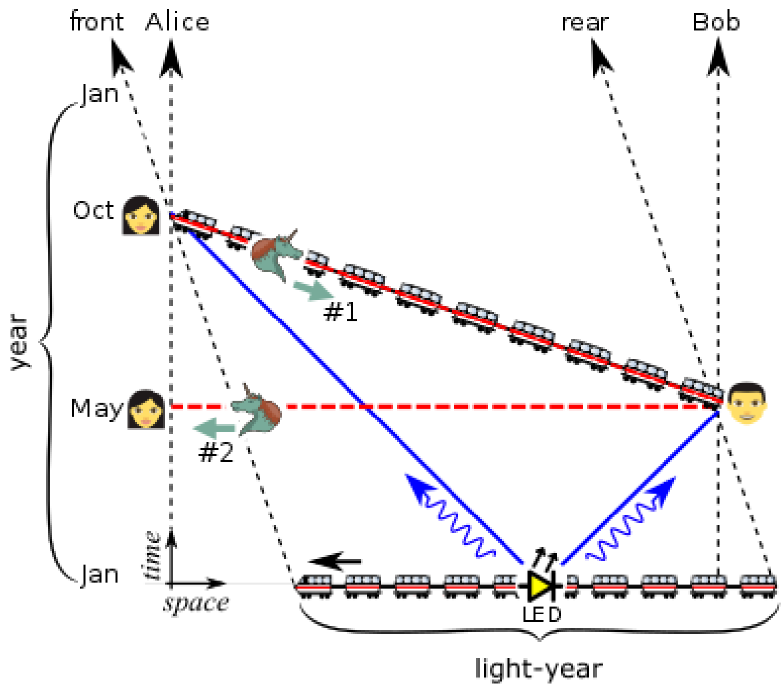


Figure 4 | "Magic phone#1" is situated on a moving train and can be used by Alice to send a message to Bob's past, which Bob relays back to Alice's past using the land-based "Magic phone #2". These magic phones transmit information with near infinite speed.

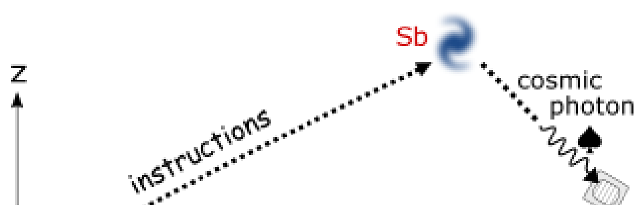
Figure 4 shows Alice and Bob slightly more than one light-year apart. The dotted world lines for each is vertical, indicating that they remain at rest for over a year. The slopes of world lines of the train's front and rear are roughly 3 years per light-year, corresponding to about 1/3 the speed of light. Both train images are a bit confusing because it is difficult to represent a moving train on a space-time diagram: A moving train can be defined by the location of each end at any train on a space-time diagram. A moving train can be defined by the location of each end at any given instant in time. This requires the concept of simultaneity, which is perceived differently in another reference frame. The horizontal image of the train at the bottom represents to location of ~~the~~ each car on the train on the first day of January, as time and simultaneity are perceived by Alice and Bob. To complicate matters, the horizontal train image is not what they would actually *see* due to the finite transit time required for light to reach their eyes. It helps to image a distant observer situated on a perpendicular to some point on the train. The transit time for light to reach this distant observer will be nearly the same for every car on the train. Many years later, this distant observer will see the horizontal train as depicted at the bottom of the figure. After the paradox has been constructed, it will be instructive to return to the perspective of this distant observer.

The slanted image of the train depicts the location of each car on the day that the (moving) passengers perceive the front to be adjacent to Alice, at the same time that the train's rear is perceived to be adjacent to Bob (in the moving reference frame.) First, we must establish that the passengers perceive the front of the train to reach Alice at the same time that the rear reaches Bob. The light-emitting-diode (LED) shown at the bottom of Fig. 4 emits two pulses from the center of the train in January. It is irrelevant whether the LED is stationary or moving because all observers will see the pulses travelling in opposite directions at the speed of light (±1 ly/yr.) Note how the backward moving pulses reaches the rear of the in May, five months before the other pulse reaches the train's front in October. But, the passengers see two light pulses created at the center of the train, directed at each end of the train, and will therefore perceive the two pulses as striking simultaneously.

To create the causality paradox, we require two "magic-phones" capable of sending messages with nearly infinite speed. Unicorn icons use arrows to depict the information's direction of travel: magic phone #1 transmits from Alice to Bob, with #2 from Bob to Alice. Magic phone #1 is situated on the moving train. When Alice shows her message through the front window as the train passes her in October, a passenger inside relays the message via magic phone #1 to the train's rear, where Bob can see it through a window. Bob immediately relays the message back to Alice via the land-based magic phone #2 in May, five months before she sent it.

Our distant observer will likely take a skeptical view of all this. The slope of the slanted train's image indicates that the distant observer will see magic phone #1 sending information from Bob to Alice, opposite to what the passengers perceive. The distant observer will first see the message inside the rear of the train (when it was adjacent to Bob in May). That message will immediately begin to travel towards of Alice, faster than the speed of light, but slow enough that Alice will not receive ~~the~~ it until October. Meanwhile, Bob sends the same message via land-based phone #2 to Alice, who receives it in May. Alice waits for almost five months, until she prepares to send the same message, showing ~~it~~ it through the front window just before the message also arrives at the front via the train-based magic phone #1. It would appear to the distant observer that the events depicted in Fig. 4 had been artificially staged.

Referee collusion:Determinism loophole



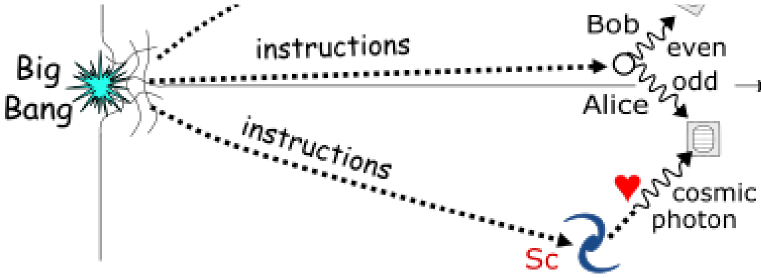


Figure 5 | In a superdeterministic universe, cosmic photons from two distant spiral galaxies were destined to arrive on Earth with properties that trigger the filters to ask the ♥ & ♠ questions of photons just prior to their arrival with a wining combination of (even/odd) answers.

Figure 5 is inspired by a comment made by Bell during a 1985 radio interview that mentioned something he called "superdeterminism".[12] The graph is a timeline that depicts the big bang, beginning at a time when space and time were too confusing for us to graph. At this beginning, "instructions" were established that would dictate the entire future of the universe, from every action taken by every human being, to the energy, path, and polarization of every photon that will ever exist. Long ago, obscure atoms in two distant galaxies (Sb and Sc) were instructed to each emit what will become "cosmic photons" that strike Earth. Meanwhile, "instructions" will call for humans to evolve on Earth and create a Bell's theorem experiment that uses the frequency and/or polarization of cosmic photons to set the polarization measurement angles while the entangled photons Alice and Bob are still en route to the detectors. Alice and Bob will arrive at their destinations already "knowing" how to respond because the cosmic photons were "instructed" to have properties that cause the questions to be "heart" and "spade". Four comments about this hypothetical scenario are in order:

1. Efforts to design an experiment similar to the one shown in Fig. 5 are underway.[13] Also, note how this experiment does not "close" the loophole, but instead greatly expands the scale of any "collusion" between the parent atom and detectors.
2. There is a reason for Bell's allusion to a completely "superdeterministic" universe where nothing happens by chance. He was attempting to exclude a class of theories associated with "hidden variables", that might replace quantum mechanics. Discussion of such theories is pedagogically inappropriate for this paper.[14]
3. Students of quantum mechanics are encouraged to refrain from attributing a specific value of a photon's polarization (♥,♣,♠) until a measurement is actually made. Before the measurement, the "answer" is neither a heart, club or spade, but something called a "mixed state".
4. It is widely believed that quantum mechanics is consistent with causality and special relativity.[15] Figure 5 can help us visualize this if the "instructions" represent the time evolution of an exotic version of Schrödinger's equation for the entire universe. If this wave equation is deterministic, future evolution of all probability amplitudes is predetermined. Viewed this way, the events depicted in Fig. 5 are just the way things happen to turn out.

The Rimstock cheat: Detector error loophole

♥ (desired)				♠ (undesired)				♣ (undesired)			
Don't stop		Don't stop		Don't stop		Do stop		Don't stop		Do stop	
+1	+1	+1	+1	+1	−3	0	0	−3	+1	0	0
win	lose	win	lose	win	lose	win	lose	win	lose	win	lose

Figure 6 | The Rimstock cheat: Bob flips a coin to determine whether to play the cheat on this round. Alice will play "even" to hearts and spades, and "odd" to clubs.

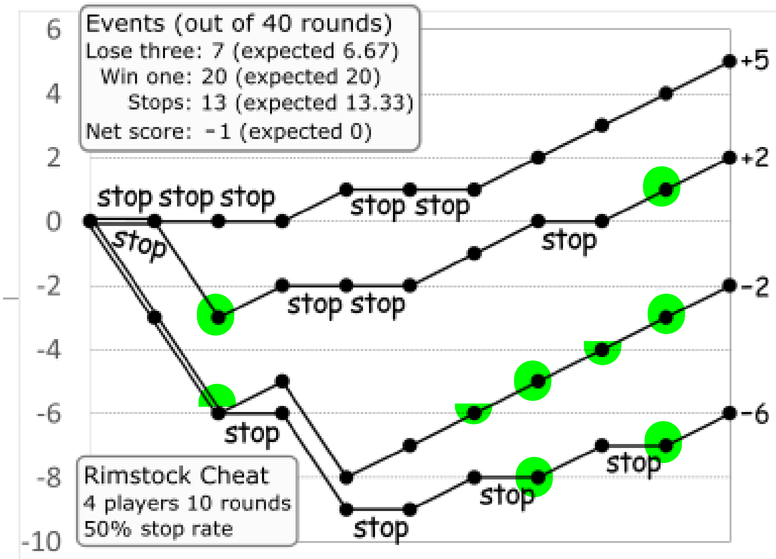


Figure 7 | Four teams of players engaging in the detector error cheat. Each connected dot represents a hand in which different questions were asked, and the horizontal dots simulate a detector error that coincided with a player receiving an unfavorable question.

The following variation of the α-strategy allows the team to match the performance of entangled particles by achieving an average score of zero: Alice preselects three answers, and informs Bob. But Bob will either answer in the same fashion, or he might abruptly stop the hand upon seeing his question card, perhaps requesting that the team take a brief break while another pair of students play the role of Alice and Bob. In a card game, this request to stop and replay a hand would require the cooperation of a gullible scorekeeper. But no detector in an actual Bell's theorem experiment is 100% efficient, and this complicates the analysis of a Bell's theorem experiment in a way that requires both careful calibration of the detector's efficiency, as well as detailed mathematical analysis.

Since this strategy never calls for Alice and Bob to give different (even/odd) answers to the same question, we may consider only rounds where the players get different questions. To understand why Bob might refuse to play a card, suppose Alice plans to answer "even" to hearts and "odd" to clubs and spades, as shown in the top row of Fig. 6. Bob is certain they will win if he sees the (favorable) heart. But if Bob sees an (unfavorable) club or spade, he knows that their chances of wining are reduced from 2/3 to only 50%. To avoid raising suspicion, Bob does not stop the game each time he sees an unfavorable question. Instead, he stops with a 50% probability upon seeing an unfavorable card. To calculated the average score, we construct a probability space consisting of equally probable outcomes, beginning with the three possible suits that Bob might see. We quadruple the size of this probability space by treating the following two pairs of events as independent and occurring with equal probability:

1. Bob will either stop the hand, or play round (*Do stop* or *Don't stop.)*
2. After seeing his question, Bob knows that Alice might receive one of only two possible questions (ignoring rounds with different questions.)

Figure 6 shows that Bob will stop the game with a probability of 1/3. But if Bob and Alice randomly share this role of stopping the game, each player will stop a given round with only 1/6, yielding an apparent detector efficiency of 5/6 = 83.$\bar{3}$%.[9]

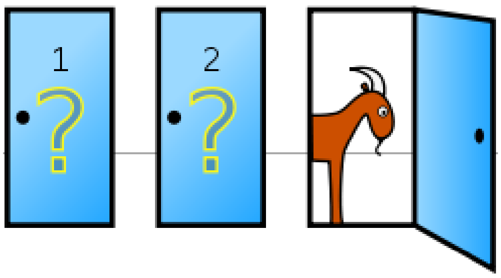
of 5/6 = 83.3%.[illegible] Typical results for a team playing this ruse are illustrated in Fig. 7. Ten rounds are played on four different occasions. The vertical axis represents ~~in~~ the team's net score, with upward steps corresponding to winning one point, and downward corresponding to losing three points. The horizontal lines showing no change in score indicate occasions where Bob or Alice refused to play an answer card (it was never necessary to ask both partners the same question in this simulation.)

Pedagogical issues

To make sixteen solitaire decks, purchase three identical standard 52 card decks. Remove only one suit (hearts, clubs, spaces) from each deck to create four solitaire sets. Each group should contain 3-5 people, and two solitaire decks (for "biased" scoring.) To avoid confusion of an ace (question card) with an (even/odd) answer card, reserve the ace for groups ~~with~~ with large even/odd number cards. For example, one group might have solitaire sets with (ace,8,9) and (king, 7,8). In a small classroom, the entire audience can observe or even give advice to one pair playing the partners' version at the front of the room. Placing the question cards adjacent to the players at the start will permit the instructor and entire class to join the partners' discussion regarding strategy during phase 1. For "neutral" scoring the instructor can either borrow question cards from the class, or convert unused "10" cards into questions. Since cheating will come so naturally, this game is not suitable for gambling (even for pennies).

Bell's theorem can lead to topics ranging from baseless pseudoscience to legitimate (but pedagogically unnecessary) speculation regarding alternatives to the theory of quantum mechanics. While few physicists are experts in such topics, all teachers will eventually face such issues in the classroom. The authors of this paper claim no expertise in any of this, and our intent is to illustrate the "spookiness" of Bell's theorem, show how one can use simple logic to prove that superluminal communication violates special relativity,[11] and ~~perhaps~~ introduce students to the concept of a "deterministic" theory or model.[15]

Appendix: The car and the goats



In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player pick door 2 instead of door 1.[16]
Cepheus, public domain

Purpose and free-will represent an important distinction between humans and elementary particles in this simulation of a Bell's theorem experiment. Unlike entangled particles, partners in the card game are attempting to maximize a score. And, unlike entangled particles, humans have the freedom to give different answers to the same question. This adds mathematical complexity arising from the knowledge gained by a player upon seeing one of the question cards in phase 2. Consider, for example, the Monty Hall problem: A contestant in a game show is shown three doors and will win whatever is behind the door he or she selects. A new car is behind one door, while the other two doors hide (less desirable) goats. After the contestant selects a door, the host shows that a goat was behind a door not selected. The host then gives the guest the opportunity to change his or her selection by instead choosing the other unopened door. Should the contestant accept this offer?

Under certain circumstances the answer is "yes".[17]

Here we argue that in the partners' version of the card game, there is no advantage to modifying an answer choice after seeing the question. There is one caveat: We must assume that the referee selects randomly among the three card suits. Any discernible pattern by the referee to favor questions of one suit would give players an advantage that we shall not analyze. This equality among all three suits permits us to study just two cases: Either both players get the same question; or they get different questions. The probability that the referee asks different questions of both players is,

$$P_D = 1 - P_S, \qquad (2)$$

where P_S is the probability that the referee selects two question cards of the same suit.

One of Alices answers is different (preselected): ♥ → o ♣ → e ♠ → e			Bob selects majority answer: $a_B = e$	Bob selects minority answer: $a_B = o$
Case 1: $P_1=2/3$ that Bob gets the majority question: q_B = ♣ (♠)				
P_D	(1a) Alice gets minority question: q_A = ♥	$a_A = o$	+ 1 point	− 3 points
P_D	(1b) Alice gets other question: q_A = ♠ (♣)	$a_A = e$	− 3 points	+ 1 point
P_S	(1c) Alice gets Bob's question: q_A = ♣ (♠)	$a_A = e$	0 points	− Q points
Case 2: $P_2=1/3$ that Bob gets the minority question: q_B = ♥				
P_D	(2a) Alice gets other question: q_A = ♠ (♣)	$a_A = e$	− 3 points	+ 1 point
P_D	(2b) Alice gets other question (2): q_A = ♣ (♠)	$a_A = e$	− 3 points	+ 1 point
P_S	(2c) Alice also gets minority question: q_A = ♥	$a_A = o$	− Q points	0 points

Figure 8 | All possible outcomes options for Bob, if Alice selects α-strategy. The minority question is ♥ because Alice will answer that one differently, and the minority question is "o" (odd) because that is Alice's answer to the minority question. Bob should select the majority question in case 1, and the minority question in case 2.

α-strategy

We begin our argument by assuming that Alice announces that she will follow an α-strategy, for example by informing Bob that hearts will be "even" (e) while clubs and spades are "odd" (o), as shown in Fig. 8.[18] To facilitate the changes of variables that allow symmetry arguments to establish to equivalent situations involving permutations of Fig. 8, it is helpful to refer to the "even" answer as the "majority" answer (since more answers are "even"), and "odd" as the "minority" answer. Likewise, "heart" is the "minority" question (since only heart has the answer that is different), while "spades" and "club" are the "majority" questions. Bob is clearly hoping that his question card will be "heart" because that guarantees a non-negative outcome if Bob also follows the same α-strategy This leads us to the following question:

> *What should Bob do if his question is not the heart?*

The answer depends on the penalty for the partners giving different answers to the same

The answer depends on the penalty for the partners giving different answers to the same question, as well as the probability that they will be asked the same question. We will show that Bob's best strategy is to also play the same α-strategy ("even" to hearts and "odd" to spades and clubs.) Each case (favorable and unfavorable) must be considered separately, and expectation values for all possible strategies must be calculated:

Case 1: Bob sees the "unfavorable" card (spade or club). Since Bob does not know which question Alice will see, he must calculated the expected score for each three subcases (1a, 1b, 1c). And he must do this calculation for each (even/odd) option at his disposal. The 2×3 array the upper right corner shows all the outcomes associated with all subcases. Since Bob cannot know which subcase will occur, he must chose between the first or second column. In case 1, it is clear that the first column (e="even") is the better choice. Keep in mind that these subcases are not equally probable. The probability of 1c is P_D, and we refer to this as the "majority" subcase, since the referee has (unknowingly) selected the suit associated with the "majority" of Alice's answers (i.e., the "even" spades/clubs because Alice selected them to be the same.) The probability of either 1a or 1b occurring is P_S. Since we have already assumed that 1a and 1b are equally probable, Bob's expectation value for each choice is:

$$\begin{aligned} E^{\alpha 1}_{maj} &= \frac{1-3}{2}P_D && = -\frac{1}{2}P_D && \text{(majority:even)} \\ E^{\alpha 1}_{min} &= \frac{1-3}{2}P_D - QP_S && = -\frac{1}{2}P_D - QP_S && \text{(minority:odd)} \end{aligned} \tag{3}$$

Here we have used the subscripts (maj/min) to denote the (majority/minority) card. It is clear that "even" is Bob's better choice whenever the penalty is positive $(Q \geq 0)$. This is the same answer Alice would give to a club or spade, and therefore we have concluded that in case 1, Bob should also follow the same α-strategy that Alice chose.

Case 2: Bob sees the (favorable) heart. The expectation values for both possible answers that Bob might give are easily shown to be:

$$\begin{aligned} E^{\alpha 2}_{maj.} &= -3P_D - QP_S && \text{(majority:even)} \\ E^{\alpha 2}_{min.} &= \quad P_D && \text{(minority:odd)} \end{aligned} \tag{4}$$

Here it is clear that Bob's best choice is also to play the same (even/odd) answer to the heart that Alice would have played.

In conclusion, if Alice selects the α-strategy, Bob's optimal strategy is to also follow the same α-strategy. Combining the the best strategies of (3) and (4) to obtain the expectation value if the team uses the α-strategy we have:

$$E_\alpha = -\frac{1}{3}P_D \tag{5}$$

β-strategy

Earlier we pointed out that one strategy was for Bob to give the opposite (even/odd) answer to each choice made by Alice. Here we also include the possibility that Bob attempts to override Alice's decision by giving the same answer as Alice. The expectation values for both of Bob's options are:

$$\begin{aligned} E^{\beta}_{opposite} &= P_D - QP_S \qquad \text{(opposite Alice)} \\ E^{\beta}_{same} &= -3P_D \qquad \text{(same as Alice)} \end{aligned} \tag{6}$$

The "neutral" scoring system associated with the equality in (1) is obtained by equating E_α to $E^{\beta}_{opposite}$ in (5) and (6). The other strategy associated with (6) is mathematically unsound, but psychologically feasible. Why would Bob opt for E^{β}_{same} and select a strategy that is guaranteed to lose three points? Perhaps Alice is convinced that the referee will ask different questions and announces that all her answers will be "even". Bob disagrees and overrides Alice's decision because he is certain that the same question will be asked. From Bob's perspective, it is better to lose 3 points than incur the penalty of Q points.

Completing the proof

There are eight ways Alice can select (even/odd) answers to each suit. Six of them are covered by Fig. 8, by interchanging the symbols (e,o) and/or (♥,♣,♠).[18] The other two fall under the β-strategy. But what if Alice also changes her mind? This cannot improve the score because Bob has already optimized his response for each choice Alice might make; assigning probabilities to Alice's choices will not increase the expected score.

Footnotes

1. Bell, John S. (1964). "On the Einstein Podolsky Rosen Paradox". *Physics* **1** (3): 195–200. doi:10.1103/physicsphysiquefizika.1.195 https://wikiversity.miraheze.org/wiki/File:Bell_1964_on_EPR_paradox.pdf
2. Mermin, N. David (1981). "Bringing home the atomic world: Quantum mysteries for anybody". *American Journal of Physics* **49** (10): 940–943. doi:10.1119/1.12594. https://wikiversity.miraheze.org/wiki/File:Mermin_AJP_mysteries_for_everybody_1981.pdf.
3. Vandegrift, Guy (1995). "Bell's theorem and psychic phenomena". *The Philosophical Quarterly* **45** (181): 471–476. doi:10.2307/2220310. http://www.wright.edu/~guy.vandegrift/shortCV/Papers/bell.pdf
4. Larsson, Jan-Åke (2014). "Loopholes in Bell inequality tests of local realism". *Journal of Physics A: Mathematical and Theoretical* **47** (42): 424003. doi:10.1088/1751-8113/47/42/424003
5. In most experiments electro-optical modulators are used instead of polarizing filters, and often it is necessary to rotate one set of orientations by 90°. Giustina, Marissa; Versteegh, Marijn A. M.; Wengerowsky, Sören; Handsteiner, Johannes; Hochrainer, Armin; Phelan, Kevin; Steinlechner, Fabian; Kofler, Johannes *et al*. (2015). "Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons". *Physical Review Letters* **115** (25): 250401. doi:10.1103/physrevlett.115.250401
6. Maccone, Lorenzo (2013). "A simple proof of Bell's inequality". *American Journal of Physics* **81** (11): 854–859. doi:10.1119/1.4823600.
7. The player can lose more than 1/3 of a point per round by adopting the obviously bad strategy of making all three answers the same (all even or all odd.) This is closely related to the fact that Bell's *"inequality"* is not Bell's *"equation"*.
8. User:Rimstock
9. Garg, Anupam; Mermin, N. David (1987). "Detector inefficiencies in the Einstein-Podolsky-Rosen experiment". *Physical Review D* **35** (12): 3831–3835. doi:10.1103/physrevd.35.3831
10. Aspect, Alain; Dalibard, Jean; Roger, Gérard (1982). "Experimental Test of Bell's Inequalities Using Time- Varying Analyzers". *Physical Review Letters* **49** (25): 1804–1807. doi:10.1103/physrevlett.49.1804
11. Liberati, Stefano; Sonego, Sebastiano; Visser, Matt (2002). "Faster-than-c Signals,

Special Relativity, and Causality". *Annals of Physics* **298** (1): 167–185. doi:10.1006/aphy.2002.6233.
12. Kleppe, A. (2011). "Fundamental Nonlocality: What Comes Beyond the Standard Models". *Bled Workshops in Physics* **12**. pp. 103–111.
13. Gallicchio, Jason; Friedman, Andrew S.; Kaiser, David I. (2014). "Testing Bell's Inequality with Cosmic Photons: Closing the Setting-Independence Loophole". *Physical Review Letters* **112** (11): 110405. doi:10.1103/physrevlett.112.110405
14. Bell, John S. (2004). "Introduction to hidden-variable question". *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy*. Cambridge University Press. pp. 29–39. doi:10.1017/cbo9780511815676.006
15. See also Ballentine, Leslie E.; Jarrett, Jon P. (1987). "Bell's theorem: Does quantum mechanics contradict relativity?". *American Journal of Physics* **55** (8): 696–701. doi:10.1119/1.15059.
16. wikipedia:special:permalink/818327855
17. Gillman, Leonard (1992). "The Car and the Goats". *The American Mathematical Monthly* **99** (1): 3–7. doi:10.2307/2324540.
18. In Fig. 8 we have purposefully maintained two parallel notations. Referring to "even/odd" as "e/o" will permit us to later reverse the meanings of the variables by allowing "o" and "e" to instead signify "odd" and "even", respectively. Likewise, we can later imagine using the symbol ♣ to instead represent spades

---