Background – Laplace Transform (3A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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An Improper Integration

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Complex Number — Real Number — Real Number

$$s = \sigma + i\omega$$

σ: $\Re\{s\}$ real part

ω: $\Im\{s\}$ imag part

t Integration Variable

The improper integral **converges** if the limit defining it exists.

F(s) : a function of s

For a given function f(t)

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

$$g(s,t) = f(t) e^{-st}$$

$$\frac{\partial}{\partial t} G(s,t) = g(s,t)$$

G: an antiderivative of **g** with respect to **t**

$$\int_0^\infty g(s,t)\, dt = \lim_{b \to \infty} \left[G(s,t) \right]_0^b$$

$$= \lim_{b \to \infty} \left[G(s,b) - G(s,0) \right]$$

During integration, complex variable **s** is treated as a constant

In the result, the literal ***t*** vanishes

$$\int_0^\infty g(s,t)\, dt = F(s)$$ *a function of* ***s***

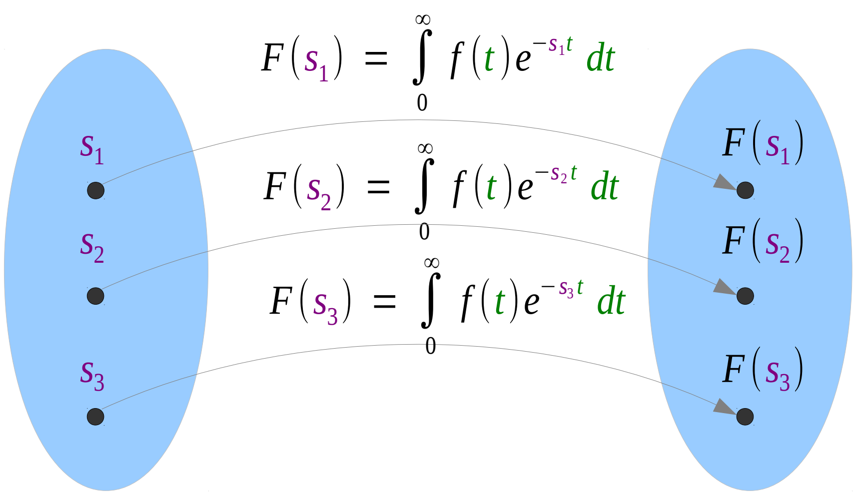
For a given function f(t)

$$F(s) = \int_0^{\infty} f(t) e^{-st} \, dt$$

Complex Number — Real Number

$$s = \sigma + i\omega$$

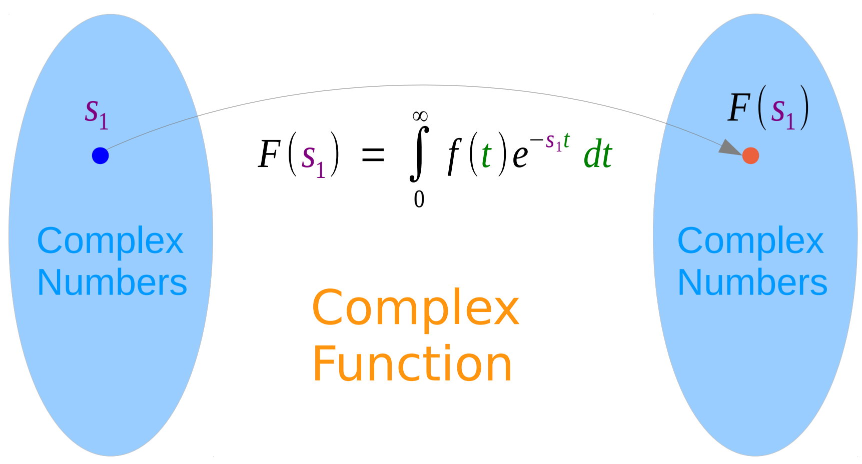
t — Real Number, Integration Variable



Complex Numbers — Complex Numbers

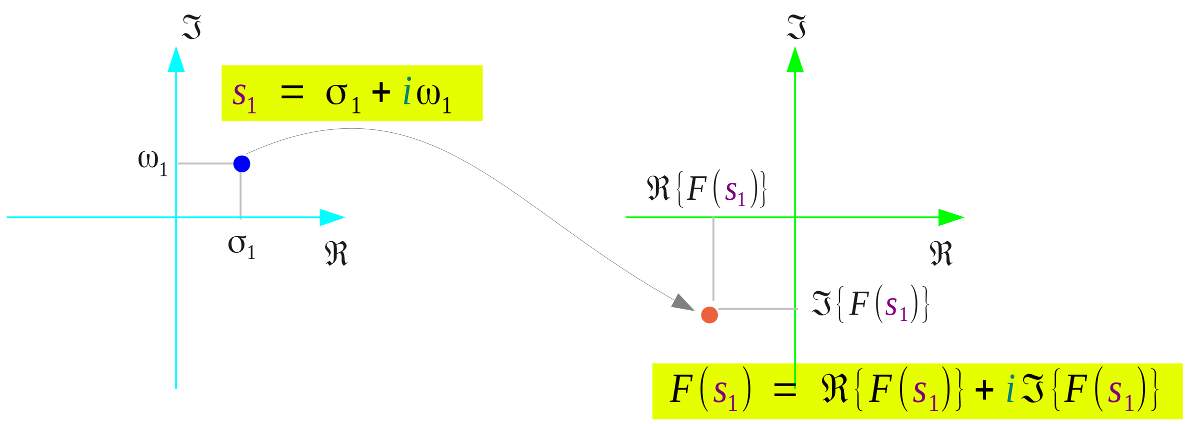
F(s) : a Complex Function

For a given function f(t)

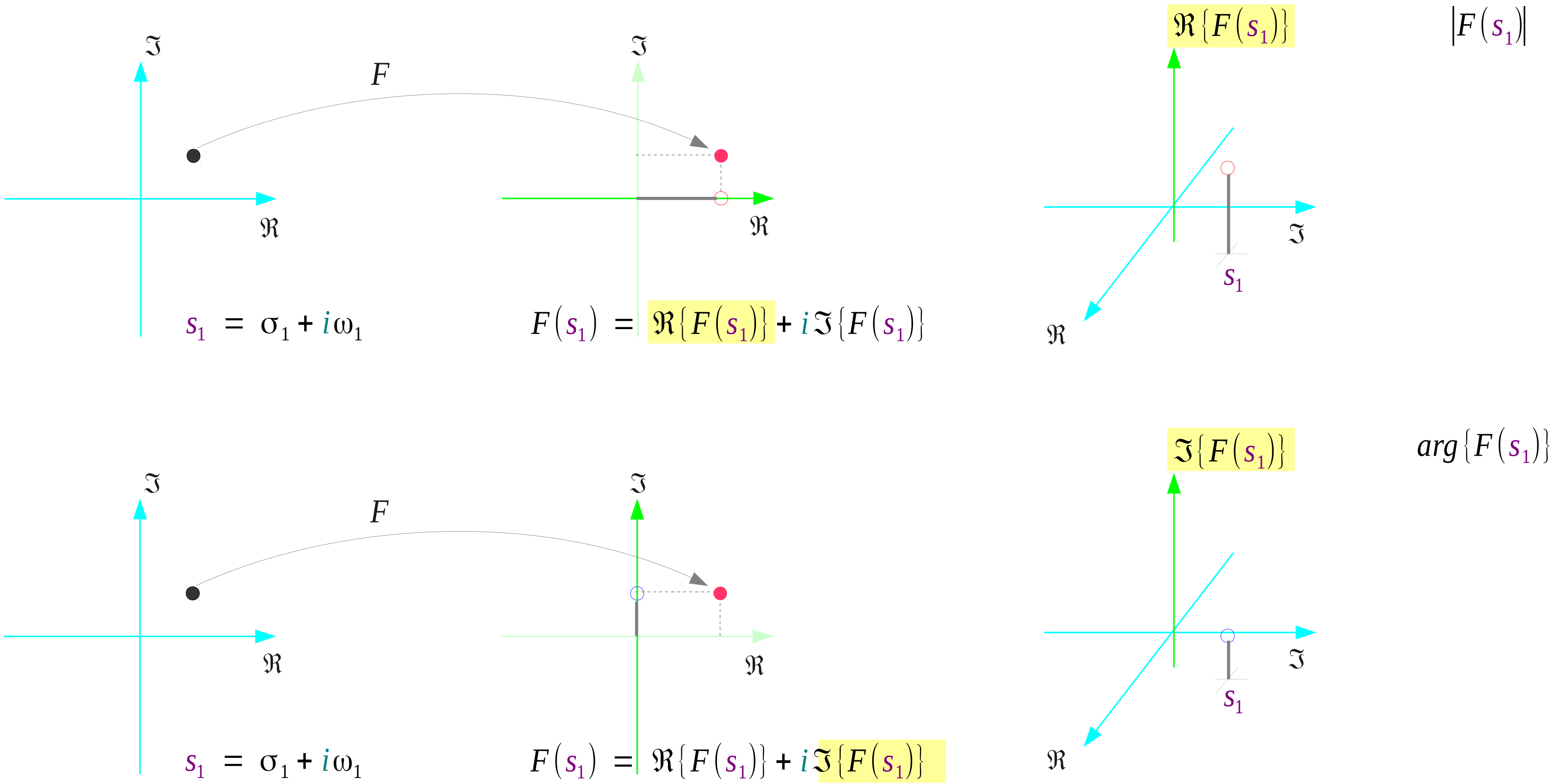


Complex Number — Real Number

$$s = \sigma + i\omega$$



Complex Function Plot



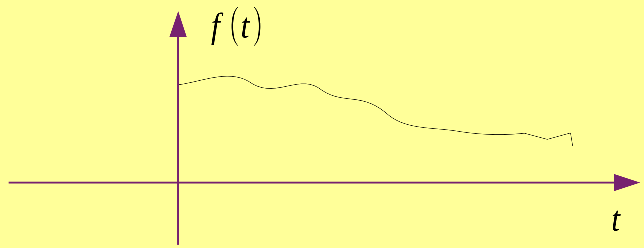
Two Functions: f(t) & F(s)

For a given function f(t)

there exits a unique F(s)

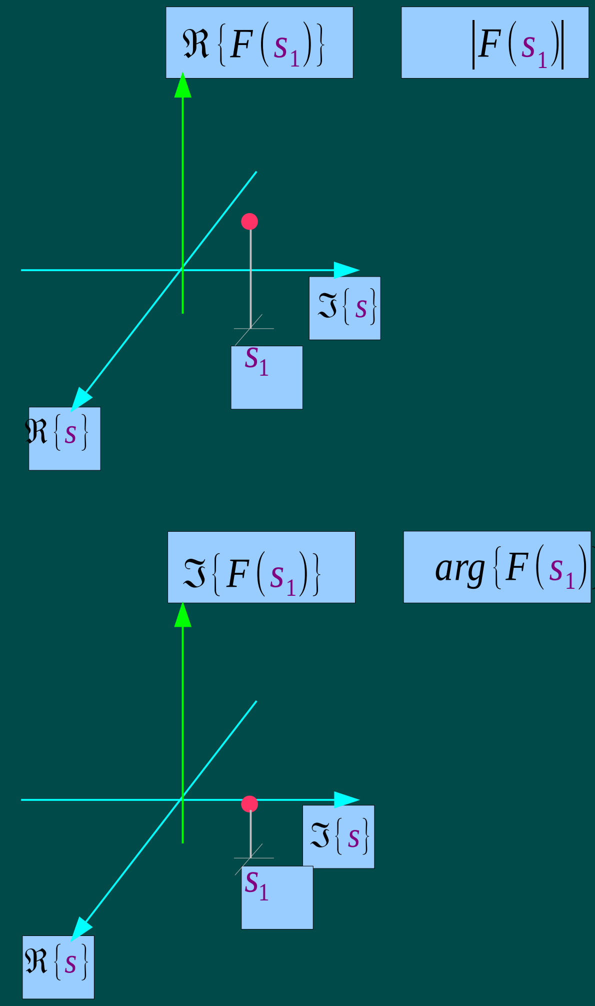
$$f(t) \Longleftrightarrow F(s)$$

t-domain function **f(t)**



Real number domain function f(t)

s-domain function **F(s)**



Complex number domain function F(s)

Laplace Transform

$f_a(x) \rightarrow$ **L** $\rightarrow F_A(s)$

$f_b(t) \rightarrow$ **L** $\rightarrow F_B(s)$

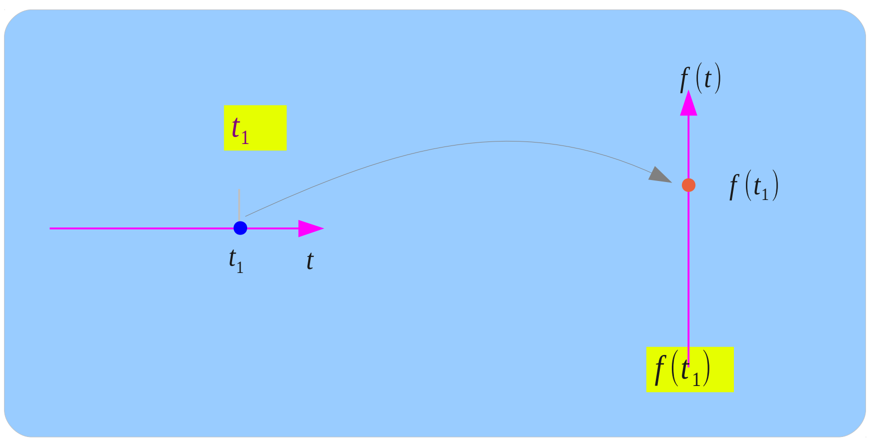
$f_c(t) \rightarrow$ **L** $\rightarrow F_C(s)$

$1 \rightarrow \int_0^\infty f_1(t) e^{-st}\, dt = F_1(s) \rightarrow \frac{1}{s}$

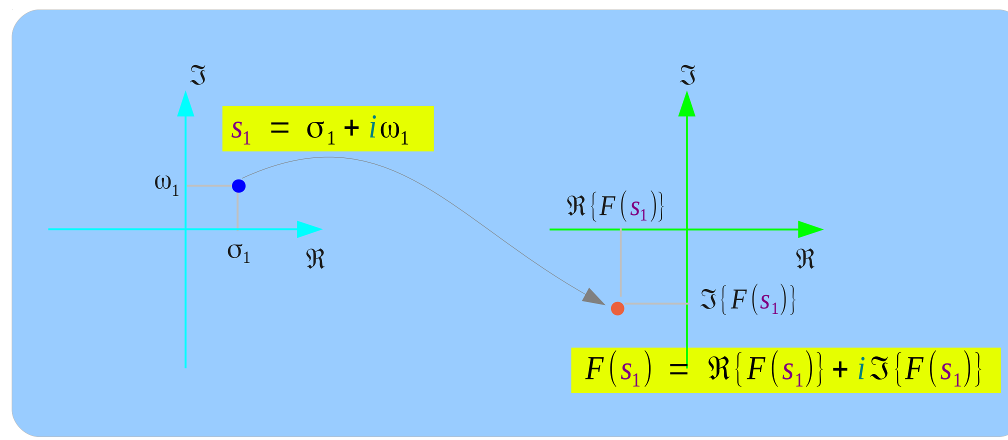
$e^{-at} \rightarrow \int_0^\infty f_2(t) e^{-st}\, dt = F_2(s) \rightarrow \frac{1}{s+a}$

$\cos(kt) \rightarrow \int_0^\infty f_3(t) e^{-st}\, dt = F_3(s) \rightarrow \frac{s}{s^2+k^2}$

$f_a(x)$ Real-valued Function



$F_A(s)$ Complex Function



Laplace transforms of 1 and exp(–at)

$1 \rightarrow$ **L** $\rightarrow \dfrac{1}{s}$

$$F(s) = \int_0^\infty 1 \cdot e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s0}\right]$$

$$-s < 0 \Rightarrow \lim_{b\to\infty} e^{-sb} = 0 \qquad s > 0 \Rightarrow F(s) = \frac{1}{s}$$

$e^{-at} \rightarrow$ **L** $\rightarrow \dfrac{1}{s+a}$

$$F(s) = \int_0^\infty e^{-at} \cdot e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)t}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{(s+a)}e^{-(s+a)b} + \frac{1}{(s+a)}e^{-(s+a)0}\right]$$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

Laplace transforms of exp(+at) and exp(–at)

$e^{-at} \;\rightarrow\; \boxed{L} \;\rightarrow\; \dfrac{1}{s+a}$

$$F(s) = \int_0^\infty e^{-at} \cdot e^{-st}\, dt = \lim_{b \to \infty} \left[-\frac{1}{(s+a)} e^{-(s+a)t} \right]_0^b = \lim_{b \to \infty} \left[-\frac{1}{(s+a)} e^{-(s+a)b} + \frac{1}{(s+a)} e^{-(s+a)0} \right]$$

$-(s+a) < 0 \Rightarrow \lim_{b \to \infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \dfrac{1}{(s+a)}$

$e^{+at} \;\rightarrow\; \boxed{L} \;\rightarrow\; \dfrac{1}{s-a}$

$$F(s) = \int_0^\infty e^{+at} \cdot e^{-st}\, dt = \lim_{b \to \infty} \left[-\frac{1}{(s-a)} e^{-(s-a)t} \right]_0^b = \lim_{b \to \infty} \left[-\frac{1}{(s-a)} e^{-(s-a)b} + \frac{1}{(s-a)} e^{-(s-a)0} \right]$$

$-(s-a) < 0 \Rightarrow \lim_{b \to \infty} e^{-(s-a)b} = 0 \qquad s > +a \Rightarrow F(s) = \dfrac{1}{(s-a)}$

Laplace transforms of cosh(kt) and sinh(kt)

$\cosh(kt) \rightarrow \boxed{L} \rightarrow \dfrac{s}{s^2-k^2}$

$$\cosh(kt) = \frac{(e^{+kt} + e^{-kt})}{2}$$

$$F(s) = \int_0^\infty \frac{(e^{+kt}+e^{-kt})}{2} \cdot e^{-st}\, dt = \frac{1}{2}\int_0^\infty e^{+kt} \cdot e^{-st}\, dt + \frac{1}{2}\int_0^\infty e^{-kt} \cdot e^{-st}\, dt$$

$s > +k \rightarrow \dfrac{1}{(s-k)}$ $\quad$ $s > -k \rightarrow \dfrac{1}{(s+k)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-k)} + \frac{1}{(s+k)}\right) = \frac{s}{(s^2-k^2)}$$

$\sinh(kt) \rightarrow \boxed{L} \rightarrow \dfrac{k}{s^2-k^2}$

$$\sinh(kt) = \frac{(e^{+kt} - e^{-kt})}{2}$$

$$F(s) = \int_0^\infty \frac{(e^{+kt}-e^{-kt})}{2} \cdot e^{-st}\, dt = \frac{1}{2}\int_0^\infty e^{+kt} \cdot e^{-st}\, dt - \frac{1}{2}\int_0^\infty e^{-kt} \cdot e^{-st}\, dt$$

$s > +k \rightarrow \dfrac{1}{(s-k)}$ $\quad$ $s > -k \rightarrow \dfrac{1}{(s+k)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-k)} - \frac{1}{(s+k)}\right) = \frac{k}{(s^2-k^2)}$$

Laplace transforms of cos(kt) and sin(kt)

$$\cos(kt) \xrightarrow{\;\mathcal{L}\;} \frac{s}{s^2+k^2}$$

$$\cos(kt) = \frac{\left(e^{+jkt} + e^{-jkt}\right)}{2}$$

$$F(s) = \int_0^\infty \frac{\left(e^{+jkt}+e^{-jkt}\right)}{2} \cdot e^{-st}\, dt = \frac{1}{2}\int_0^\infty e^{+jkt} \cdot e^{-st}\, dt + \frac{1}{2}\int_0^\infty e^{-jkt} \cdot e^{-st}\, dt$$

$s > 0 \Rightarrow \dfrac{1}{(s-j\omega)}$ $\quad$ $s > 0 \Rightarrow \dfrac{1}{(s+j\omega)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-j\omega)} + \frac{1}{(s+j\omega)}\right) = \frac{s}{(s^2+k^2)}$$

$$\sin(kt) \xrightarrow{\;\mathcal{L}\;} \frac{k}{s^2+k^2}$$

$$\sin(kt) = \frac{\left(e^{+jkt} - e^{-jkt}\right)}{2j}$$

$$F(s) = \int_0^\infty \frac{\left(e^{+jkt}-e^{-jkt}\right)}{2j} \cdot e^{-st}\, dt = \frac{1}{2j}\int_0^\infty e^{+jkt} \cdot e^{-st}\, dt - \frac{1}{2j}\int_0^\infty e^{-jkt} \cdot e^{-st}\, dt$$

$s > 0 \Rightarrow \dfrac{1}{(s-j\omega)}$ $\quad$ $s > 0 \Rightarrow \dfrac{1}{(s+j\omega)}$

$$F(s) = \frac{1}{2}\left(\frac{1}{(s-j\omega)} - \frac{1}{(s+j\omega)}\right) = \frac{k}{(s^2+k^2)}$$

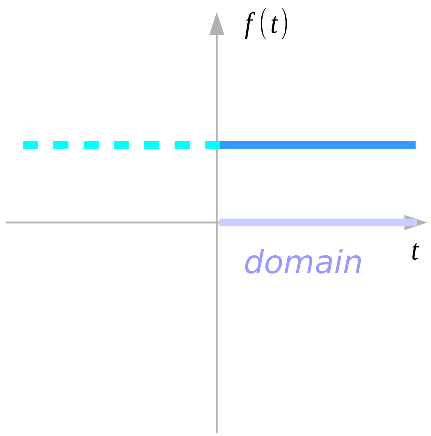
Region of Convergence

$1 \rightarrow \boxed{L} \rightarrow \frac{1}{s}$

$$\int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s0}\right] = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-sb} + \frac{1}{s}\right]$$

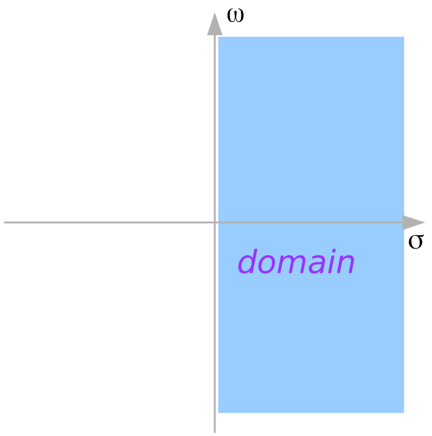
$$-s < 0 \iff -\Re\{s\} < 0$$

$$-(\sigma + i\omega) < 0 \iff -\sigma < 0$$



$$\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} e^{-(\sigma+i\omega)b} = \lim_{b\to\infty} e^{-b\sigma} e^{+ib\omega} = 0$$

$$\left|e^{+ib\omega}\right| = 1$$



right-sided function $\quad t > 0$

right-sided ROC $\quad \sigma > 0$

$$-(s+a) < 0 \Rightarrow \lim_{b\to\infty} e^{-(s+a)b} = 0 \qquad s > -a \Rightarrow F(s) = \frac{1}{(s+a)}$$

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Forward and Inverse Laplace Transform

Forward Laplace Transform

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

Integration with a real variable t

$$F(s) = \int_0^{\infty} f(t) e^{-st}\, dt$$

Inverse Laplace Transform

$$f(t) \xleftarrow{\mathcal{L}^{-1}} F(s)$$

Integration with a complex variable s

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{+st}\, ds$$

Some Laplace Transform Pairs

1	$\Longleftrightarrow$	$\dfrac{1}{s}$	e^{-at}	$\Longleftrightarrow$	$\dfrac{1}{s+a}$
t	$\Longleftrightarrow$	$\dfrac{1}{s^2}$	$\sin(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
t^2	$\Longleftrightarrow$	$\dfrac{2}{s^3}$	$\cos(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
t^3	$\Longleftrightarrow$	$\dfrac{6}{s^4}$	$\sinh(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2-\omega^2}$
t^n	$\Longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$	$\cosh(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2-\omega^2}$

Cover-up Method (1)

$$\frac{2s-1}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$$

$$\frac{2s-1}{(s+3)(s-2)}(s+3) \Rightarrow A = \frac{2s-1}{(s-2)}\Big|_{s=-3} = \frac{-6-1}{-3-2} = +\frac{7}{5}$$

$$\frac{2s-1}{(s+3)(s-2)}(s-2) \Rightarrow B = \frac{2s-1}{(s+3)}\Big|_{s=2} = \frac{4-1}{2+3} = +\frac{3}{5}$$

Cover-up Method (2)

$$\frac{2s-1}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$$

$$\left.\frac{2s-1}{(s+3)(s-2)}(s+3)\right|_{s=-3} = \left.\frac{A}{(s+3)}(s+3) + \frac{B}{(s-2)}(s+3)\right|_{s=-3} = A$$

$$\left.\frac{2s-1}{(s+3)(s-2)}(s-2)\right|_{s=2} = \left.\frac{A}{(s+3)}(s-2) + \frac{B}{(s-2)}(s-2)\right|_{s=2} = B$$

Cover-up Method : Repeated Poles (1)

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{(s+1)}$$

$$\frac{1}{s^2(s+1)} s^2 \Rightarrow A_1 = \frac{1}{(s+1)}\Big|_{s=0} = \frac{1}{1} = +1$$

$$\frac{d}{ds}\left(\frac{1}{s^2(s+1)} s^2\right) \Rightarrow A_2 = \frac{d}{ds}\left(\frac{1}{(s+1)}\right)\Big|_{s=0} = -\frac{1}{(s+1)^2}\Big|_{s=0} = -\frac{1}{1} = -1$$

$$\frac{1}{s^2(s+1)} (s+1) \Rightarrow B = \frac{1}{s^2}\Big|_{s=-1} = \frac{1}{1} = +1$$

Cover-up Method : Repeated Poles (2)

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{(s+1)}$$

$$\left.\frac{1}{s^2(s+1)} s^2 \right|_{s=0} = \left.\frac{A_1}{s^2} s^2 + \frac{A_2}{s} s^2 + \frac{B}{(s+1)} s^2 \right|_{s=0} = \left. A_1 + A_2 s + \frac{B}{(s+1)} s^2 \right|_{s=0} = A_1$$

$$\left.\frac{d}{ds}\left(\frac{1}{s^2(s+1)} s^2\right) \right|_{s=0} = \left.\frac{d}{ds}\left(A_1 + A_2 s + \frac{B}{(s+1)} s^2\right) \right|_{s=0} = \left. A_2 + \left(\frac{2Bs}{(s+1)} - B\frac{s^2}{(s+1)^2}\right) \right|_{s=0} = A_2$$

$$\left.\frac{1}{s^2(s+1)}(s+1) \right|_{s=-1} = \left.\frac{A_1}{s^2}(s+1) + \frac{A_2}{s}(s+1) + \frac{B}{(s+1)}(s+1) \right|_{s=-1} = B$$

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Cover-up Method : Inverse Laplace Transform

$$\frac{1}{s^2(s+1)} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)}$$

$$\Longleftrightarrow \quad t - 1 + e^{-t}$$

$$1 \Longleftrightarrow \frac{1}{s}$$

$$t \Longleftrightarrow \frac{1}{s^2}$$

$$e^{-at} \Longleftrightarrow \frac{1}{s+a}$$

Examples

$$\frac{2s+5}{(s-3)^2} = \frac{A_1}{(s-3)} + \frac{A_2}{(s-3)^2}$$

$$\frac{2s+5}{(s-3)^2}(s-3)^2 = \frac{A_1}{(s-3)}(s-3)^2 + \frac{A_2}{(s-3)^2}(s-3)^2 = A_1(s-3) + A_2$$

$$\left.\frac{2s+5}{(s-3)^2}(s-3)^2\right|_{s=3} = (2s+5)\big|_{s=3} = 11 = A_2$$

$$\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right) = \frac{d}{ds}\left(A_1(s-3) + A_2\right) = A_1$$

$$\left.\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right)\right|_{s=3} = 2 = A_1$$

Examples

$$X(s) = \frac{P(s)}{(s+p)(s+r)^k} = \frac{K}{(s+p)} + \frac{A_0}{(s+r)^k} + \frac{A_1}{(s+r)^{k-1}} + \cdots + \frac{A_{k-1}}{(s+r)^1}$$

$$A_0 = X(s)(s+r)^k \big|_{s=-r}$$

$$A_1 = \frac{d}{ds}[X(s)(s+r)^k] \big|_{s=-r}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{ds^2}[X(s)(s+r)^k] \big|_{s=-r}$$

$$A_m = \frac{1}{m!}\frac{d^m}{ds^m}[X(s)(s+r)^k] \big|_{s=-r}$$

Partial Fraction Methods

$$\frac{1}{\cdots(ax+b)\cdots} \Rightarrow \cdots + \frac{A}{(ax+b)} + \cdots$$

$$\frac{1}{\cdots(ax+b)^k\cdots} \Rightarrow \cdots + \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)\cdots} \Rightarrow \cdots + \frac{Ax+b}{(ax^2+bx+c)} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)^k\cdots} \Rightarrow \cdots + \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k} + \cdots$$

Differentiation in the s-domain

$$f(t) \quad \Longleftrightarrow \quad F(s) \qquad F(s) = \int_0^\infty f(t) \cdot e^{-st}\,dt$$

$$-t\,f(t) \quad \Longleftrightarrow \quad F'(s) \qquad \frac{d}{ds}F(s) = \int_0^\infty \frac{\partial}{\partial s}\left[f(t) \cdot e^{-st}\right] dt = \int_0^\infty (-t) f(t) \cdot e^{-st}\,dt$$

$$+t^2 f(t) \quad \Longleftrightarrow \quad F''(s) \qquad \frac{d^2}{ds^2}F(s) = \int_0^\infty \frac{\partial^2}{\partial s^2}\left[f(t) \cdot e^{-st}\right] dt = \int_0^\infty (-t)^2 f(t) \cdot e^{-st}\,dt$$

$$-t^3 f(t) \quad \Longleftrightarrow \quad F^{(3)}(s) \qquad \frac{d^3}{ds^3}F(s) = \int_0^\infty \frac{\partial^3}{\partial s^3}\left[f(t) \cdot e^{-st}\right] dt = \int_0^\infty (-t)^3 f(t) \cdot e^{-st}\,dt$$

$$t^n f(t) \quad \Longleftrightarrow \quad (-1)^n \frac{d^n}{ds^n} F(s)$$

Differentiation in the t-domain (1)

$f(t) \longleftrightarrow F(s)$ $\qquad F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$

$$\int_0^\infty f'(t) \cdot e^{-st}\, dt = \left[f(t) \cdot e^{-st}\right]_0^\infty - \int_0^\infty (-s) f(t) \cdot e^{-st}\, dt$$

$$= -f(0) + s\int_0^\infty f(t) \cdot e^{-st}\, dt = sF(s) - f(0)$$

$f'(t) \longleftrightarrow sF(s) - f(0)$

$f''(t) \longleftrightarrow s(sF(s) - f(0)) - f'(0)$

$f^{(3)}(t) \longleftrightarrow s(s(sF(s) - f(0)) - f'(0)) - f''(0)$

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3/21/15

Differentiation in the t-domain (2)

$$f(t) \Longleftrightarrow F(s)$$

$$f'(t) \Longleftrightarrow sF(s) - f(0)$$

$$f''(t) \Longleftrightarrow s^2F(s) - sf(0) - f'(0)$$

$$f^{(3)}(t) \Longleftrightarrow s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$f^{(n)}(t) \Longleftrightarrow s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$$

Differentiation in the t-domain (3)

$f^{(n)}(t) \Longleftrightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^1 f^{(n-2)}(0) - f^{(n-1)}(0)$

s^n $F(s)$	s^{n-1} $f^{(0)}(0)$	s^{n-2} $f^{(1)}(0)$	s^1 $f^{(n-2)}(0)$	s^0 $f^{(n-1)}(0)$
	$n-1+0$	$n-2+1$	$1+n-2$	$0+n-1$

$$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}$$

$$s(\ \cdots\ s(s(sF(s) - f(0)) - f'(0)) - f''(0)\ \cdots\ ) - f^{(n-1)}(0)$$

Differentiation Properties

$$\frac{d^n}{dt^n} f(t) \quad \Longleftrightarrow \quad s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ \cdots \ - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$t^n f(t) \quad \Longleftrightarrow \quad (-1)^n \cdot \frac{d^n}{ds^n} F(s)$$

Integration in the t-domain

$f(t) \Longleftrightarrow F(s)$

$\int_0^t f(\tau)\,d\tau \Longleftrightarrow \dfrac{F(s)}{s}$

$g(t) \Longleftrightarrow G(s)$

$g(t) = \int_0^t f(\tau)\,d\tau$, $G(s) = ?$

$f(t) \Longleftrightarrow F(s)$

$f(t) = g'(t) \Longleftrightarrow F(s) = sG(s) - g(0)$

$$f(t) = \frac{d}{dt}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{d}{dt} g(t)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$g(0) = \int_0^0 f(\tau)\,d\tau = 0$$

$$F(s) = \frac{F(s)}{s}$$

Translation in the s-domain

$f(t) \Longleftrightarrow F(s)$ $\qquad F(s) = \int_0^{\infty} f(t) \cdot e^{-st}\, dt$

$e^{+at} f(t) \Longleftrightarrow F(s-a)$ $\qquad F(s-a) = \int_0^{\infty} f(t) \cdot e^{-(s-a)t}\, dt = \int_0^{\infty} \left[e^{+at} f(t)\right] e^{-st}\, dt$

$e^{\pm at} f(t) \Longleftrightarrow F(s \mp a)$

Translation in the t-domain

$f(t) \longleftrightarrow F(s)$

$$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

$f(t-a)u(t-a) \longleftrightarrow e^{-as}F(s)$

$f(t \mp a)u(t \mp a) \longleftrightarrow e^{\mp as}F(s)$

$$\int_0^\infty f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_0^a f(t-a)u(t-a) \cdot e^{-st}\, dt + \int_a^\infty f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_a^\infty f(t-a) \cdot e^{-st}\, dt$$

$$= \int_0^\infty f(\nu) \cdot e^{-s(\nu+a)}\, d\nu$$

$$= e^{-as} \cdot \int_0^\infty f(\nu) \cdot e^{-s\nu}\, d\nu$$

$$= e^{-as} \cdot F(s)$$

$\nu = t - a \qquad d\nu = dt$

$0 = a - a$

shift right : always o.k.
shift left: only when no information is lost during improper integration by the left shift

Translation Properties

$$e^{\pm a t} f(t) \quad \Longleftrightarrow \quad F(s \mp a)$$

$$f(t \mp a) u(t \mp a) \quad \Longleftrightarrow \quad e^{\mp a s} F(s)$$

shift right : always o.k.
shift left: only when no information is lost during improper integration by the left shift

Initial Value Theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} s F(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st} \, dt$$

$$s F(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) \cdot e^{-st} \, dt$$

$$\lim_{s \to \infty} s F(s)$$

$$= f(0^-) + \lim_{s \to \infty} \int_{0^-}^{\infty} f'(t) \cdot e^{-st} \, dt$$

$$= f(0^-) + f(0^+) - f(0^-)$$

$$= f(0^+)$$

Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t)\cdot e^{-st}\, dt$$

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t)\cdot e^{-st}\, dt$$

$$\lim_{s\to 0}\left[sF(s) - f(0^-)\right] = \lim_{s\to 0}\int_{0^-}^{\infty} f'(t)\cdot e^{-st}\, dt$$

$$\lim_{s\to\infty} sF(s) = f(0^-) + \lim_{s\to 0}\int_{0^-}^{\infty} f'(t)\cdot e^{-st}\, dt$$

$$= f(0^-) + f(\infty) - f(0^-)$$

$$= f(\infty)$$

Laplace Transform and ODE's

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \Longleftrightarrow Y(s)$$

$$y'(x) \Longleftrightarrow sY(s) - y(0)$$

$$y''(x) \Longleftrightarrow s^2 Y(s) - s\,y(0) - y'(0)$$

$$e^{-3x} \Longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + \frac{1}{s+3}$$

Partitioning

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[s\,Y(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)Y(s) = (k_1 s + k_2 + 3k_1) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2

depends only on input e^{-3x}

Decomposed Y(s)

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y_{zi}(s) = \left(k_1 s + k_2 + 3k_1\right)$$

depends only on initial conditions k_1, k_2 — **No Input**

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y_{zs}(s) = \frac{1}{s+3}$$

depends only on input e^{-3x} — **No State**

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y(s) = \left(k_1 s + k_2 + 3k_1\right) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2 | **depends only on input** e^{-3x}

ZIR & ZSR

$$y_{zi}(x) \Longleftrightarrow Y_{zi}(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

Zero Input Response

$$y_{zs}(x) \Longleftrightarrow Y_{zs}(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

Zero State Response

$$y(x) \Longleftrightarrow Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)(s+3)}$$

$$y(x) \Longleftrightarrow Y(s) = Y_{zi}(s) + Y_{zs}(s)$$

Laplace Transform and IVP's

$$y'' + 3y' + 2y = 0 \qquad y(0) = k_1, \quad y'(0) = k_2$$

ZIR IVP

$$y_{zi}(x) \longleftrightarrow Y_{zi}(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = 0, \quad y'(0) = 0$$

ZSR IVP

$$y_{zs}(x) \longleftrightarrow Y_{zs}(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

Unilateral and Bilateral Laplace Transforms

Unilateral Laplace Transform

Including an Impulse at the origin

$$F_-(s) = \int_{0^-}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Leftrightarrow s F_-(s) - f(0^-)$$

Excluding an Impulse at the origin

$$F_+(s) = \int_{0^+}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Leftrightarrow s F_+(s) - f(0^+)$$

Bilateral Laplace Transform

$$F_2(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Leftrightarrow s F_2(s) - f(0)$$

To include impulse inputs

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \longleftrightarrow Y(s)$$

$$y'(x) \longleftrightarrow sY(s) - y(0^-)$$

$$y''(x) \longleftrightarrow s^2 Y(s) - s\,y(0^-) - y'(0^-)$$

$$e^{-3x} \longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

ODEs with an input g(x)

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

usually known i.c. $\quad y(0^-) = k_1, \quad y'(0^-) = k_2$

i.c. to be calculated $\quad y(0^+) = m_1, \quad y'(0^+) = m_2$

solution to be found $\quad y(t) \quad (t > 0)$

ODEs with an input g(x)

Non-homogeneous Eq

$$y'' + 3y' + 2y = e^{+x}$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \ -2$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = A e^{+x}$$

$$Ae^{+x} + 3Ae^{+x} + 2Ae^{+x} = e^{+x}$$

$$6Ae^{+x} = e^{+x} \qquad A = 1/6$$

$$y_p = \frac{1}{6} e^{+x}$$

General Solution

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6} e^{+x}$$

IVP with nonzero IC

$$y(0) = 1, \ y'(0) = 2$$

$$1 = c_1 e^0 + c_2 e^0 + \frac{1}{6} e^0$$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x} + \frac{1}{6} e^{+x}$$

$$2 = -c_1 e^0 - 2c_2 e^0 + \frac{1}{6} e^0$$

$$c_1 + c_2 + \frac{1}{6} = 1$$

$$-c_1 - 2c_2 + \frac{1}{6} = 2$$

$$c_2 = -\frac{8}{3}$$

$$c_1 = \frac{5}{6} + \frac{8}{3} = \frac{21}{6} = \frac{7}{2}$$

$$y = \frac{7}{2} e^{-x} - \frac{8}{3} e^{-2x} + \frac{1}{6} e^{+x}$$

Zero State Response

$$y(0) = 0, \ y'(0) = 0$$

$$c_1 + c_2 + \frac{1}{6} = 0$$

$$-c_1 - 2c_2 + \frac{1}{6} = 0$$

$$c_2 = +\frac{1}{3}$$

$$c_1 = -\frac{1}{3} - \frac{1}{6} = -\frac{1}{2}$$

$$y = -\frac{1}{2} e^{-x} + \frac{1}{3} e^{-2x} + \frac{1}{6} e^{+x}$$

ODEs without an input g(x)

Homogeneous Eq

$$y'' + 3y' + 2y = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \;\; -2$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

Homogeneous Solution

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Zero Input Response

$$y(0) = 1, \;\; y'(0) = 2$$

$$1 = c_1 e^0 + c_2 e^0$$

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$2 = -c_1 e^0 - 2c_2 e^0$$

$$c_1 + c_2 = 1$$

$$-c_1 - 2c_2 = 2$$

$$c_2 = -3$$

$$c_1 = 1 - (-3) = 4$$

$$y = 4e^{-x} - 3e^{-2x}$$

$$y(0) = 0, \;\; y'(0) = 0$$

$$c_1 + c_2 = 0$$

$$-c_1 - 2c_2 = 0$$

$$c_2 = 0$$

$$c_1 = 0$$

$$y = 0$$

ODEs with an input g(x)

$$y'' + 3y' + 2y = e^{+x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2Y(s) = \frac{1}{s-1}$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + \frac{1}{s-1}$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

IVP with nonzero IC

$$y(0) = 1, \quad y'(0) = 2$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)} + \frac{1}{(s-1)(s+1)(s+2)}$$

$$= -\frac{8}{3}\frac{1}{(s+2)} + \frac{7}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\longleftrightarrow \quad y = \frac{7}{2}e^{-x} - \frac{8}{3}e^{-2x} + \frac{1}{6}e^{+x}$$

Zero State Response

$$y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{1}{(s-1)(s+1)(s+2)}$$

$$= +\frac{1}{3}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\longleftrightarrow \quad y = -\frac{1}{2}e^{-x} + \frac{1}{3}e^{-2x} + \frac{1}{6}e^{+x}$$

Homogeneous Solution

$$y'' + 3y' + y = 0 \qquad y(0) = k_1, \ \ y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = 0$$

$$(s^2 + 3s + 2)\,Y(s) = k_1 s + k_2 + 3k_1$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} \qquad 0$$

homogeneous solution

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$\Longleftrightarrow \quad Y_h(s) = \frac{c_1}{(s+1)} + \frac{c_2}{(s+2)}$$

$$= \frac{c_1(s+2) + c_2(s+1)}{(s+1)(s+2)}$$

$$= \frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)}$$

$$=$$

for every initial value of y_h

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

$$= \frac{y(0)s + y'(0) + 3y(0)}{(s+1)(s+2)}$$

$$= \frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)}$$

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_h' = -c_1 e^{-x} - 2c_2 e^{-2x}$$

$$y_h(0) = c_1 + c_2$$

$$y_h'(0) = -c_1 - 2c_2$$

$$y(0) \leftarrow y_h(0)$$

$$y'(0) \leftarrow y_h'(0)$$

$$(s^2+3s+2)\,Y_h(s) - y_h(0)s - y_h'(0) - 3y_h(0) = 0 \quad \Rightarrow \quad (s+1)(s+2)\frac{(c_1+c_2)s + (2c_1+c_2)}{(s+1)(s+2)} - (c_1+c_2)s + (2c_1+c_2) = 0$$

Homogeneous Solution

$$y'' + 3y' + y = 0 \qquad y(0) = k_1, \;\; y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = 0$$

$$(s^2 + 3s + 2)\,Y(s) = k_1 s + k_2 + 3k_1 = 0$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

Zero Input Response

$$y(0) = 1, \;\; y'(0) = 2$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)}$$

$$= +4\frac{1}{(s+1)} - 3\frac{1}{(s+2)}$$

$$\Longleftrightarrow \quad y = 4e^{-x} - 3e^{-2x}$$

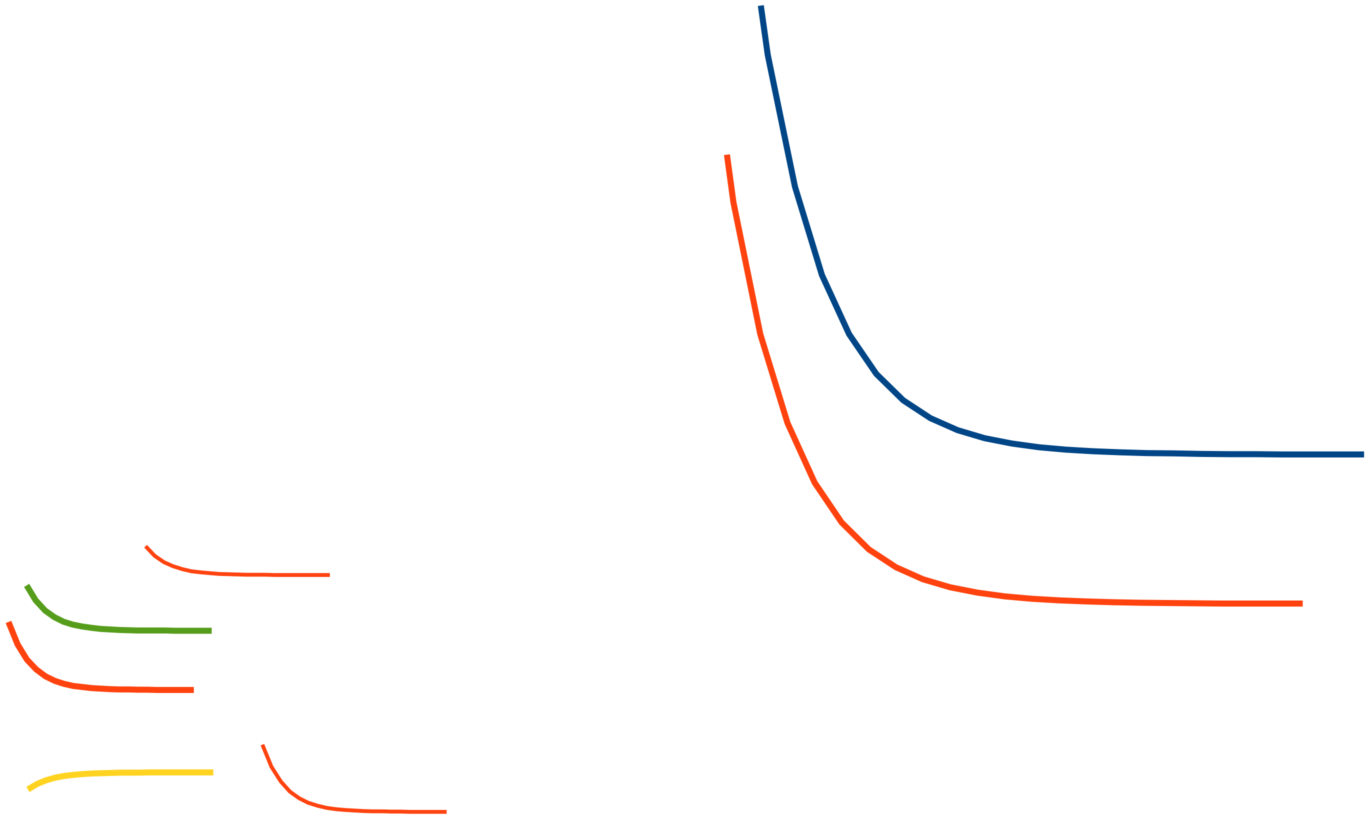
Zero Input & Zero State Response

$$y(0) = 0, \;\; y'(0) = 0$$

$$Y(s) = 0$$

$$\Longleftrightarrow \quad y = 0$$

Impulse Response h(t)



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