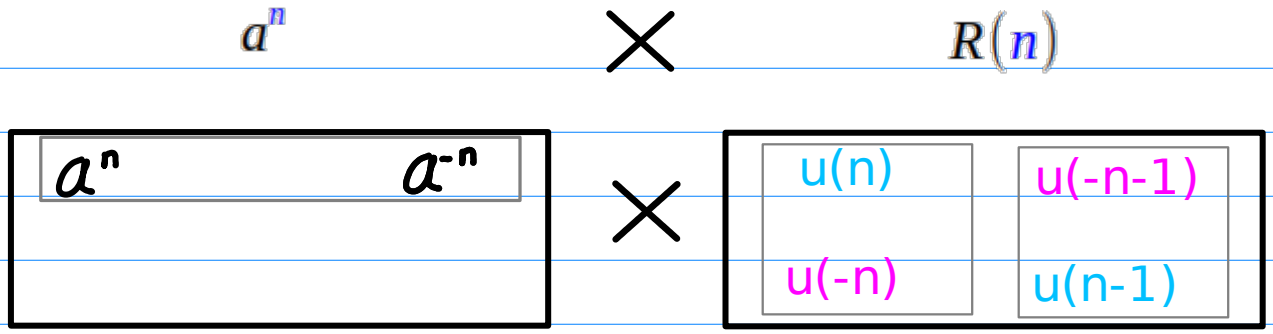
# Laurent Series and z-Transform - Geometric Series Permutations B

20240723 Tue



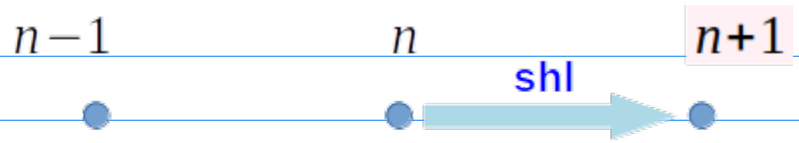

| | | | | |
|---|---|---|---|---|
| (1) | $a^n$ u(n) | $a^{-n}$ u(n) | (2) |
| (3) | $a^n$ u(-n) | $a^{-n}$ u(-n) | (4) |
| (5) | $a^n$ u(-n-1) | $a^{-n}$ u(-n-1) | (6) |
| (7) | $a^n$ u(n-1) | $a^{-n}$ u(n-1) | (8) |

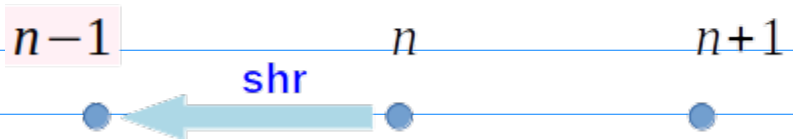
| $b^n$ | $shl(b^n)$ |
|---|---|
| $a^n$ | $a^{(n+1)}$ |
| $a^{-n}$ | $a^{-(n+1)}$ |

| $b^n$ | $shr(b^n)$ |
|---|---|
| $a^n$ | $a^{(n-1)}$ |
| $a^{-n}$ | $a^{-(n-1)}$ |

| $R(n)$ | $shl(R(n))$ |
|---|---|
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-(n+1))$ |

| $R(n)$ | $shr(R(n))$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(-(n+1))$ | $u(-n)$ |

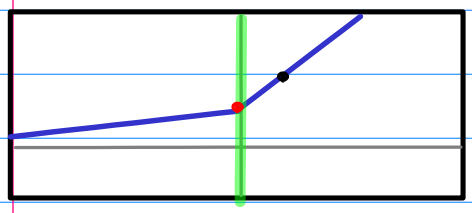


**shl(b^n)**
**shr(b^n)**

**shl(n) = n + 1**
**shr(n) = n - 1**

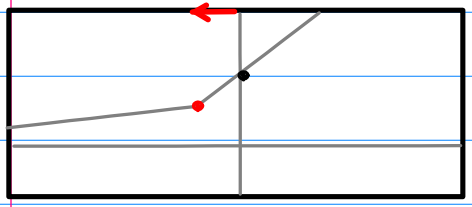
$2^n$



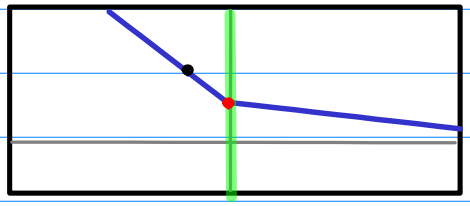
shift left

**n ← n+1**

$2^{n+1}$



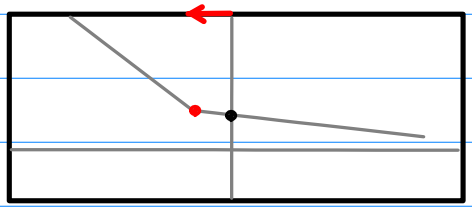
$2^{-n}$



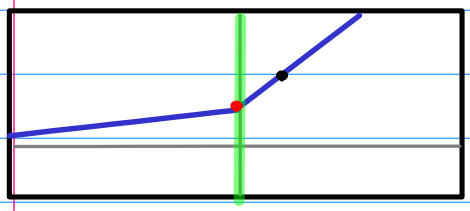
shift left

**n ← n+1**

$2^{-(n+1)}$



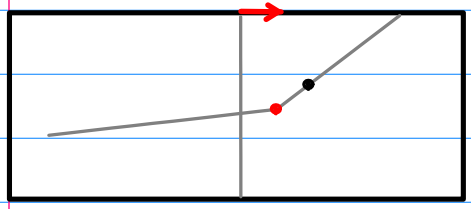
$2^n$



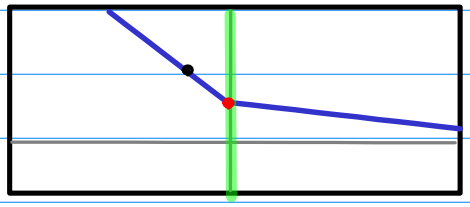
shift right

**n ← n-1**

$2^{n-1}$



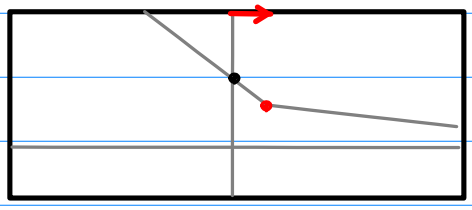
$2^{-n}$



shift right

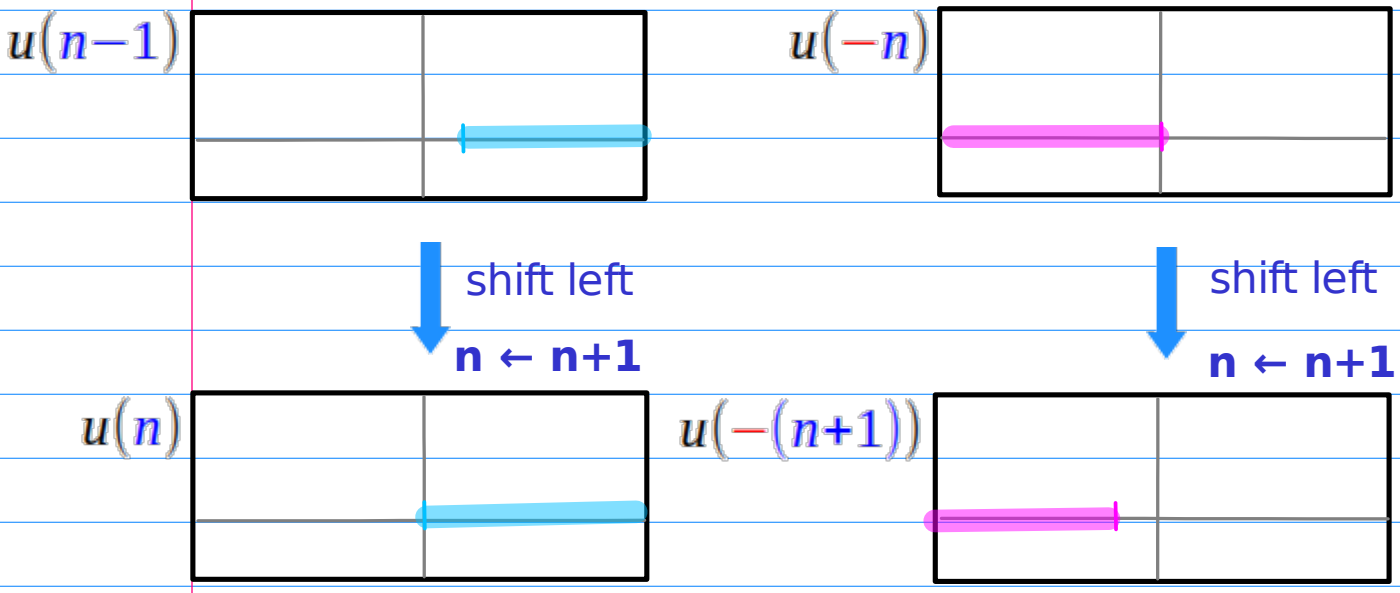
**n ← n-1**

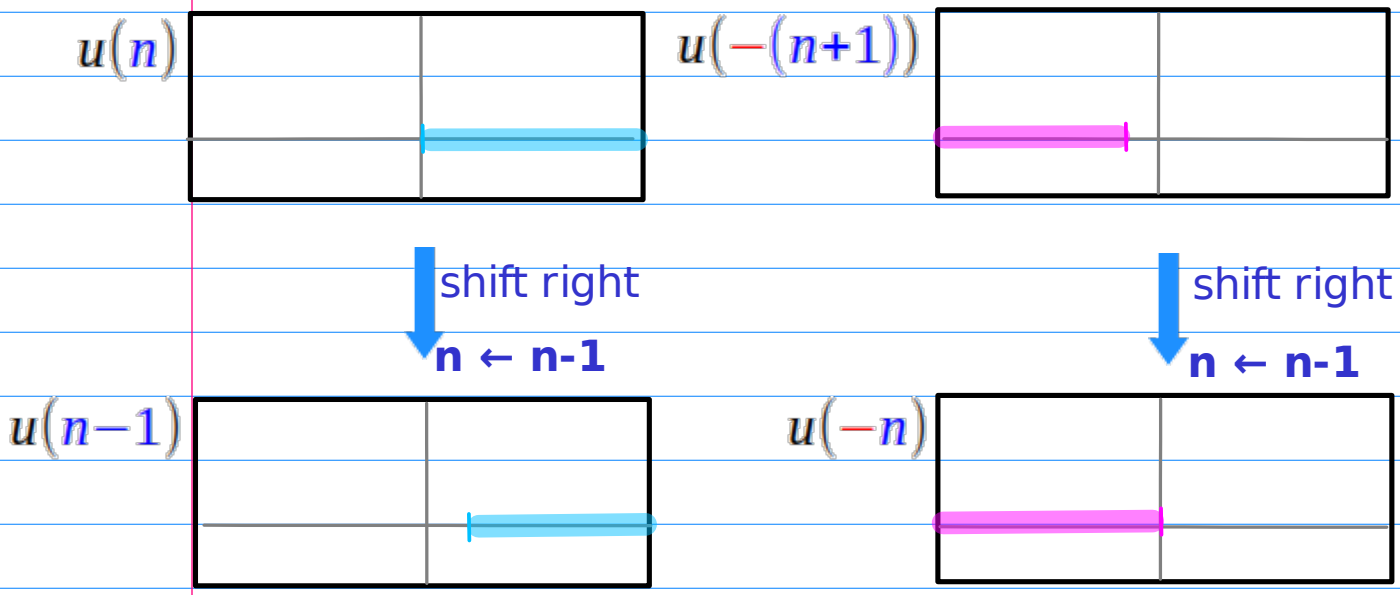
$2^{-(n-1)}$



**shl(R(n))**
**shr(R(n))**

**shl(n) = n + 1**
**shr(n) = n - 1**



$a^n$ × $R(n)$

| | |
|---|---|
| $a^{n+1}$ | $a^{-n-1}$ |
| $a^{n-1}$ | $a^{-n+1}$ |

×

| | |
|---|---|
| u(n) | u(-n-1) |
| u(-n) | u(n-1) |

| | | | |
|---|---|---|---|
| (1') | $a^{n+1}$u(n) | $a^{-n-1}$u(n) | (2') |
| (3') | $a^{n-1}$u(-n) | $a^{-n+1}$u(-n) | (4') |
| (5') | $a^{n+1}$u(-n-1) | $a^{-n-1}$u(-n-1) | (6') |
| (7') | $a^{n-1}$u(n-1) | $a^{-n+1}$u(n-1) | (8') |

| | | | |
|---|---|---|---|
| (1'') | $a^{n-1}$u(n) | $a^{-n+1}$u(n) | (2'') |
| (3'') | $a^{n+1}$u(-n) | $a^{-n-1}$u(-n) | (4'') |
| (5'') | $a^{n-1}$u(-n-1) | $a^{-n+1}$u(-n-1) | (6'') |
| (7'') | $a^{n+1}$u(n-1) | $a^{-n-1}$u(n-1) | (8'') |

$a^n$ × $R(n)$

| $a^{n+1}$ | $a^{-n-1}$ |
|---|---|

×

| u(n) | u(-n-1) |
|---|---|
| u(-n) | u(n-1) |

| | | | |
|---|---|---|---|
| (1') | $a^{n+1}u(n)$ | $a^{-n-1}u(n)$ | (2') |
| (3') | $a^{n-1}u(-n)$ | $a^{-n+1}u(-n)$ | (4') |
| (5') | $a^{n+1}u(-n-1)$ | $a^{-n-1}u(-n-1)$ | (6') |
| (7') | $a^{n-1}u(n-1)$ | $a^{-n+1}u(n-1)$ | (8') |

$a^n$ × $R(n)$

| | |
|---|---|
| $a^{n+1}$ | $a^{-n-1}$ |
| $a^{n-1}$ | $a^{-n+1}$ |

×

| | |
|---|---|
| u(n) | u(-n-1) |
| u(-n) | u(n-1) |

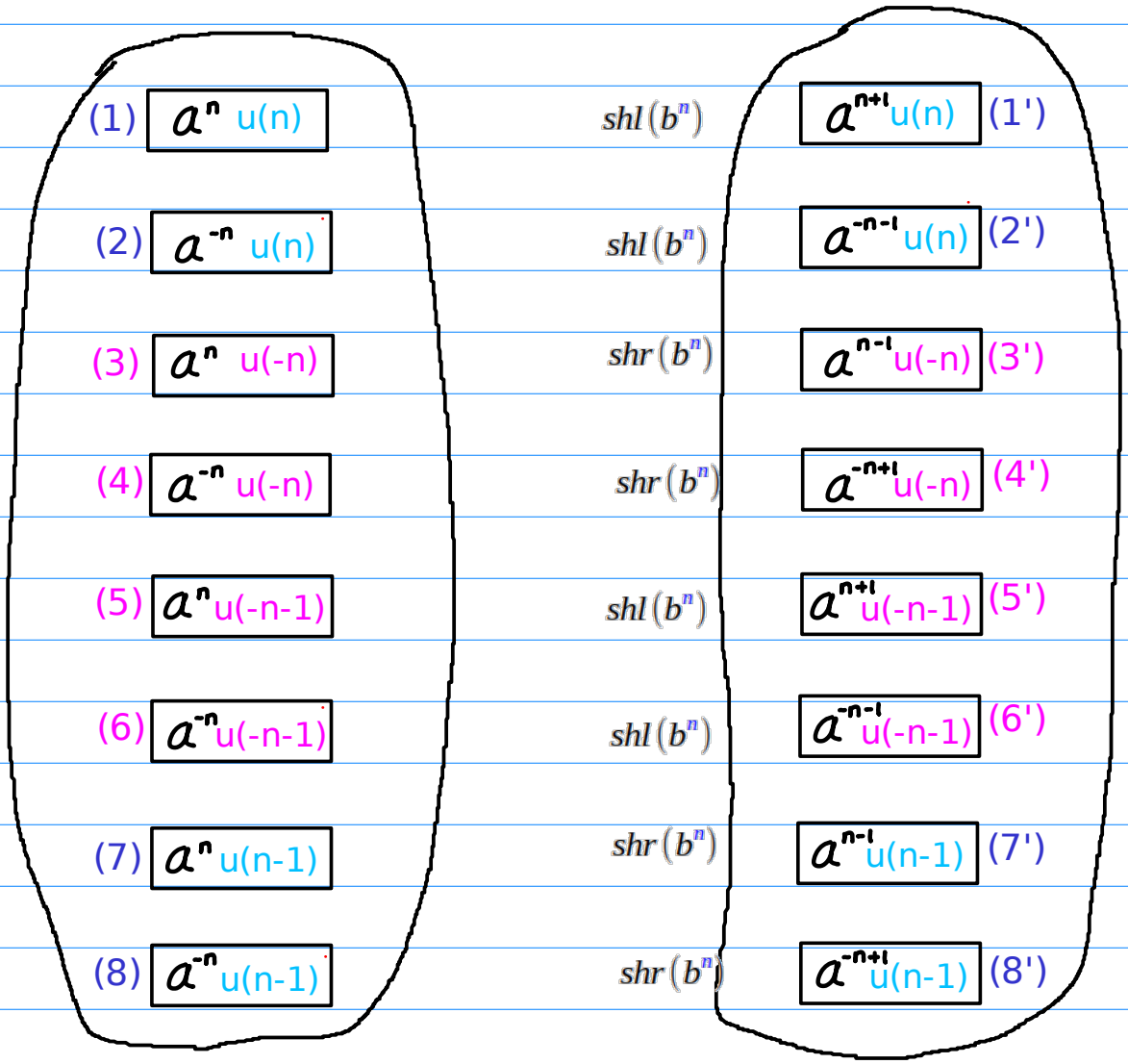
| | | | |
|---|---|---|---|
| (1'') | $a^{n-1}$ u(n) | $a^{-n+1}$ u(n) | (2'') |
| (3'') | $a^{n+1}$ u(-n) | $a^{-n-1}$ u(-n) | (4'') |
| (5'') | $a^{n-1}$ u(-n-1) | $a^{-n+1}$ u(-n-1) | (6'') |
| (7'') | $a^{n+1}$ u(n-1) | $a^{-n-1}$ u(n-1) | (8'') |

| Unshifted Sequence x | | Shifted Sequence 1 x' | | Shifted Sequence 2 x'' | | |
|---|---|---|---|---|---|---|
| (1) $a^{n}\ u(n)$ | $shl(b^{n})$ | $a^{n+1}u(n)$ (1') | $shl(b^{-n})$ | $a^{-n-1}u(n)$ (1'') | $n \leftarrow n+1$ |
| (2) $a^{-n}\ u(n)$ | $shl(b^{n})$ | $a^{-n-1}u(n)$ (2') | $shl(b^{-n})$ | $a^{n+1}u(n)$ (2'') | $n \leftarrow n+1$ |
| (3) $a^{n}\ u(-n)$ | $shr(b^{n})$ | $a^{n-1}u(-n)$ (3') | $shr(b^{-n})$ | $a^{-n+1}u(-n)$ (3'') | $n \leftarrow n-1$ |
| (4) $a^{-n}\ u(-n)$ | $shr(b^{n})$ | $a^{-n+1}u(-n)$ (4') | $shr(b^{-n})$ | $a^{n-1}u(-n)$ (4'') | $n \leftarrow n-1$ |
| (5) $a^{n}u(-n-1)$ | $shl(b^{n})$ | $a^{n+1}u(-n-1)$ (5') | $shl(b^{-n})$ | $a^{-n-1}u(-n-1)$ (5'') | $n \leftarrow n+1$ |
| (6) $a^{-n}u(-n-1)$ | $shl(b^{n})$ | $a^{-n-1}u(-n-1)$ (6') | $shl(b^{-n})$ | $a^{n+1}u(-n-1)$ (6'') | $n \leftarrow n+1$ |
| (7) $a^{n}u(n-1)$ | $shr(b^{n})$ | $a^{n-1}u(n-1)$ (7') | $shr(b^{-n})$ | $a^{-n+1}u(n-1)$ (7'') | $n \leftarrow n-1$ |
| (8) $a^{-n}u(n-1)$ | $shr(b^{n})$ | $a^{-n+1}u(n-1)$ (8') | $shr(b^{-n})$ | $a^{n-1}u(n-1)$ (8'') | $n \leftarrow n-1$ |

many possible permuations are possible
but consider these two

**Unshifted Sequence x**

**Shifted Sequence 1 x'**



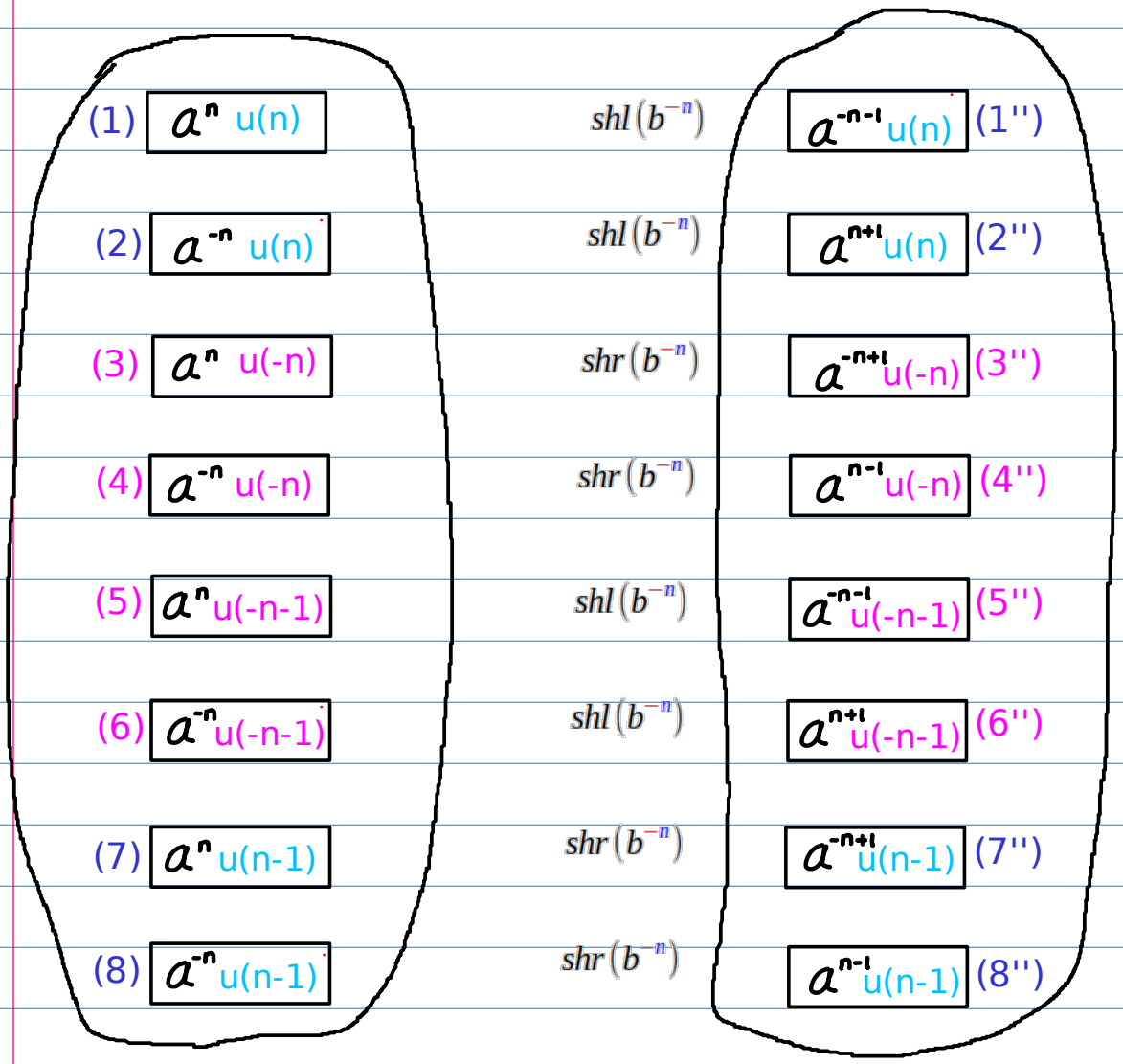

**Inter-permutations over unshifted sequnce and shifted sequence**

**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

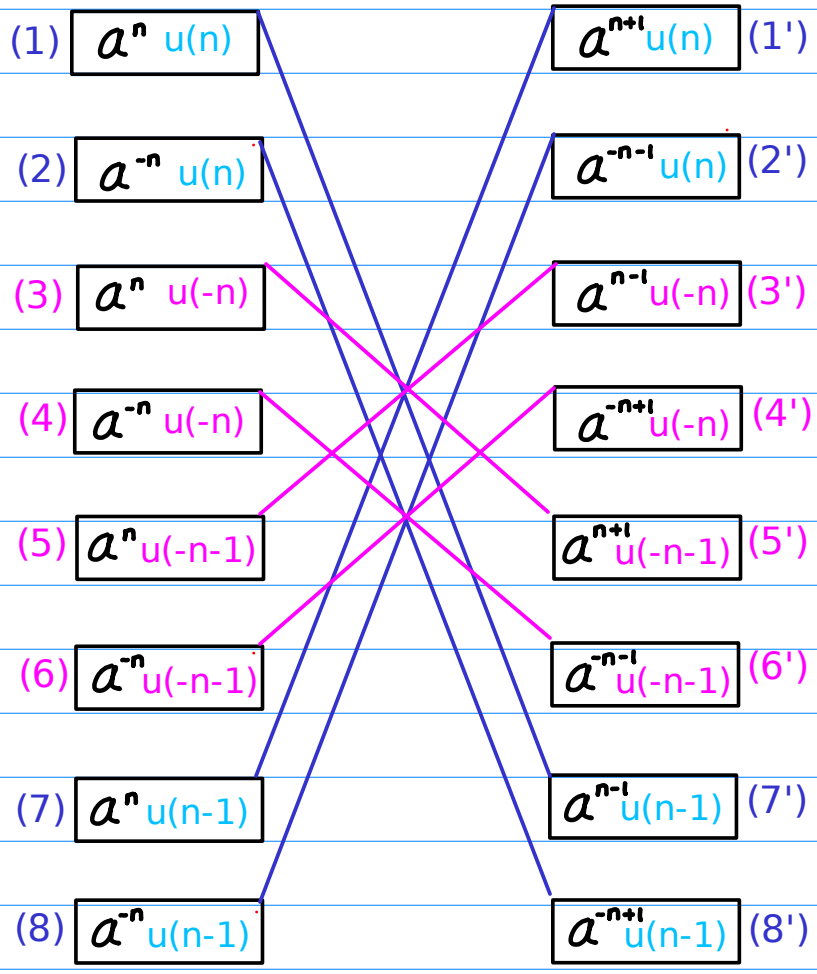
**Unshifted Sequence x**

**Shifted Sequence 2 x''**



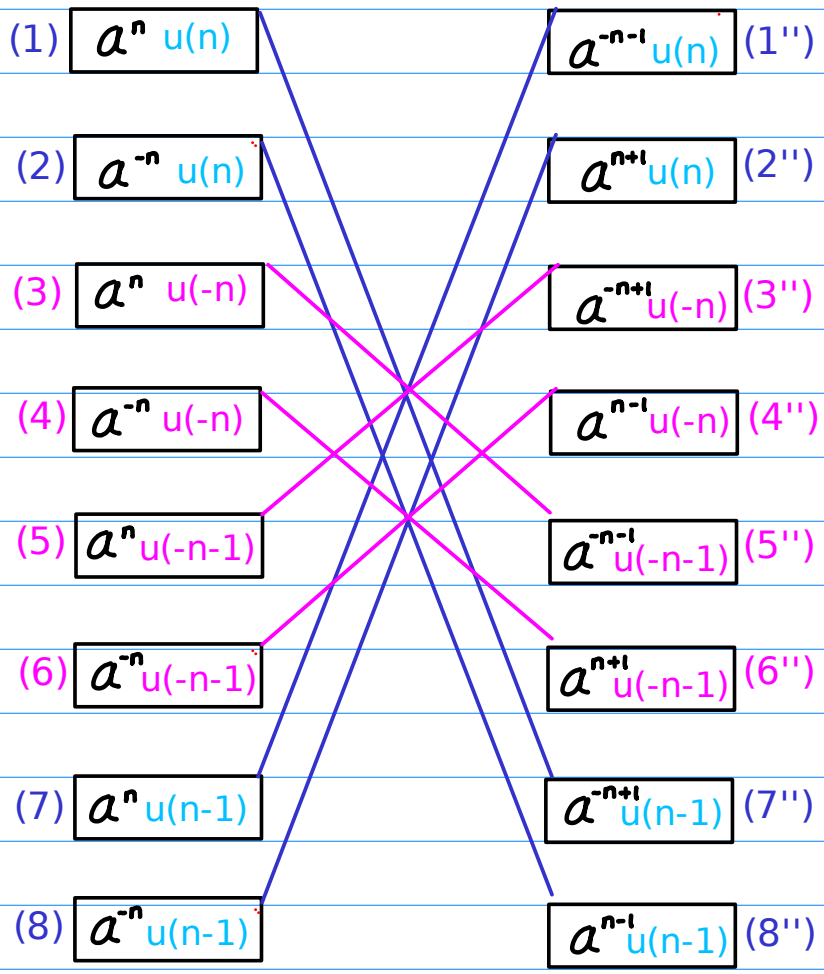
**Unshifted Sequence x** → **Shifted Sequence 1 x'**

| | | | |
|---|---|---|---|
| (1) $\frac{1}{1-2z}$ $2^n u(n)$ | (2) $\frac{1}{1-0.5z}$ $2^{-n} u(n)$ | (1') $\frac{2}{1-2z}$ $2^{n+1} u(n)$ | (2') $\frac{0.5}{1-0.5z}$ $2^{-n-1} u(n)$ |
| (3) $\frac{1}{1-0.5z^{-1}}$ $2^n u(-n)$ | (4) $\frac{1}{1-2z^{-1}}$ $2^{-n} u(-n)$ | (3') $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1} u(-n)$ | (4') $\frac{2}{1-2z^{-1}}$ $2^{-n+1} u(-n)$ |
| (5) $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^n u(-n-1)$ | (6) $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n} u(-n-1)$ | (5') $\frac{z^{-1}}{-0.5z^{-1}}$ $2^{n+1} u(-n-1)$ | (6') $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1} u(-n-1)$ |
| (7) $\frac{2z}{1-2z}$ $2^n u(n-1)$ | (8) $\frac{0.5z}{1-0.5z}$ $2^{-n} u(n-1)$ | (7') $\frac{z}{1-2z}$ $2^{n-1} u(n-1)$ | (8') $\frac{z}{1-0.5z}$ $2^{-n+1} u(n-1)$ |



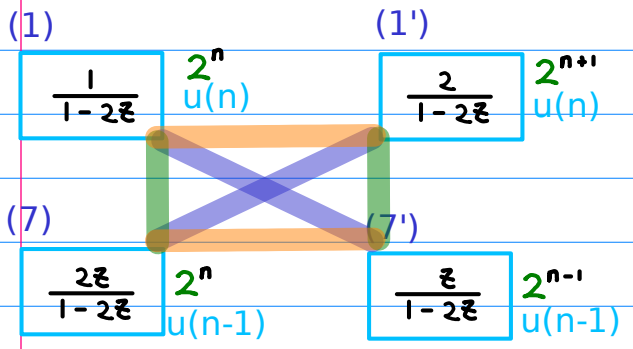
**Unshifted Sequence x** → **Shifted Sequence 2 x''**

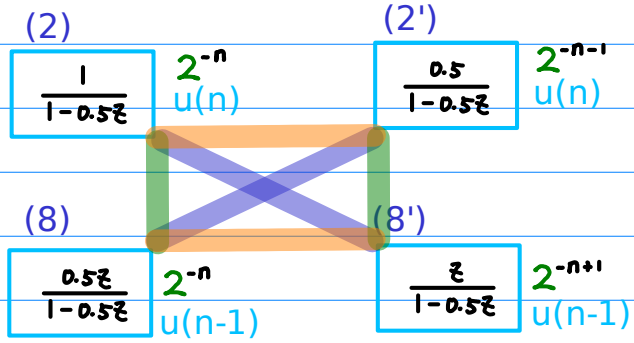
| | | | |
|---|---|---|---|
| (1) $\frac{1}{1-2z}$ $2^n u(n)$ | (2) $\frac{1}{1-0.5z}$ $2^{-n} u(n)$ | (1'') $\frac{0.5}{1-0.5z}$ $2^{-n-1} u(n)$ | (2'') $\frac{2}{1-2z}$ $2^{n+1} u(n)$ |
| (3) $\frac{1}{1-0.5z^{-1}}$ $2^n u(-n)$ | (4) $\frac{1}{1-2z^{-1}}$ $2^{-n} u(-n)$ | (3'') $\frac{2}{1-2z^{-1}}$ $2^{-n+1} u(-n)$ | (4'') $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1} u(-n)$ |
| (5) $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^n u(-n-1)$ | (6) $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n} u(-n-1)$ | (5'') $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1} u(-n-1)$ | (6'') $\frac{z^{-1}}{-0.5z^{-1}}$ $2^{n+1} u(-n-1)$ |
| (7) $\frac{2z}{1-2z}$ $2^n u(n-1)$ | (8) $\frac{0.5z}{1-0.5z}$ $2^{-n} u(n-1)$ | (7'') $\frac{z}{1-0.5z}$ $2^{-n+1} u(n-1)$ | (8'') $\frac{z}{1-2z}$ $2^{n-1} u(n-1)$ |

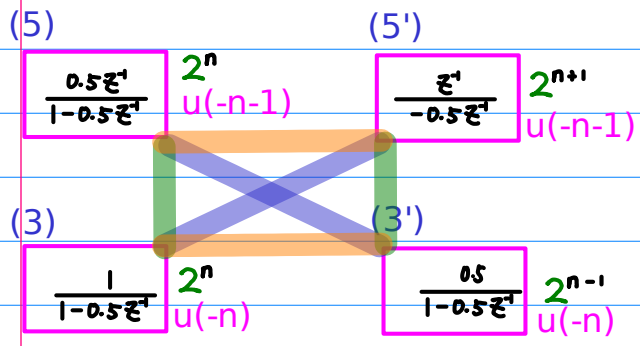


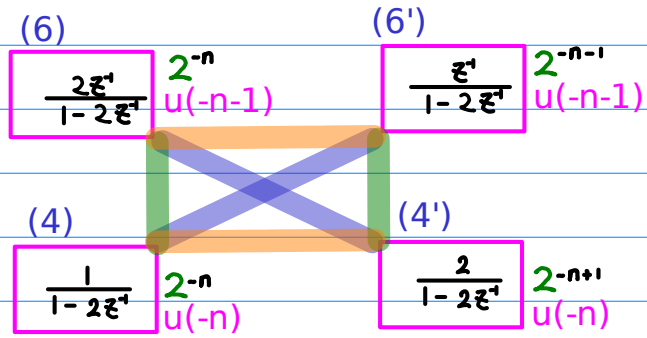
# Inter-permutation (x) → (x')

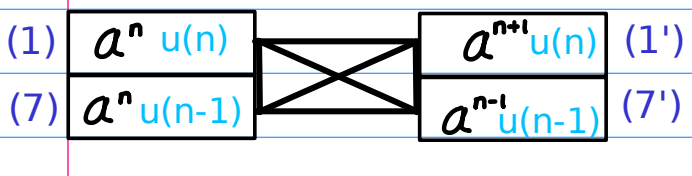
(1)~(8) (1')~(8')

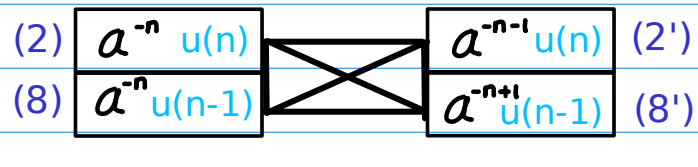


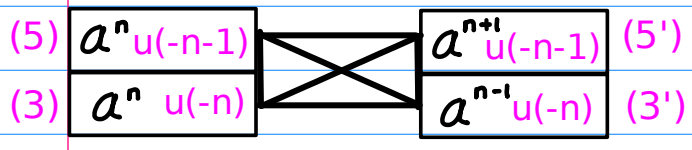



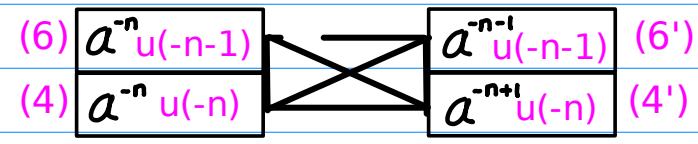



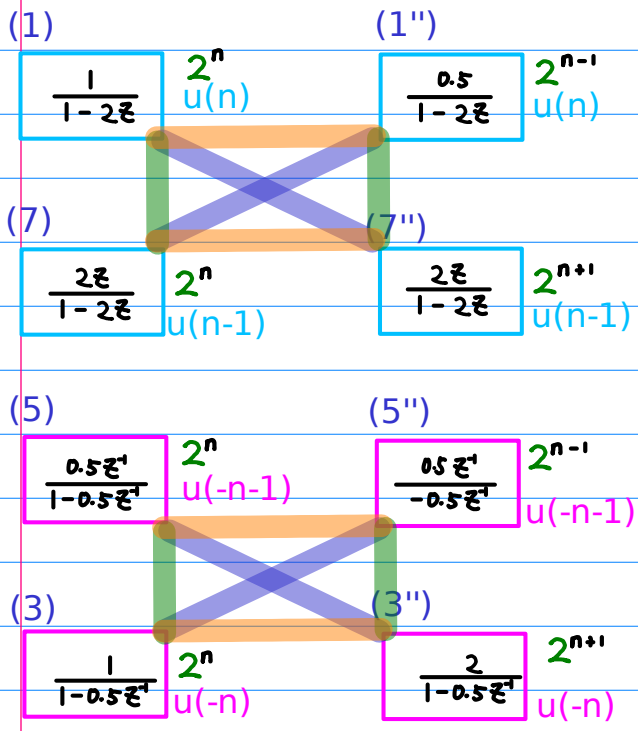



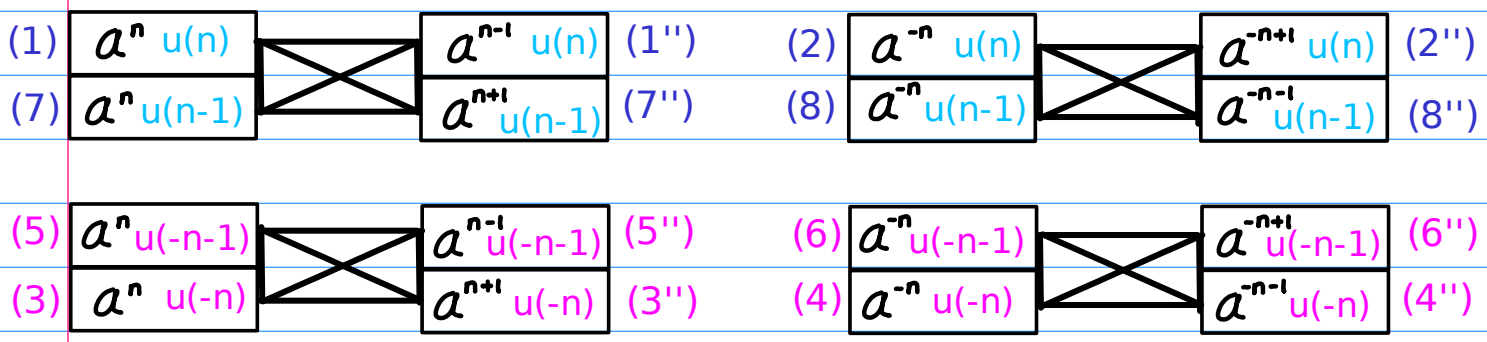



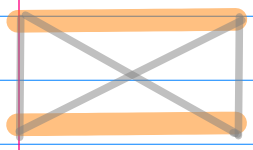
# Inter-permutation (x) → (x'')

(1)~(8) (1'')~(8'')

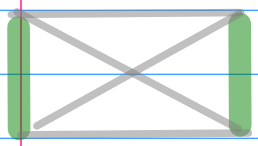


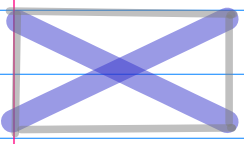
# Decomposing Shift Operations



( exponent shift, identity )



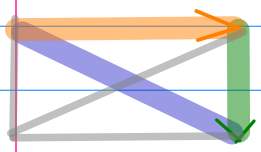
( identity, range shift )

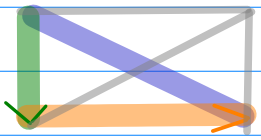


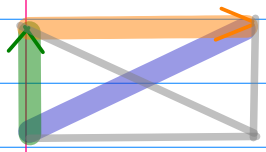
( exponent shift, range sift )

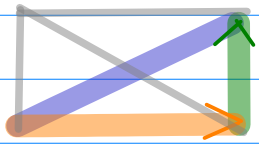
= ( exponent shift, identity )

+ ( identity, range shift )







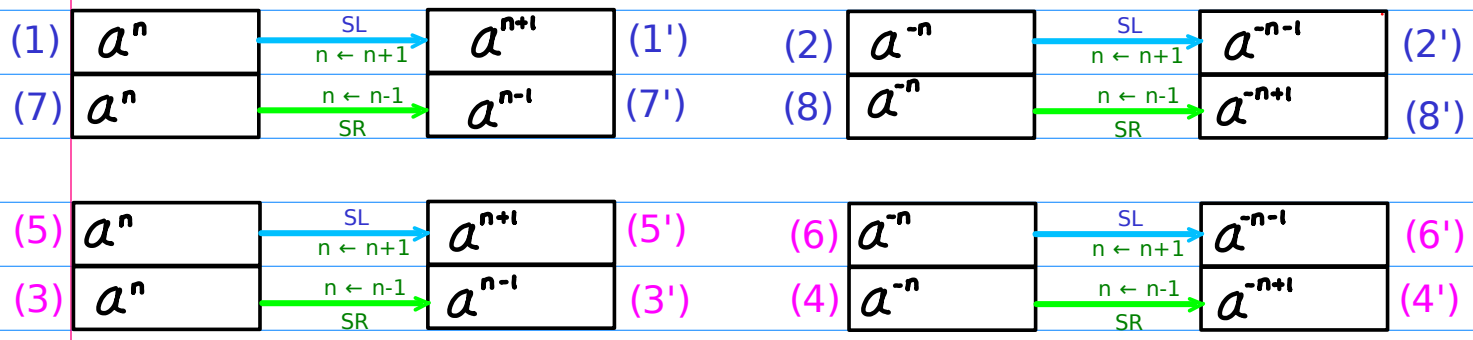


**(id, SR) + (SR, id) = (SR, SR)**
**(id, SL) + (SL, id) = (SL, SL)**

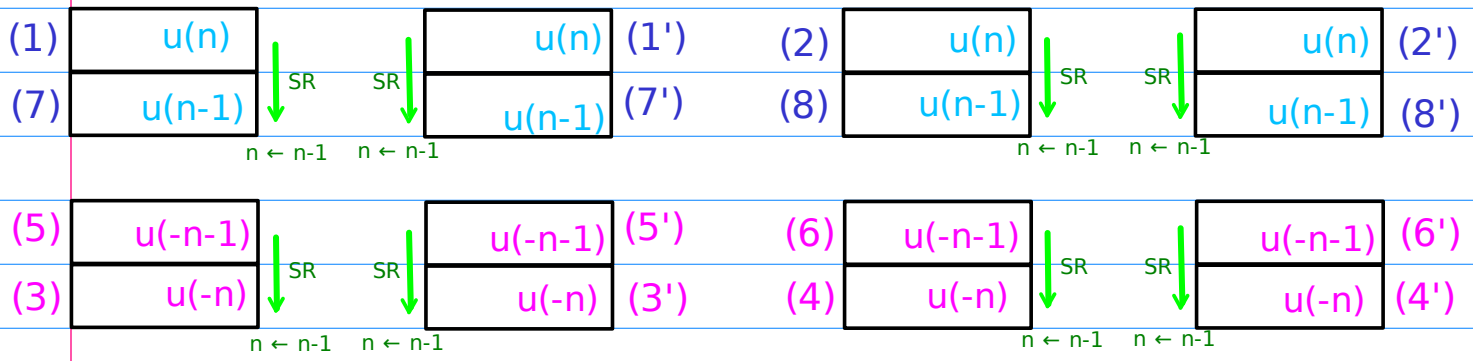
**(x)** → **(x')**

(1)~(8) (1')~(8')

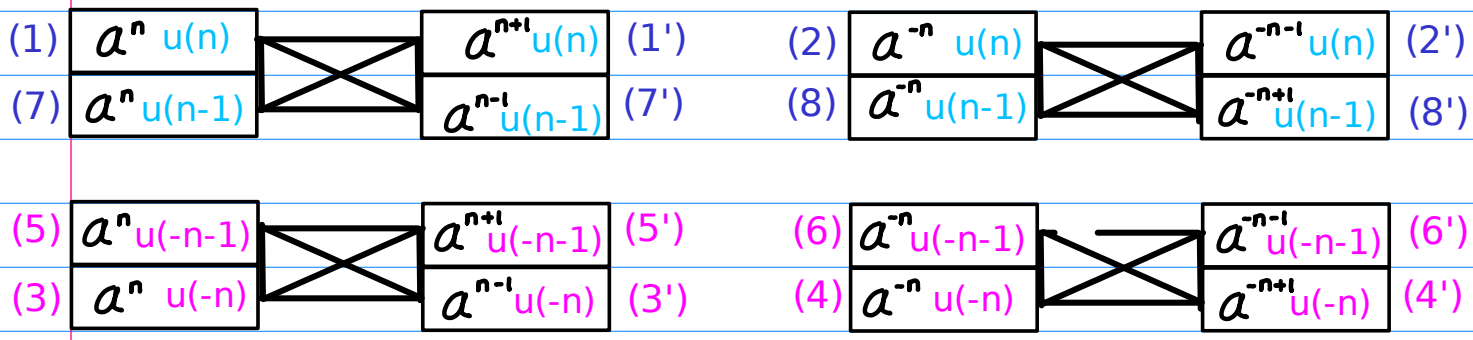
## Exponent Shifts : (SR, id) or (SL, id)




## Range Shifts : (id, SR) or (id, SL)




## Exponent & Range Permutations




## Decomposition

**(EP, RP) = (EP, id) + (id, RP)**

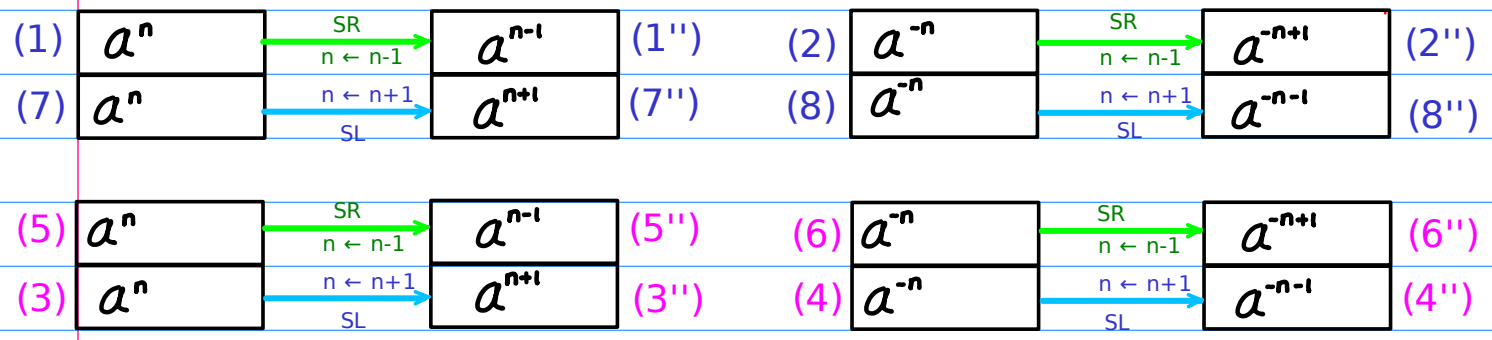
**EP : Exponent Permuations**
**RP : Range Permutations**

**(id, SR) + (SL, id) = (SL, SR)**
**(id, SL) + (SR, id) = (SR, SL)**

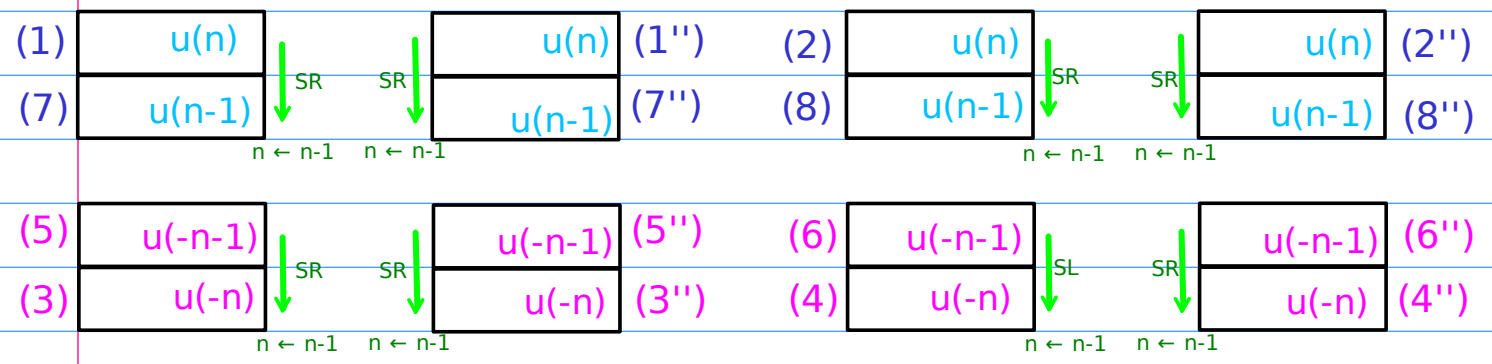
**(x)** → **(x'')**

(1)~(8) → (1'')~(8'')

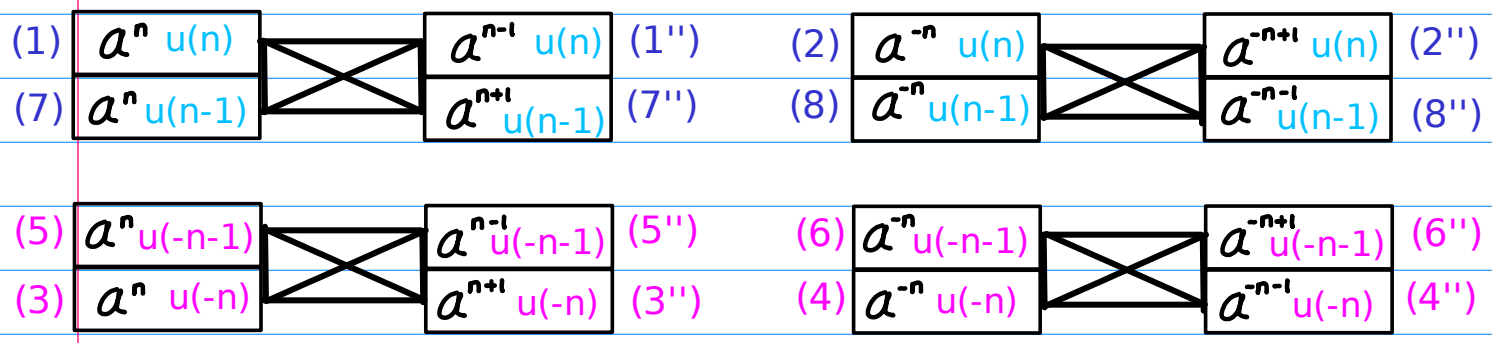
## Exponent Shifts : (SR, id) or (SL, id)




## Range Shifts : (id, SR) or (id, SL)




## Exponent & Range Permutations




## Decomposition

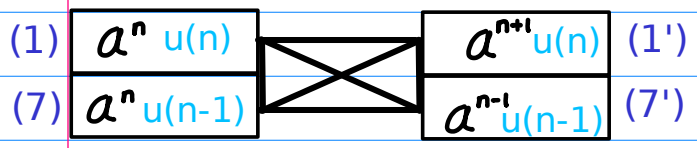
**(EP, RP) = (EP, id) + (id, RP)**

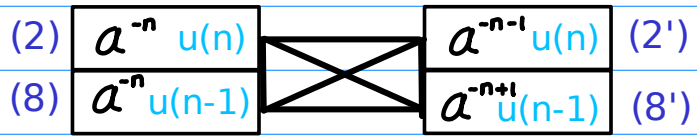
**EP : Exponent Permuations**
**RP : Range Permutations**

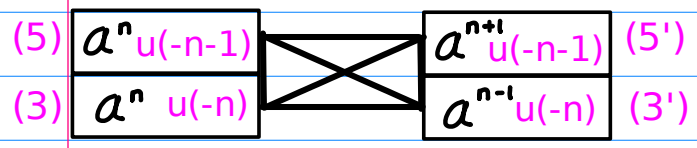
**(id, SR) + (SR, id) = (SR, SR)**
**(id, SL) + (SL, id) = (SL, SL)**

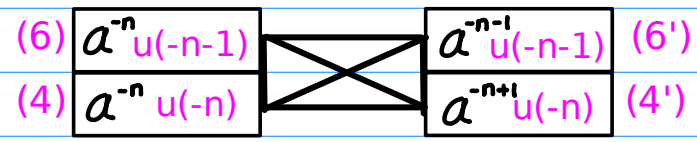
**(x)** → **(x')**

(1)~(8) (1')~(8')



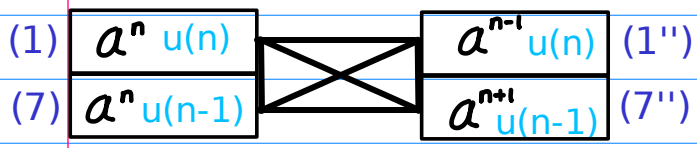



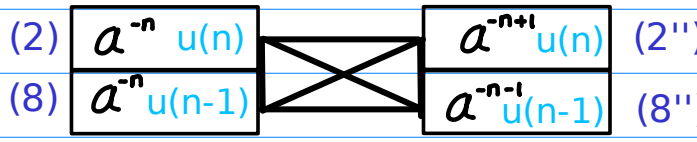



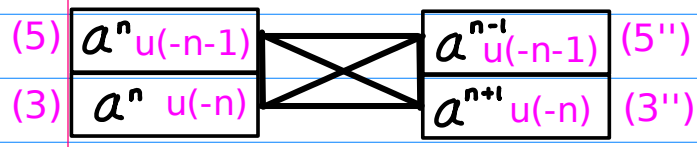

**(id, SR) + (SL, id) = (SL, SR)**
**(id, SL) + (SR, id) = (SR, SL)**

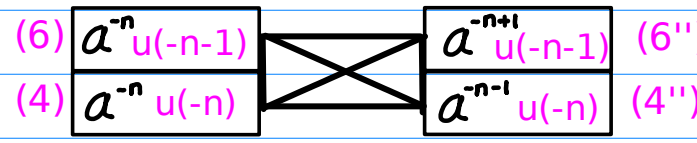
**(x)** → **(x'')**

(1)~(8) (1'')~(8'')



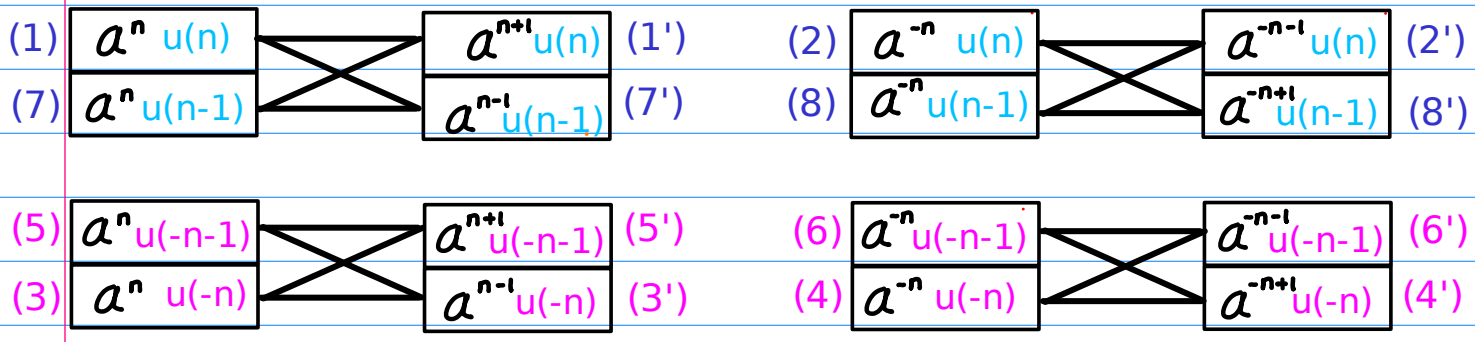



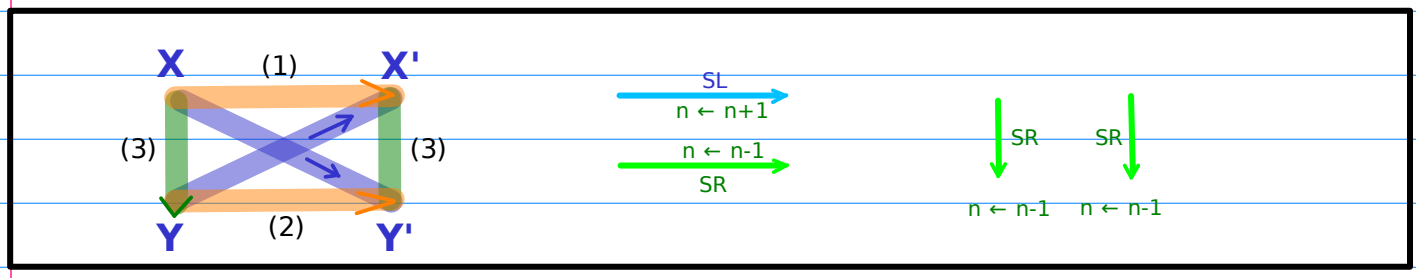



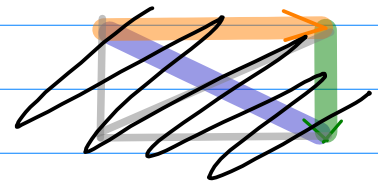
# Inter-permutation Summary

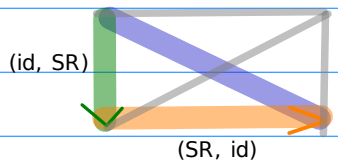
**(x)** → **(x')**

(1)~(8) → (1')~(8')



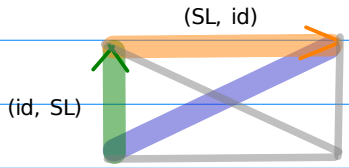



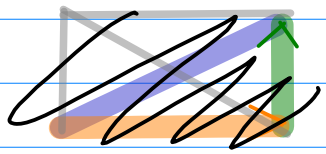



(SR, id) shift right exponent

(id, SR) shift right range

(SR, SR)



(SL, id) shift left exponent

(id, SL) shift left range

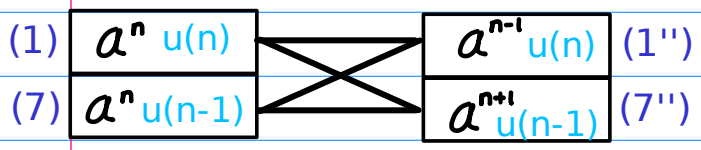
(SL, SL)

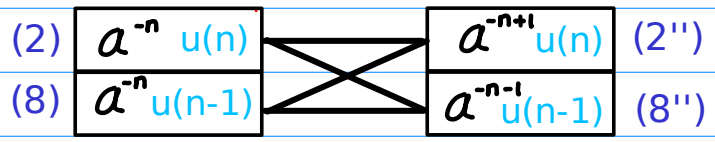
**(SR, id) + (id, SR) = (SR, SR)**
**(SL, id) + (id, SL) = (SL, SL)**

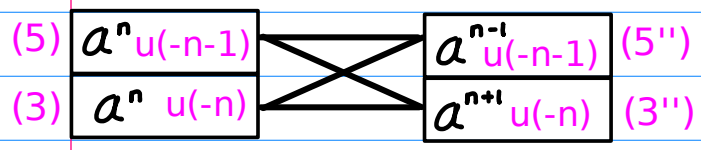
# Inter-permutation Summary

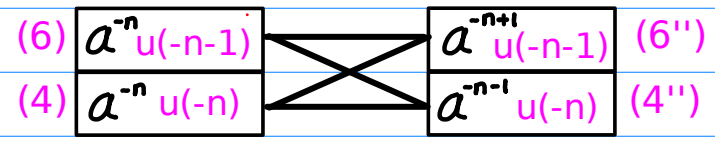
**(x)** → **(x'')**

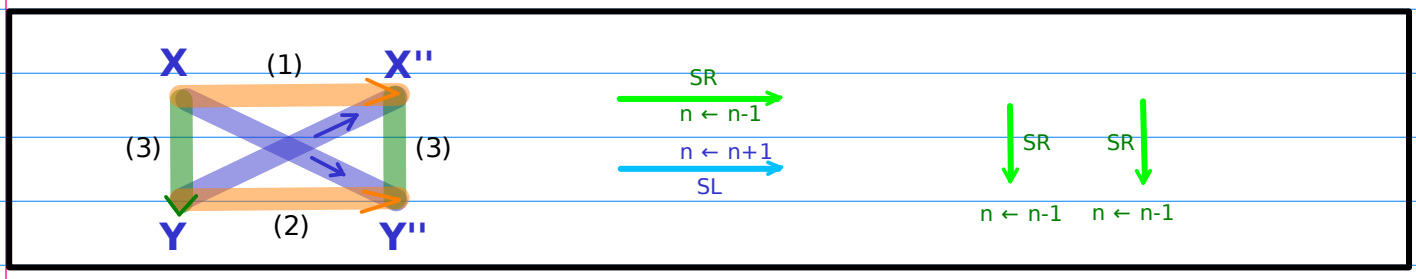
(1)~(8) → (1'')~(8'')

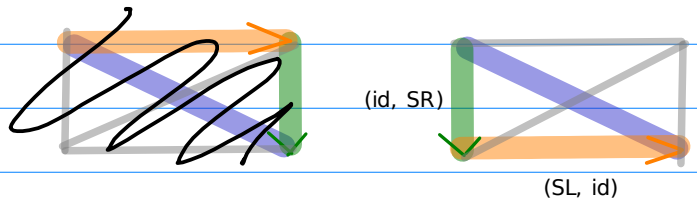


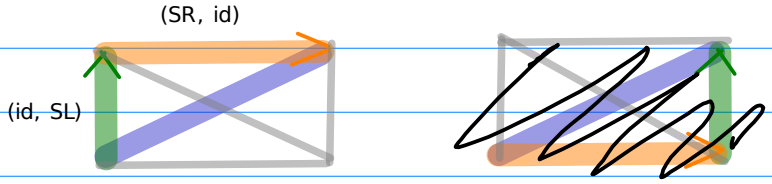




(SL, id) shift right exponent

(id, SR) shift right range

(SL, SR)




(SR, id) shift left exponent

(id, SL) shift left range

(SR, SL)

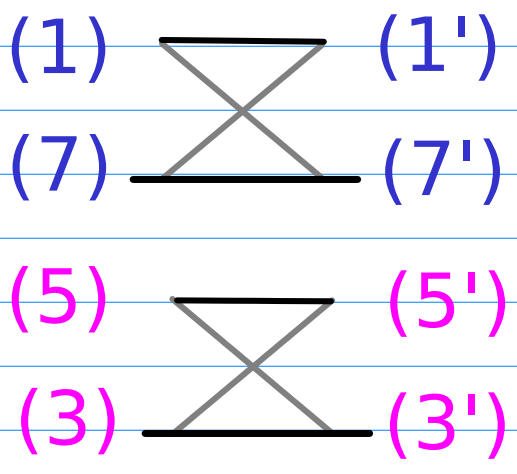
**(SL, id) + (id, SR) = (SL, SR)**
**(SR, id) + (id, SL) = (SR, SL)**

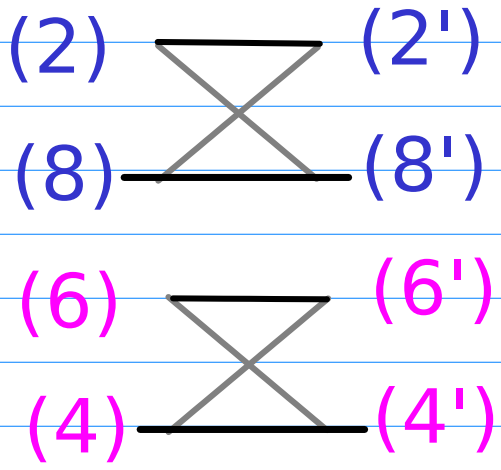
(1) - (1')
(7) - (7')
(5) - (5')
(3) - (3')

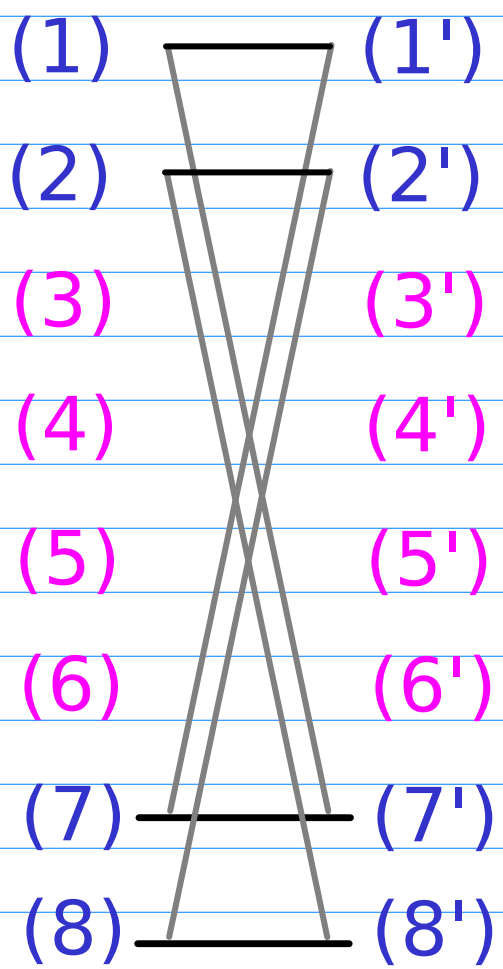
(2) - (2')
(8) - (8')
(6) - (6')
(4) - (4')

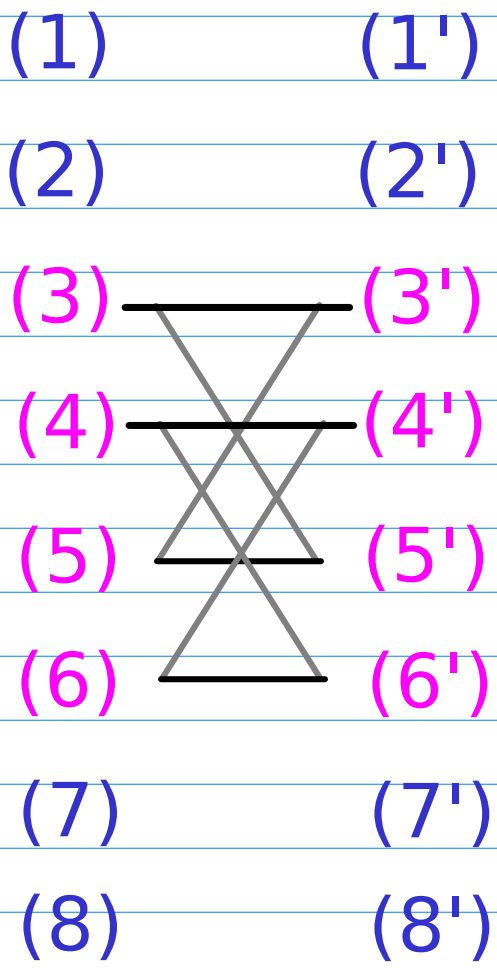
**(x)** → **(x')**

(1)~(8) → (1')~(8')

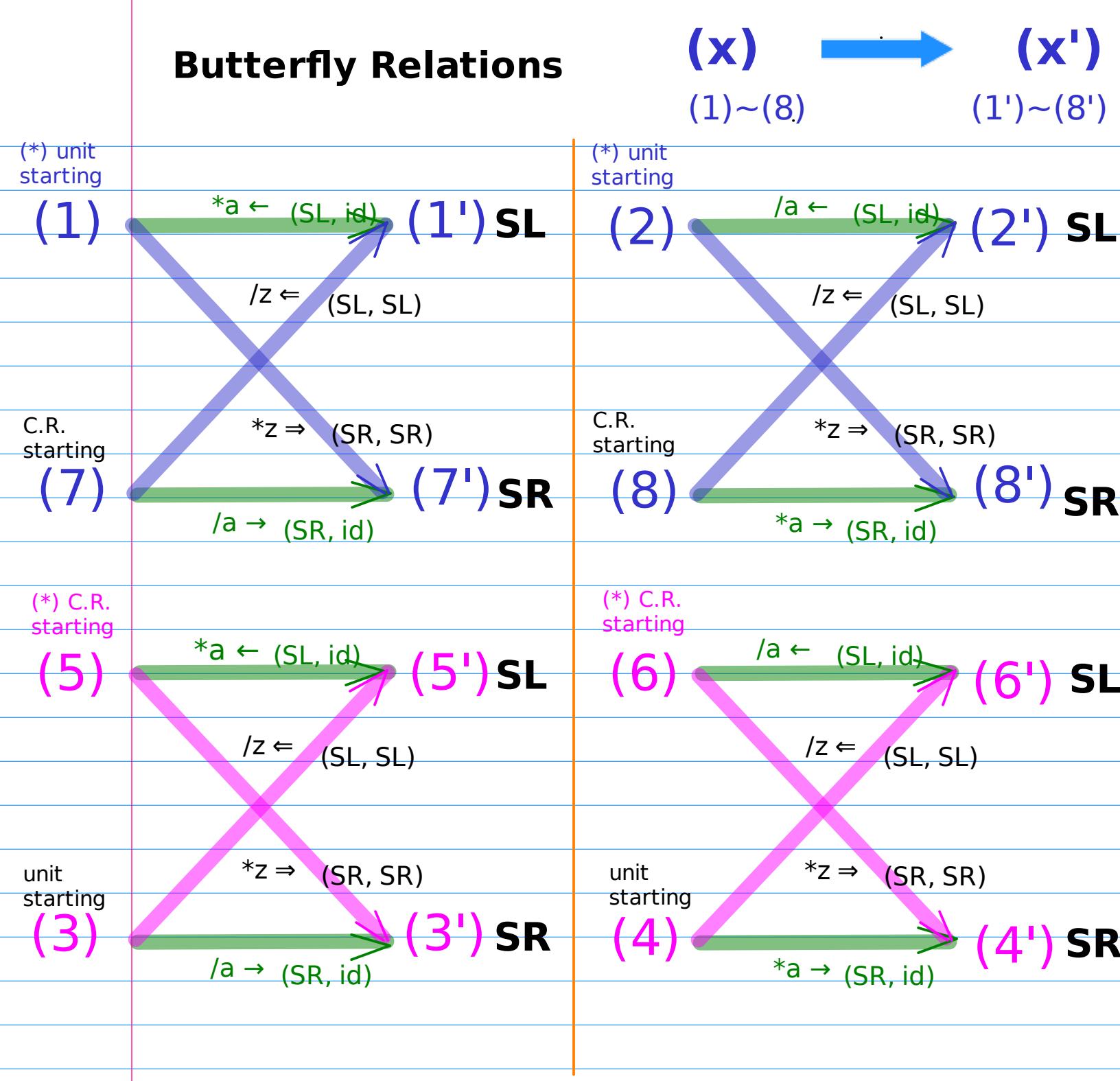


# Butterfly Relations

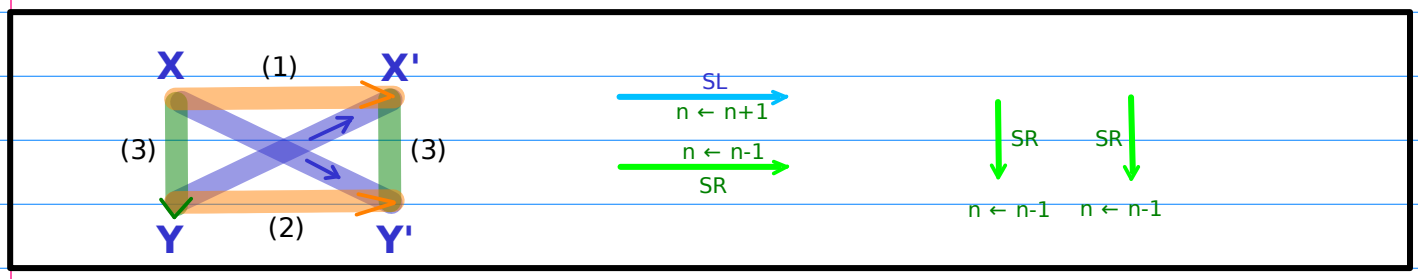


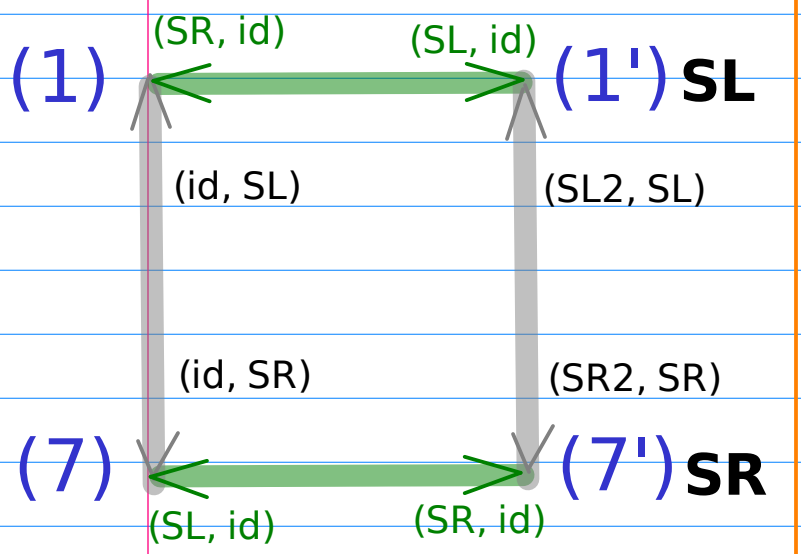

**(Exp Shift, Range Shift)**

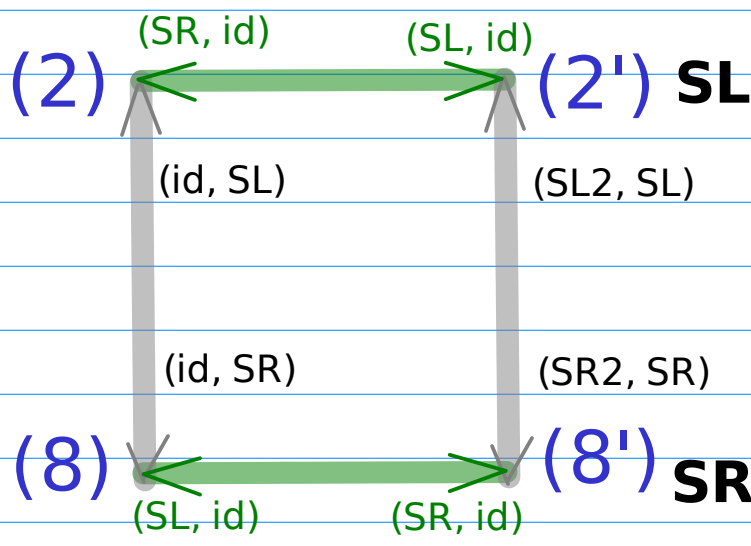
# Butterfly Relations

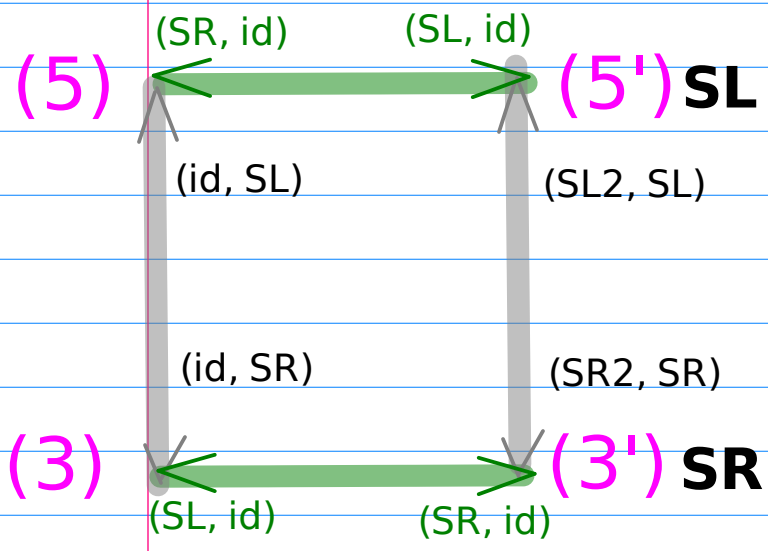
**(x)** → **(x')**

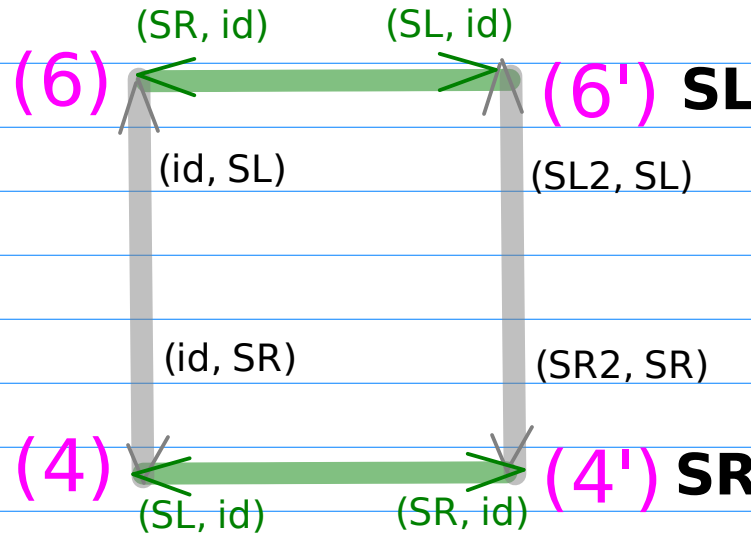
(1)~(8) (1')~(8')



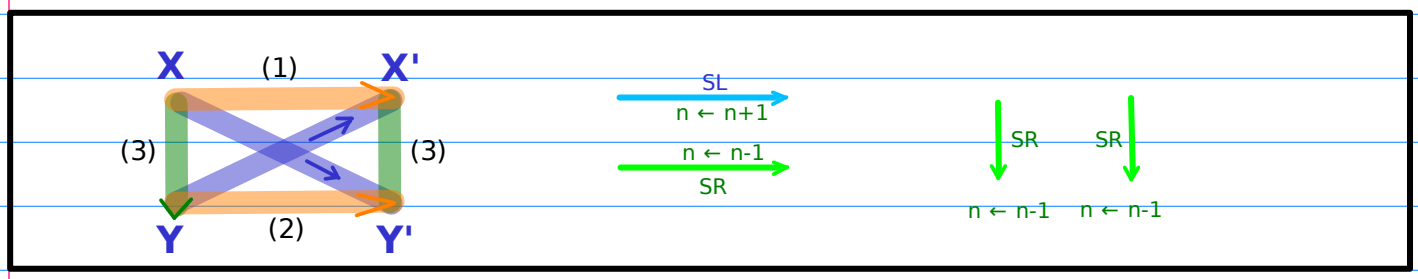



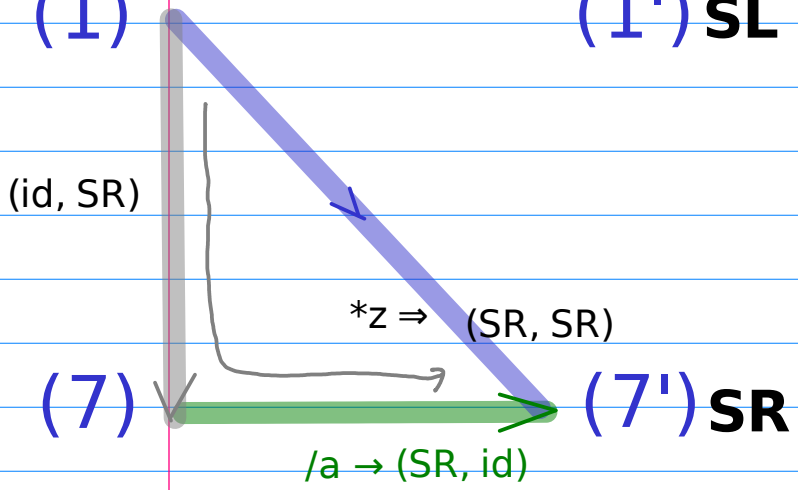
# Decomposition of Exp and Rng Shifts (1)

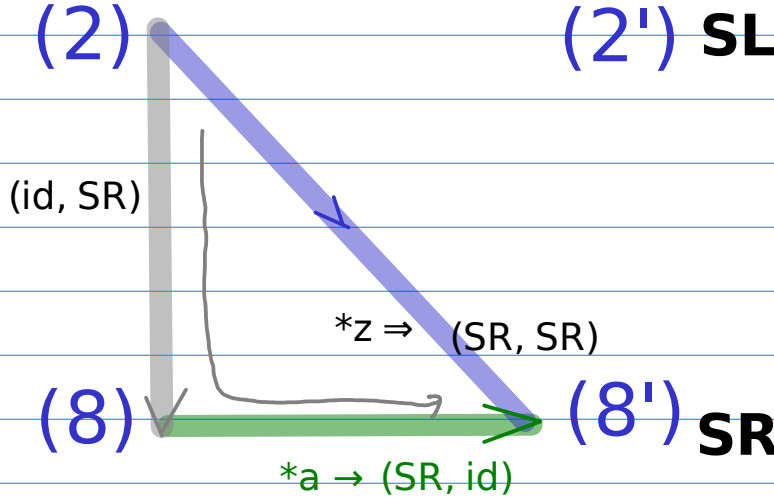
**(x)** → **(x')**

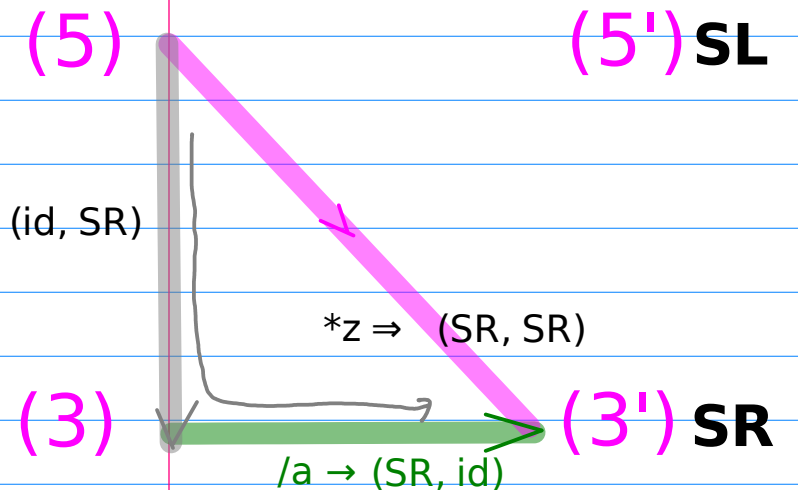
(1)~(8) (1')~(8')

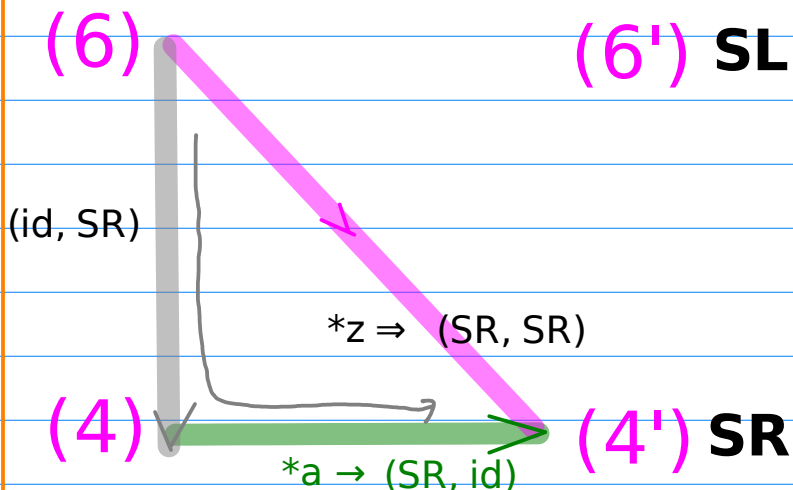
**(id, SR) + (SR, id) = (SR, SR)**



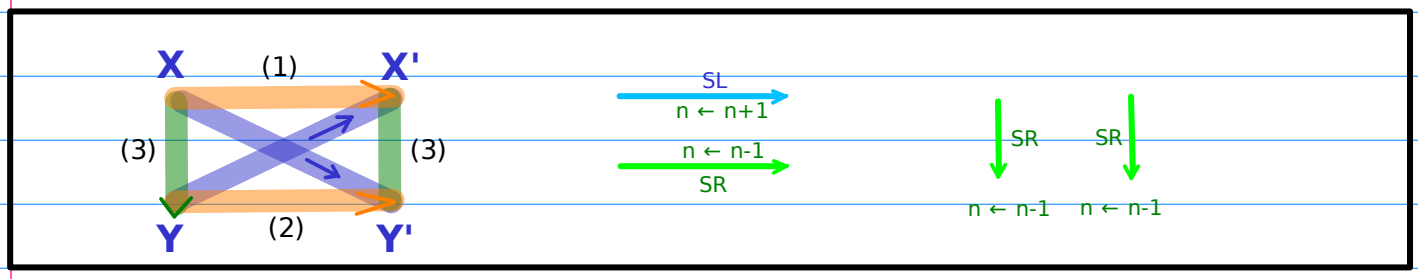



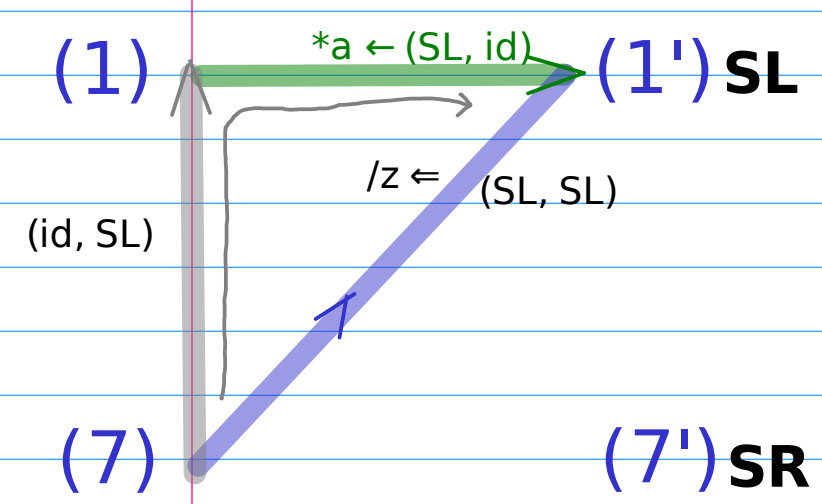
# Decomposition of Exp and Rng Shifts (2)

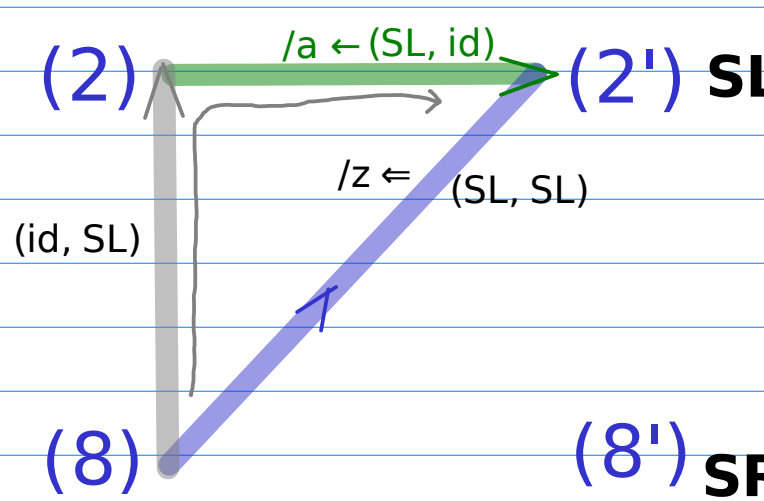
**(x)** → **(x')**

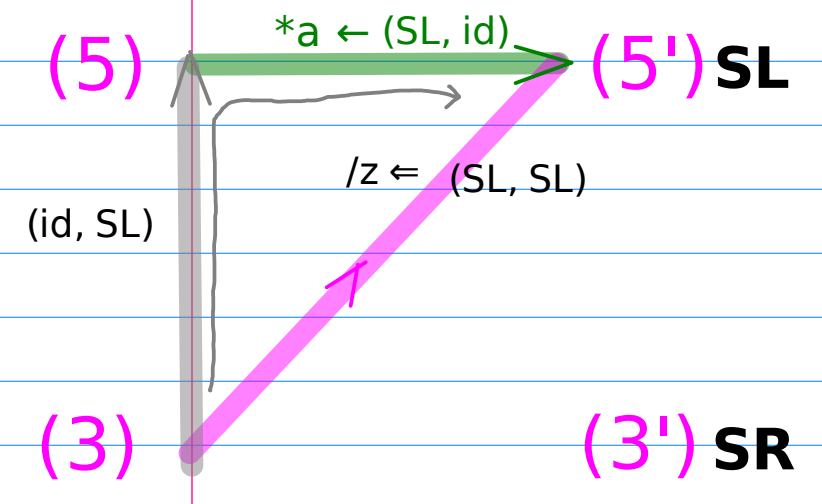
(1)~(8) → (1')~(8')

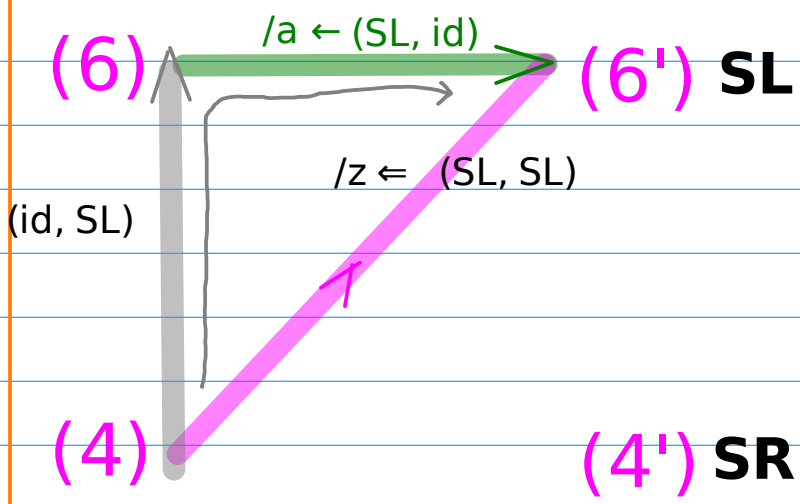
**(id, SL) + (SL, id) = (SL, SL)**



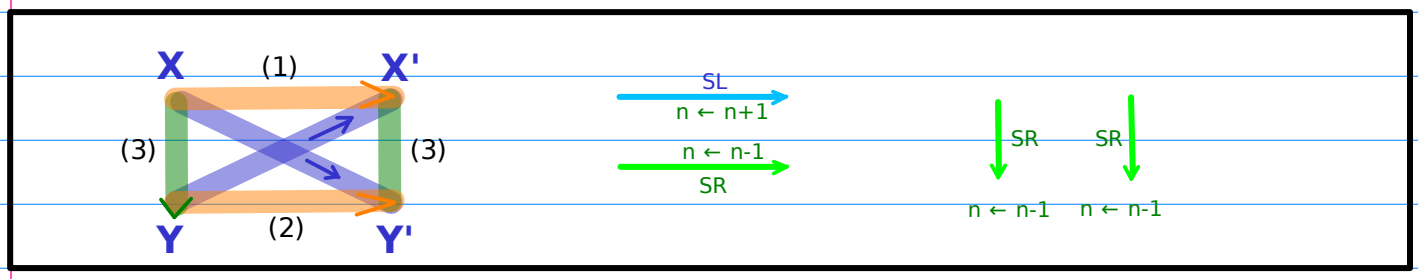



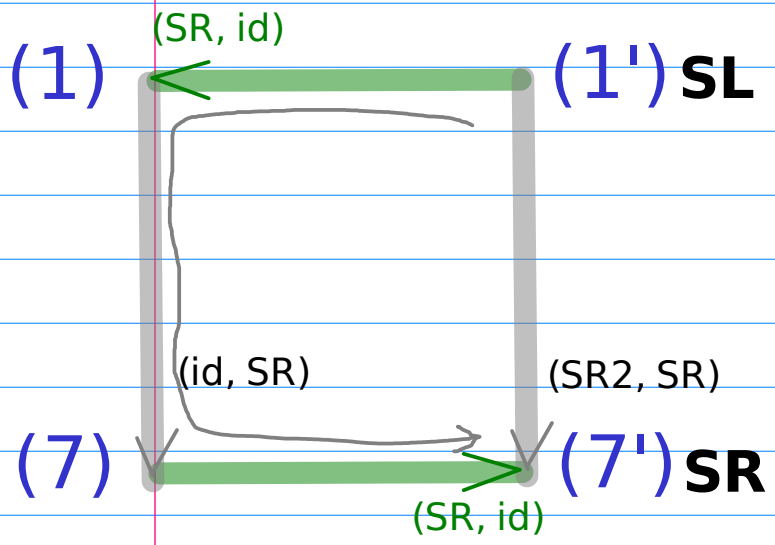
# Decomposition of Exp and Rng Shifts (3)

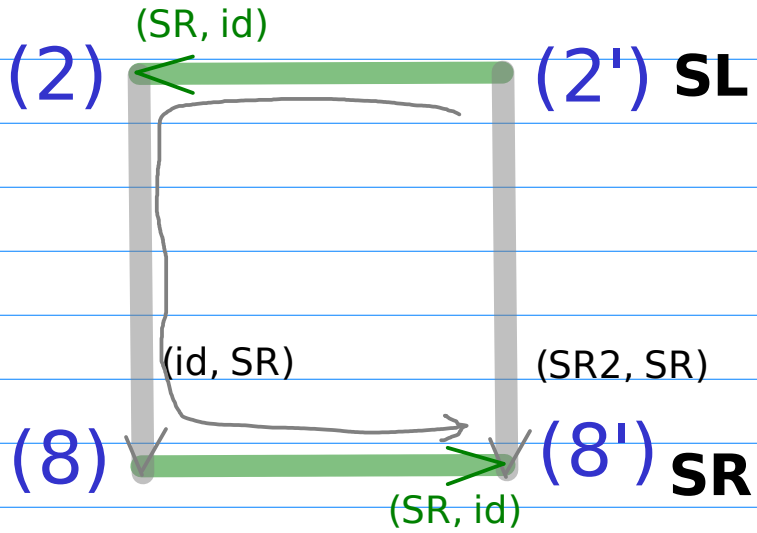
**(x)** → **(x')**

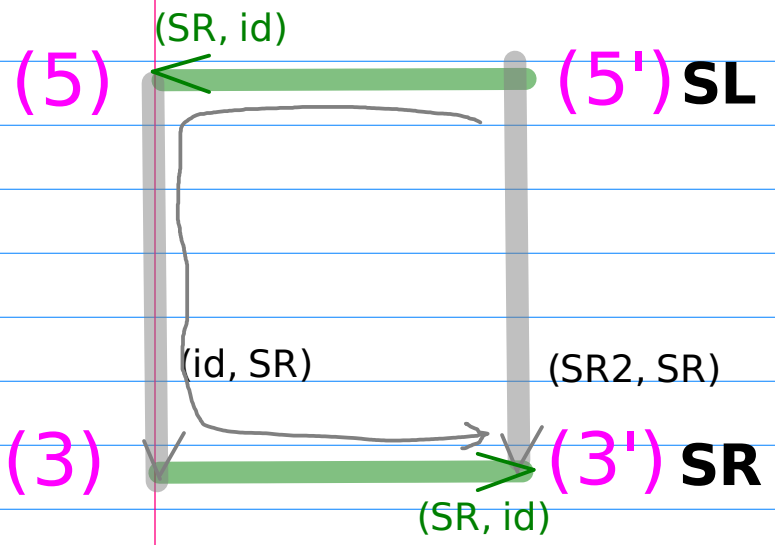
(1)~(8) → (1')~(8')

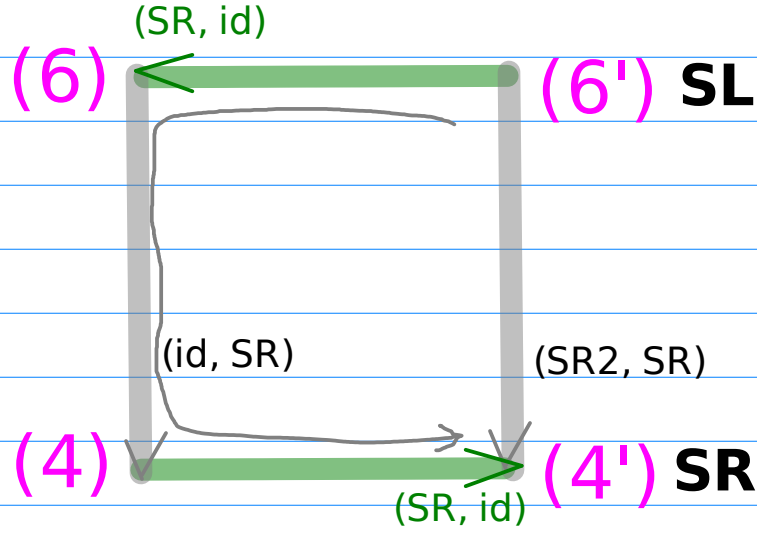
**(SR, id) + (id, SR) + (SR, id) = (SR2, SR)**

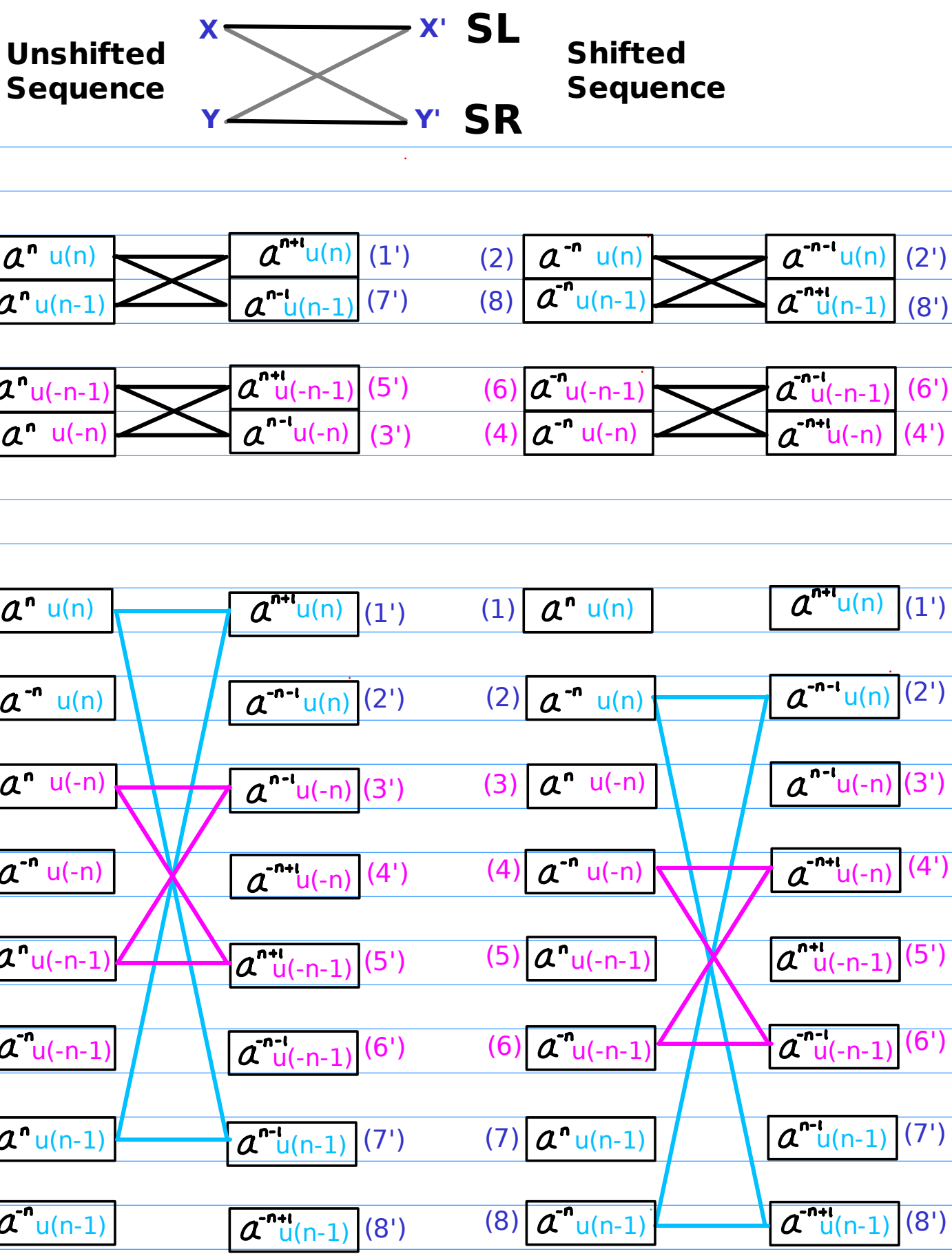


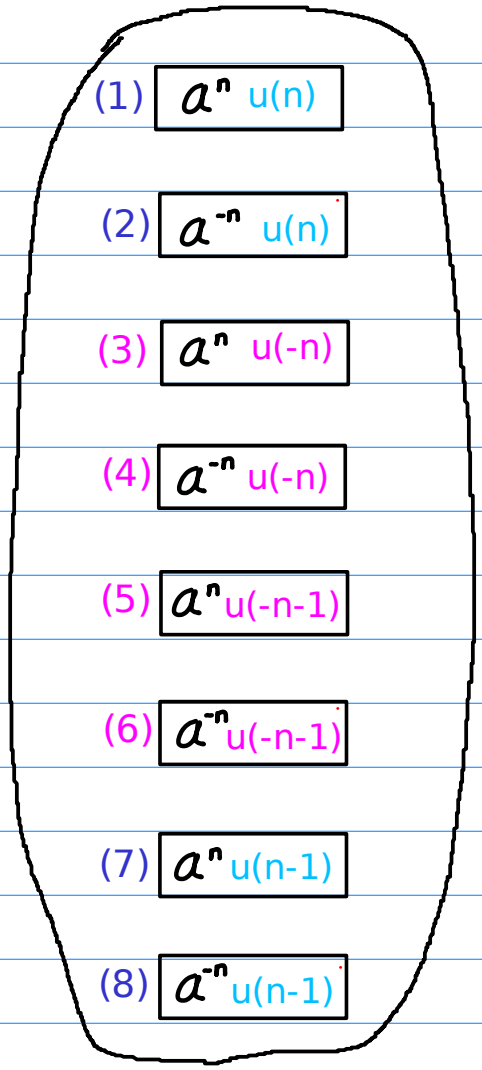




Unshifted Sequence
X
X'
Y
Y'
SL
SR
Shifted Sequence
(1) $a^{n}$ u(n)
(7) $a^{n}$ u(n-1)
(5) $a^{n}$ u(-n-1)
(3) $a^{n}$ u(-n)
$a^{n+l}$ u(n) (1')
$a^{n-l}$ u(n-1) (7')
$a^{n+l}$ u(-n-1) (5')
$a^{n-l}$ u(-n) (3')
(2) $a^{-n}$ u(n)
(8) $a^{-n}$ u(n-1)
(6) $a^{-n}$ u(-n-1)
(4) $a^{-n}$ u(-n)
$a^{-n-l}$ u(n) (2')
$a^{-n+l}$ u(n-1) (8')
$a^{-n-l}$ u(-n-1) (6')
$a^{-n+l}$ u(-n) (4')

# Intra-Permuations over (1) ~ (8)




$a^{n+1}$u(n) (1')

$a^{-n-1}$u(n) (2')

$a^{n-1}$u(-n) (3')

$a^{-n+1}$u(-n) (4')

$a^{n+1}$u(-n-1) (5')

$a^{-n-1}$u(-n-1) (6')

$a^{n-1}$u(n-1) (7')

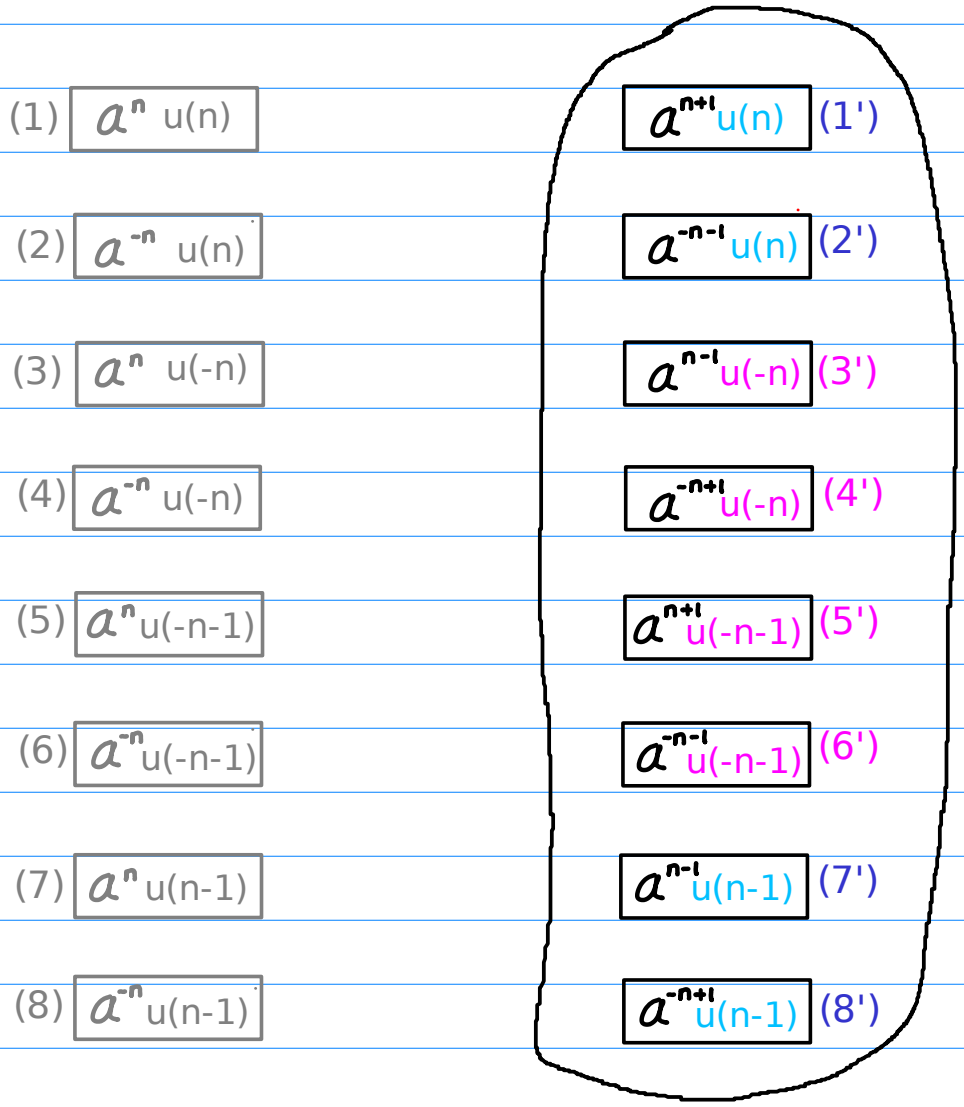
$a^{-n+1}$u(n-1) (8')

**A. Flipping**
- **Base Inverting**
- **Range Flipping**

**B. Range Shifting = Range Flipping + Range Complementing**
- **Range Flipping**
- **Range Complementing**

**C. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

# Intra-Permuations over (1') ~ (8')




**D. Flipping2**

**Base Inverting**

**Shifted Range Flipping = Exponent Shifting2 + Range Flipppi**

**E. Shifting2 = Exponent Shifting2 + Range Shifting**

**Shifted Range Flipping = Exponent Shifting2 + Range Flipping**

**Range Complementing**

**F. Complementary Inverting**

**Base Inverting**

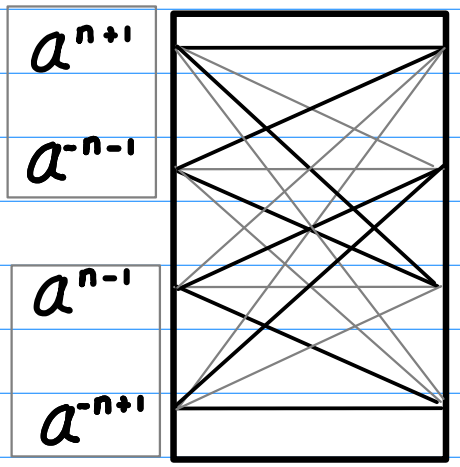
**Range Complementing**

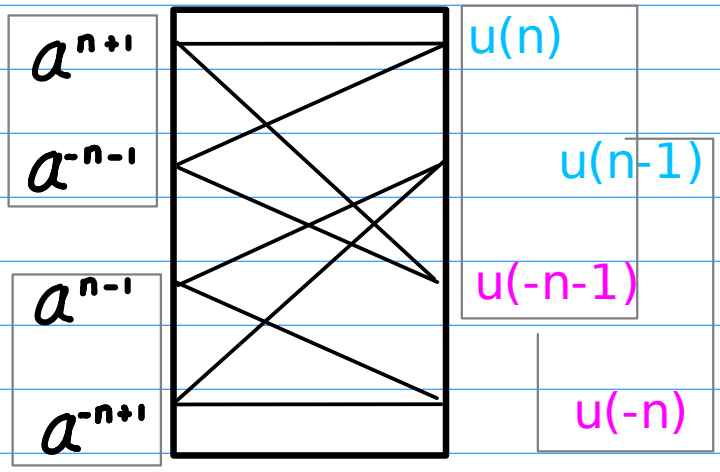
$a^n$ × $R(n)$

| $a^{n+1}$ | $a^{-n-1}$ |
|---|---|
| $a^{n-1}$ | $a^{-n+1}$ |

×

| u(n) | u(-n-1) |
|---|---|
| u(n-1) | u(-n) |




n ← n+2 or
n ← n - 2

| | | | |
|---|---|---|---|
| (1') $a^{n+1}u(n)$ | | $a^{-n-1}u(n)$ | (2') |
| (3') $a^{n-1}u(-n)$ | | $a^{-n+1}u(-n)$ | (4') |
| (5') $a^{n+1}u(-n-1)$ | | $a^{-n-1}u(-n-1)$ | (6') |
| (7') $a^{n-1}u(n-1)$ | | $a^{-n+1}u(n-1)$ | (8') |

| | | | |
|---|---|---|---|
| (1'') $a^{n-1}u(n)$ | | $a^{-n+1}u(n)$ | (2'') |
| (3'') $a^{n+1}u(-n)$ | | $a^{-n-1}u(-n)$ | (4'') |
| (5'') $a^{n-1}u(-n-1)$ | | $a^{-n+1}u(-n-1)$ | (6'') |
| (7'') $a^{n+1}u(n-1)$ | | $a^{-n-1}u(n-1)$ | (8'') |

## permutation over (1) ~ (8)

A. Flipping
    Base Inverting
    Range Flipping

B. Range Shifting = Range Flipping + Range Complementing
    Range Flipping
    Range Complementing

C. Complementary Inverting
    Base Inverting
    Range Complementing

## permutation over (1') ~ (8')

D. Flipping2
    Base Inverting
    Shifted Range Flipping = Exponent Shifting2 + Range Flippping

E. Shifting2 = Exponent Shifting2 + Range Shifting
    Shifted Range Flipping = Exponent Shifting2 + Range Flipping
    Range Complementing

F. Complementary Inverting
    Base Inverting
    Range Complementing

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flippping

Shifting2 = Shifted Range Flipping + Range Complementing
    = Exponent Shifting2 + Range (Flippping+Complementing)
    = Exponent Shfiting2 + Range Shifting

## permutation over (1) ~ (8)

**A. Flipping**
- **Base Inverting**
- **Range Flipping**

(1) - (4) (5) - (8)
(2) - (3) (6) - (7)
(3) - (2) (7) - (6)
(4) - (1) (8) - (5)

**B. Range Shifting**
- **Range Flipping**
- **Range Complementing**

(1) - (7) (5) - (3)
(2) - (8) (6) - (4)
(3) - (5) (7) - (1)
(4) - (6) (8) - (2)

**C. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

(1) - (6) (5) - (2)
(6) - (1) (6) - (1)
(2) - (5) (7) - (4)
(5) - (2) (8) - (3)

## permutation over (1') ~ (8')

**D. Flipping2**
- **Base Inverting**
- **Shifted Range Flipping**

(1') - (4') (5') - (8')
(2') - (3') (6') - (7')
(3') - (2') (7') - (6')
(4') - (1') (8') - (5')

**E. Shifting2**
- **Shifted Range Flipping**
- **Range Complementing**

(1') - (7') (5') - (3')
(2') - (8') (6') - (4')
(3') - (5') (7') - (1')
(4') - (6') (8') - (2')

**F. Complementary Inverting**
- **Base Inverting**
- **Range Complementing**

(1') - (6') (5') - (2')
(6') - (1') (6') - (1')
(2') - (5') (7') - (4')
(5') - (2') (8') - (3')

## permutation over (1) ~ (8)

| | A | B | C |
|---|---|---|---|
| Base Inverting | x | | x |
| Range Flipping | x | x | |
| Range Complementing | | x | x |

## permutation over (1') ~ (8')

| | D | E | F |
|---|---|---|---|
| Base Inverting | x | | x |
| Shifted Range Flipping | x | x | |
| Range Complementing | | x | x |

Range Shifting = Range Flipping + Range Complementing
Shifted Range Flipping = Exponent Shifting2 + Range Flippping

Shifting2 = Shifted Range Flipping + Range Complementing
= Exponent Shifting2 + Range (Flippping+Complementing)
= Exponent Shfiting2 + Range Shifting

**Over (1) ~ (8)**

**Base Inverting** $a^n \Longleftrightarrow a^{-n}$

**Range Flipping** $R(n) \Longleftrightarrow R(-n)$

**Range Complementing** $R(n) \Longleftrightarrow \overline{R(n)}$

**Over (1') ~ (8')**

**Base Inverting** $a^n \Longleftrightarrow a^{-n}$

**Shifted Range Flipping** $a^n R(n) \Longleftrightarrow a^{sh\,2(n)} R(-n)$

**Range Complementing** $R(n) \Longleftrightarrow \overline{R(n)}$

**A.I Flipping**
**Base Inverting**
**Range Flipping**

$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

**B.I Range Shifting**
**Range Flipping**
**Range Complementing**

$$R(n) \Longleftrightarrow R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

**C.I Complementary Inverting**
**Base Inverting**
**Range Complementing**

$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n} \overline{R(n)}$$

**D.I Flipping2**
**Base Inverting**
**Shifted Range Flipping**

$$a^n \Longleftrightarrow a^{-n}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

**E.I Shifting2**
**Shifted Range Flipping**
**Range Complementing**

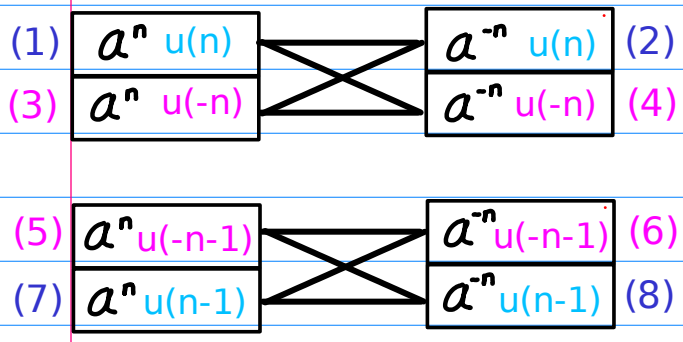
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

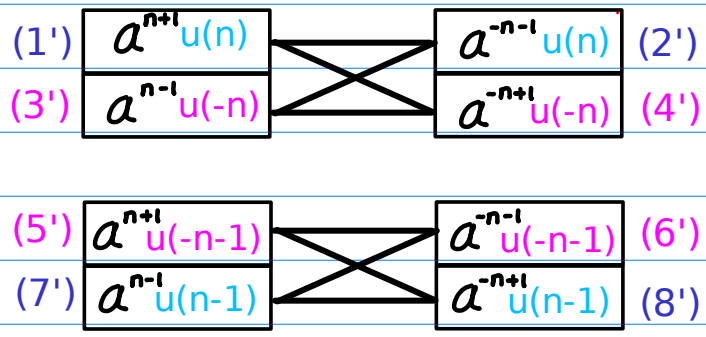
**F.I Complementary Inverting**
**Base Inverting**
**Range Complementing**

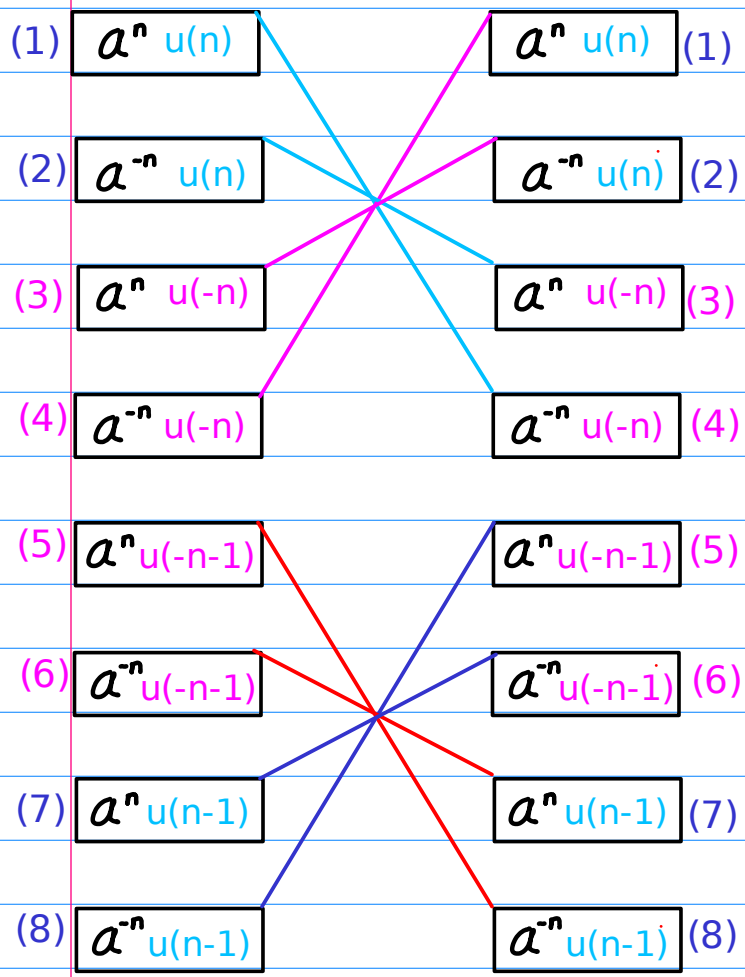
$$a^n \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n} \overline{R(n)}$$

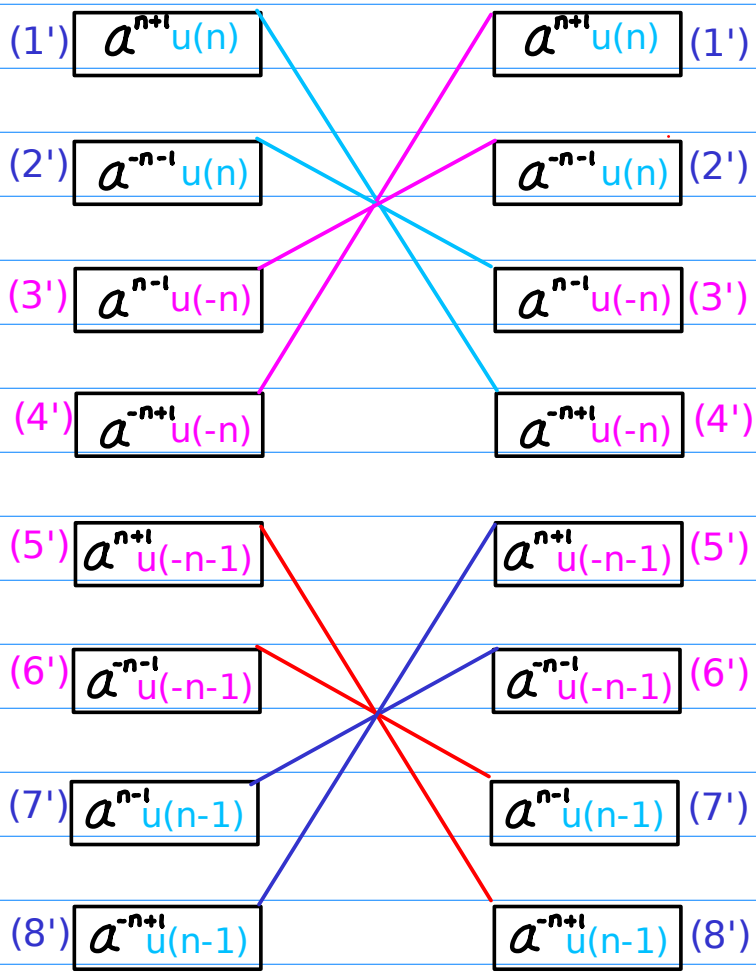
## A.I Flipping
### Base Inverting
### Range Flipping

## D.I Flipping2
### Base Inverting
### Shifted Range Flipping



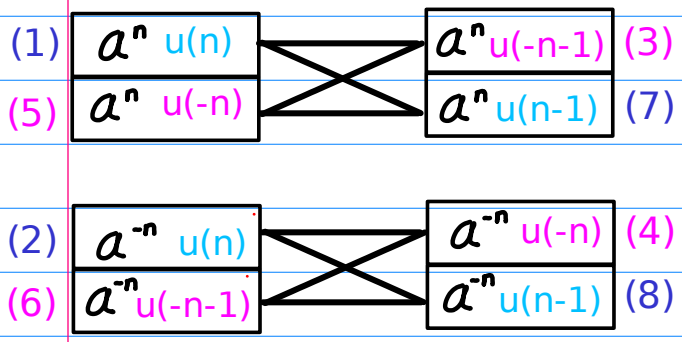

(1) - (4)
(2) - (3)
(3) - (2)
(4) - (1)

(5) - (8)
(6) - (7)
(7) - (6)
(8) - (5)

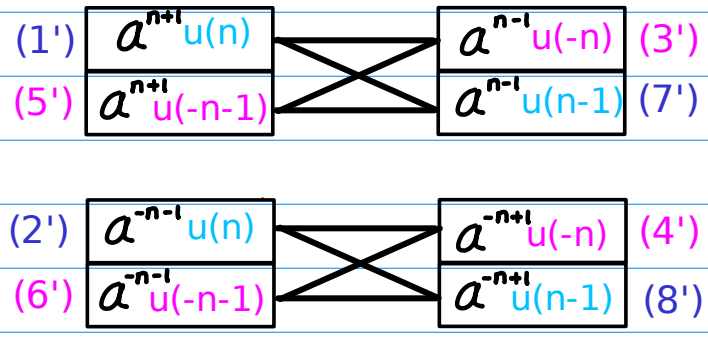
(1') - (4')
(2') - (3')
(3') - (2')
(4') - (1')

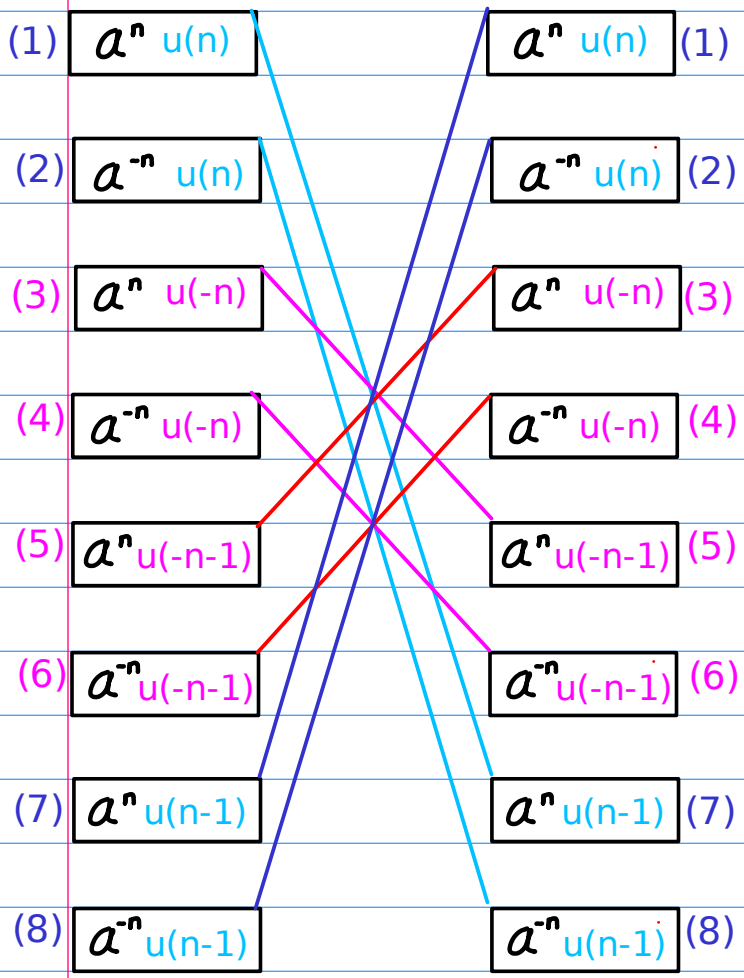
(5') - (8')
(6') - (7')
(7') - (6')
(8') - (5')

## B.I Range Shifting
## Range Flipping
## Range Complementing



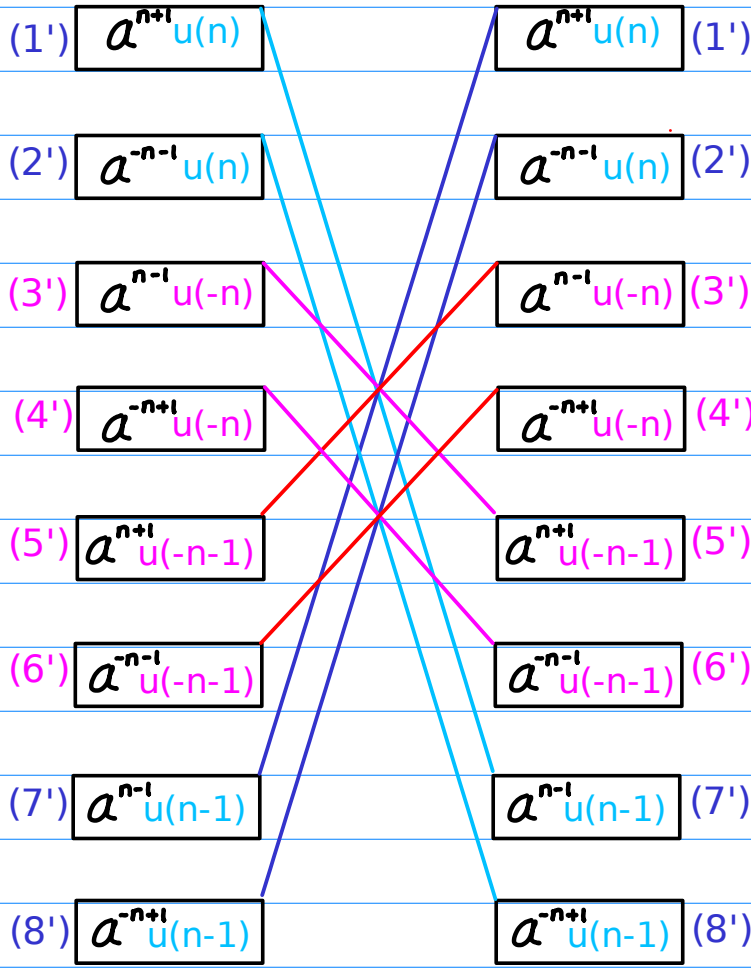

## E.I Shifting2
## Shifted Range Flipping
## Range Complementing




(1) - (7)
(2) - (8)
(3) - (5)
(4) - (6)

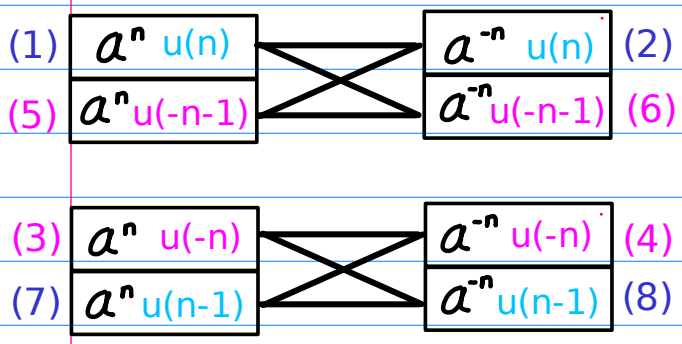
(5) - (3)
(6) - (4)
(7) - (1)
(8) - (2)



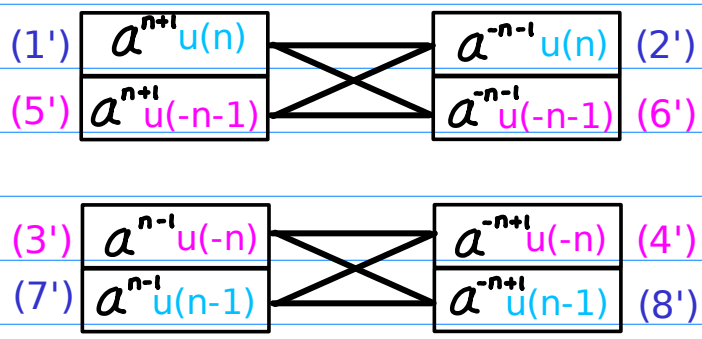

(1') - (7')
(2') - (8')
(3') - (5')
(4') - (6')

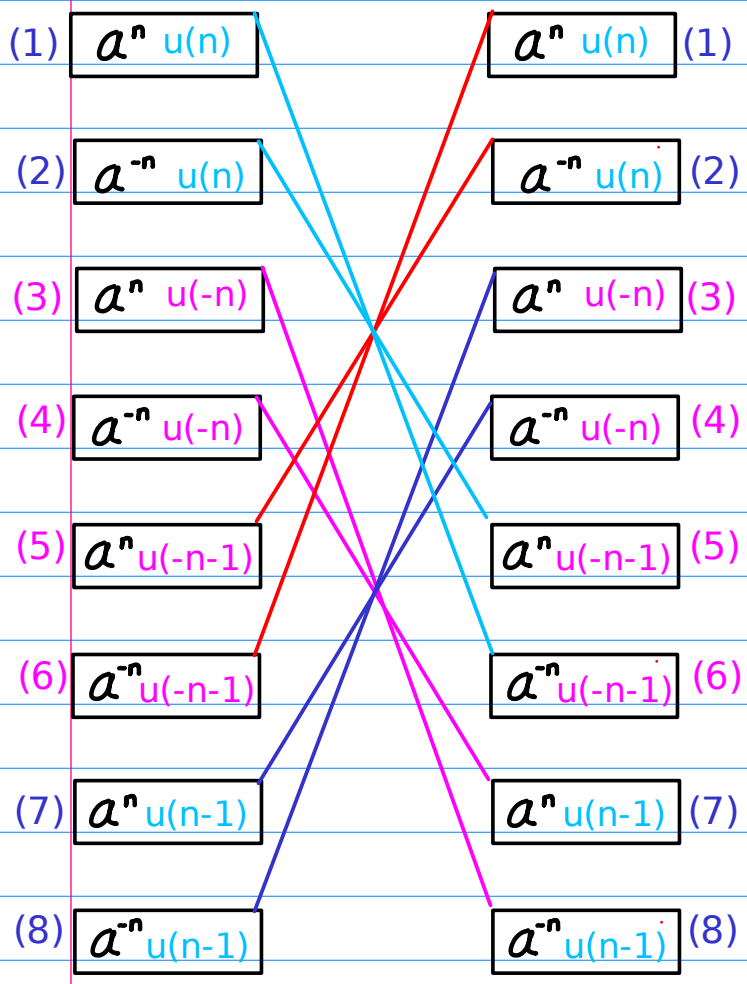
(5') - (3')
(6') - (4')
(7') - (1')
(8') - (2')

## C.I Complementary Inverting
## Base Inverting
## Range Complementing



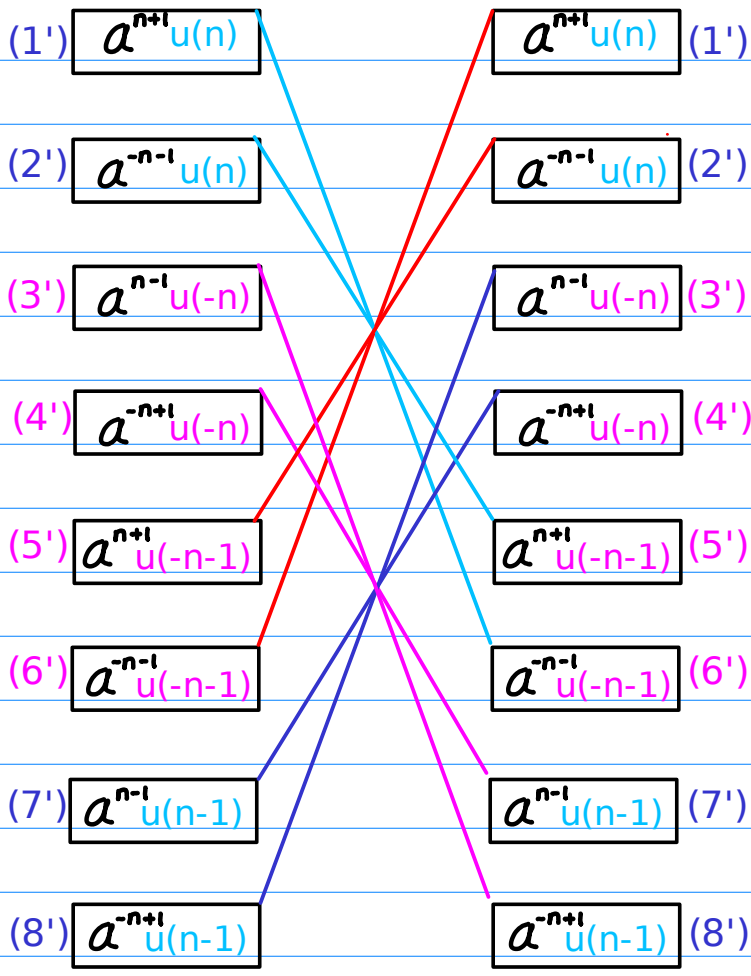

## F.I Complementary Inverting
## Base Inverting
## Range Complementing




(1) - (6)
(6) - (1)
(2) - (5)
(5) - (2)

(5) - (2)
(6) - (1)
(7) - (4)
(8) - (3)

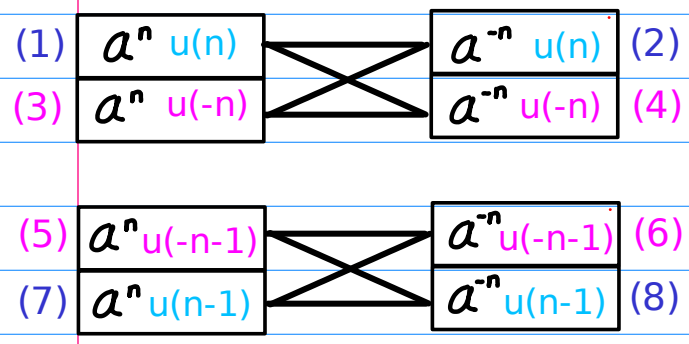


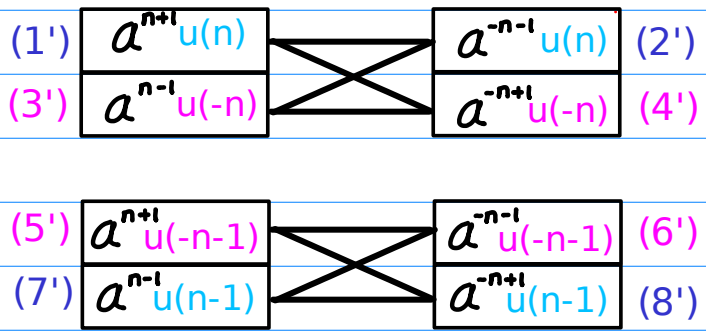

(1') - (6')
(6') - (1')
(2') - (5')
(5') - (2')

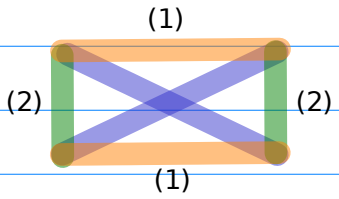
(5') - (2')
(6') - (1')
(7') - (4')
(8') - (3')

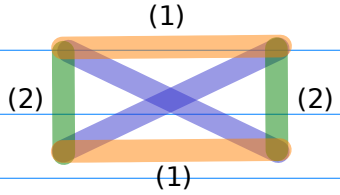
## A.I Flipping
### (1) Base Inverting
### (2) Range Flipping

## D.I Flipping2
### (1) Base Inverting
### (2) Shifted Range Flipping




$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$a^n R(n) \overset{(2)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^n$ | $a^{-n}$ |
| $a^{-n}$ | $a^n$ |

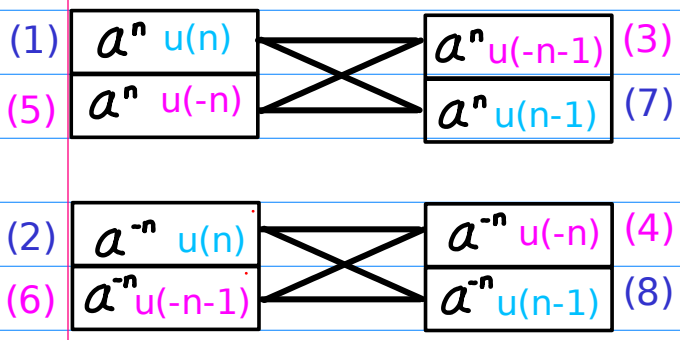
| $b^n$ | $b^{-sh2(n)}$ |
|---|---|
| $a^{(n+1)}$ | $a^{-(n-1)}$ |
| $a^{-(n+1)}$ | $a^{(n-1)}$ |
| $a^{(n-1)}$ | $a^{-(n+1)}$ |
| $a^{-(n-1)}$ | $a^{(n+1)}$ |

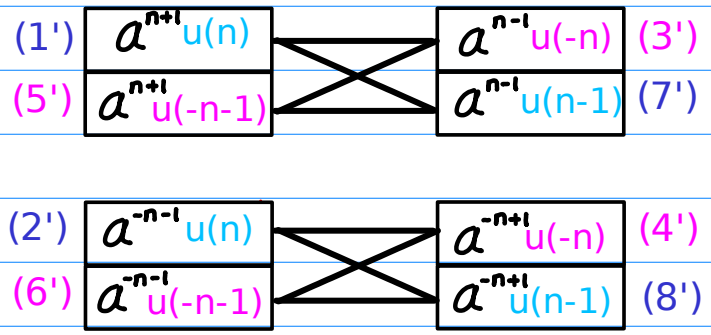
| $R(n)$ | $R(-n)$ |
|---|---|
| $u(n)$ | $u(-n)$ |
| $u(n-1)$ | $u(-n-1)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

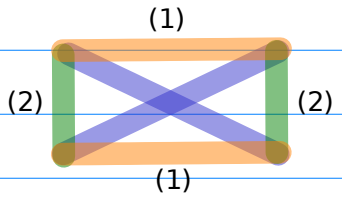
| $R(n)$ | $R(-n)$ |
|---|---|
| $u(n)$ | $u(-n)$ |
| $u(n-1)$ | $u(-n-1)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

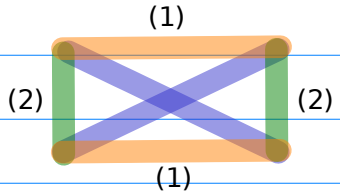
## B.I Range Shifting
### (1) Range Complementing
### (2) Range Flipping

## E.I Shifting2
### (1) Shifted Range Flippin
### (2) Range Complementing




$$R(n) \overset{(1)}{\Longleftrightarrow} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \overset{(1)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

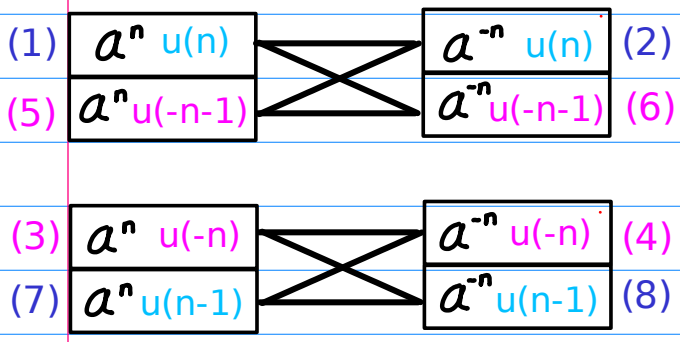
| $b^n$ | $b^{sh2(n)}$ |
|---|---|
| $a^{(n+1)}$ | $a^{(n-1)}$ |
| $a^{-(n+1)}$ | $a^{-(n-1)}$ |
| $a^{(n-1)}$ | $a^{(n+1)}$ |
| $a^{-(n-1)}$ | $a^{-(n+1)}$ |

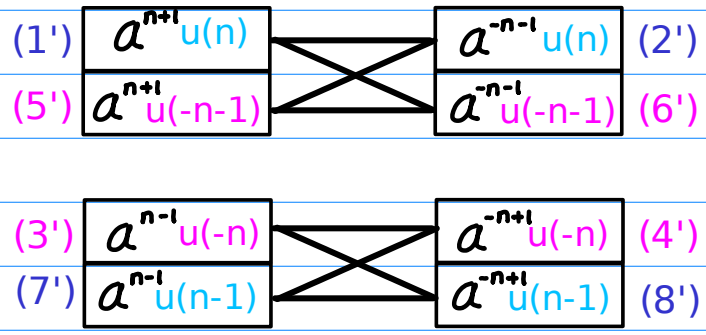
| $R(n)$ | $\overline{R(-n)}$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-n-1)$ |
| $u(-n-1)$ | $u(-n)$ |

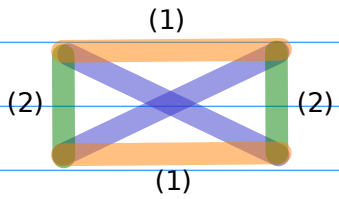
| $R(n)$ | $\overline{R(-n)}$ |
|---|---|
| $u(n)$ | $u(n-1)$ |
| $u(n-1)$ | $u(n)$ |
| $u(-n)$ | $u(-n-1)$ |
| $u(-n-1)$ | $u(-n)$ |

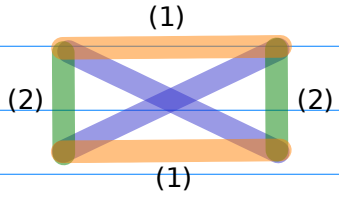
## C.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

## F.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing




$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^n$ | $a^{-n}$ |
| $a^{-n}$ | $a^n$ |

| $b^n$ | $b^{-n}$ |
|---|---|
| $a^{(n+1)}$ | $a^{-(n+1)}$ |
| $a^{-(n+1)}$ | $a^{(n+1)}$ |
| $a^{(n-1)}$ | $a^{-(n-1)}$ |
| $a^{-(n-1)}$ | $a^{(n-1)}$ |

| $R(n)$ | $\overline{R(n)}$ |
|---|---|
| $u(n)$ | $u(-n-1)$ |
| $u(n-1)$ | $u(-n)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

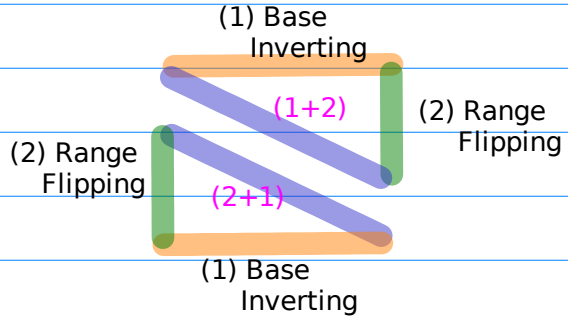
| $R(n)$ | $\overline{R(n)}$ |
|---|---|
| $u(n)$ | $u(-n-1)$ |
| $u(n-1)$ | $u(-n)$ |
| $u(-n)$ | $u(n)$ |
| $u(-n-1)$ | $u(n-1)$ |

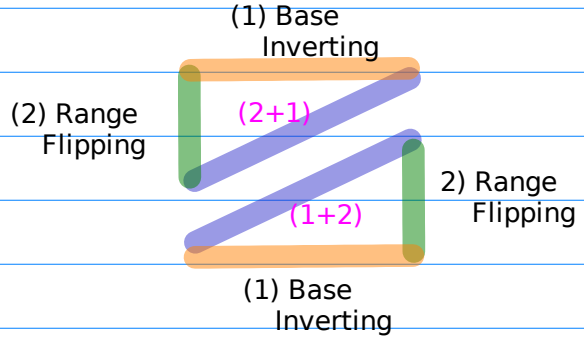
## A.I Flipping
### (1) Base Inverting
### (2) Range Flipping

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$
$$\xrightarrow{(2)} a^{-n} R(-n)$$

$$a^n R(n) \xrightarrow{(2)} a^n R(-n)$$
$$\xrightarrow{(1)} a^{-n} R(-n)$$



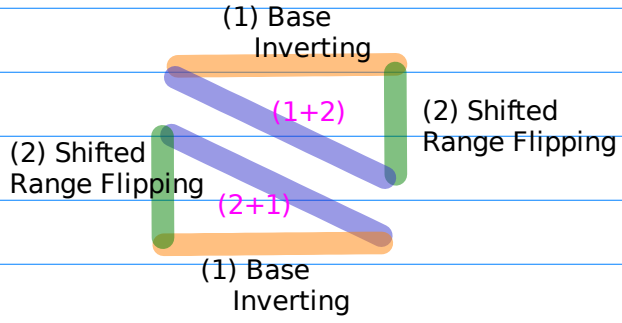



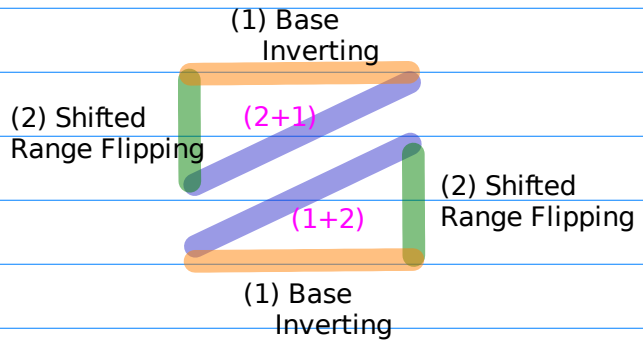

## D.I Flipping2
### (1) Base Inverting
### (2) Shifted Range Flippin

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$a^n R(n) \overset{(2)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$
$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## B.I Range Shifting
## (1) Range Complementing
## (2) Range Flipping

$$R(n) \overset{(1)}{\Longleftrightarrow} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

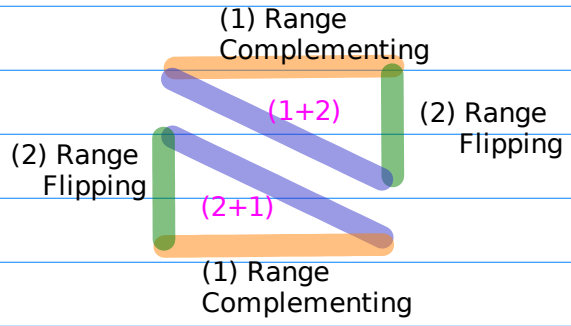
$$a^n R(n) \overset{(1)}{\Longleftrightarrow} a^{sh2(n)} R(-n)$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

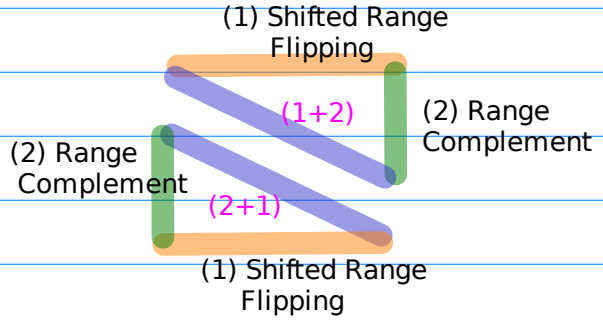
$$a^n R(n) \overset{(1)}{\rightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\rightarrow} a^{-n} \overline{R(n)}$$

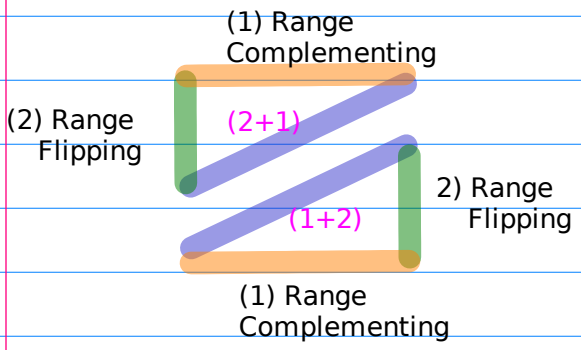
$$a^n R(n) \overset{(1)}{\rightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\rightarrow} a^{-sh2(n)} R(-n)$$

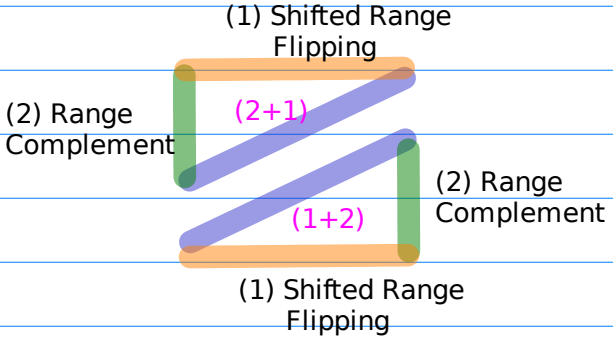
$$a^n rng(n) \overset{(2)}{\rightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\rightarrow} a^{-n} \overline{R(n)}$$

$$a^n R(n) \overset{(2)}{\rightarrow} a^{sh2(n)} R(-n)$$
$$\overset{(1)}{\rightarrow} a^{-sh2(n)} R(-n)$$



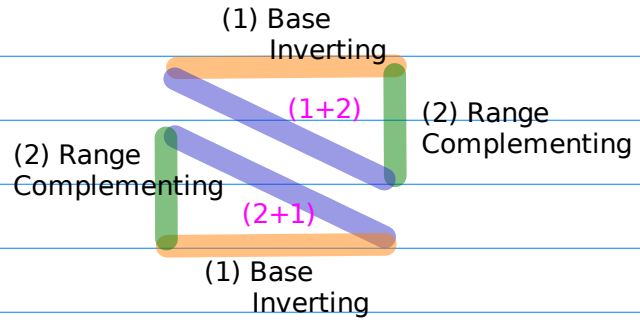

## E.I Shifting2
## (1) Shifted Range Flippin
## (2) Range Complementing

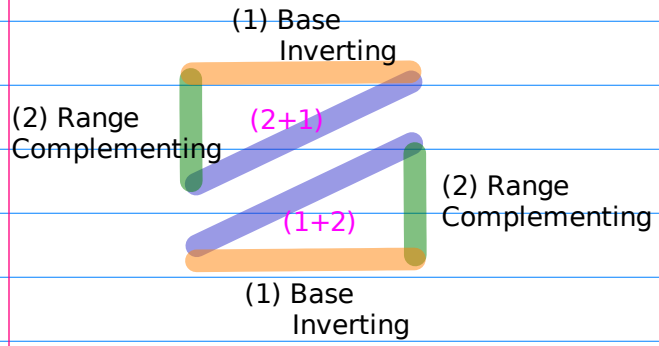
## C.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^n R(n) \overset{(1)}{\Longrightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\Longrightarrow} a^{-n}\overline{R(n)}$$

$$a^n R(n) \overset{(2)}{\Longrightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\Longrightarrow} a^{-n}\overline{R(n)}$$



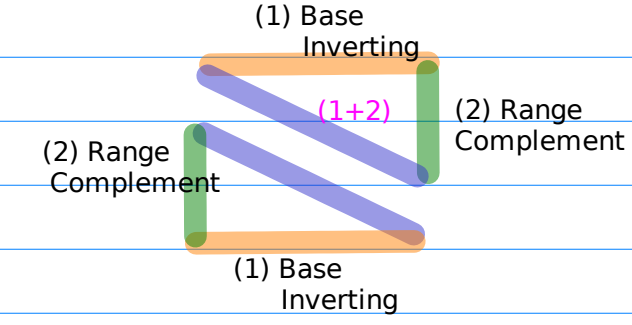



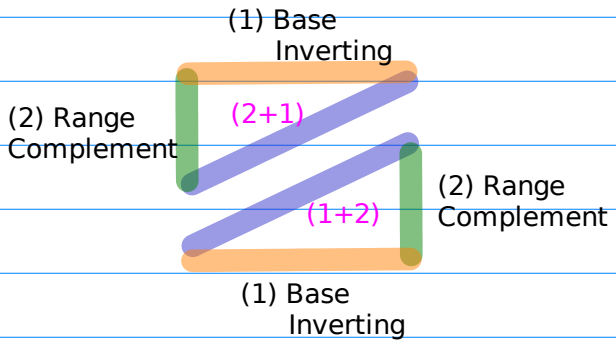

## F.I Complementary Inverting
### (1) Base Inverting
### (2) Range Complementing

$$a^n \overset{(1)}{\Longleftrightarrow} a^{-n}$$
$$R(n) \overset{(2)}{\Longleftrightarrow} \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

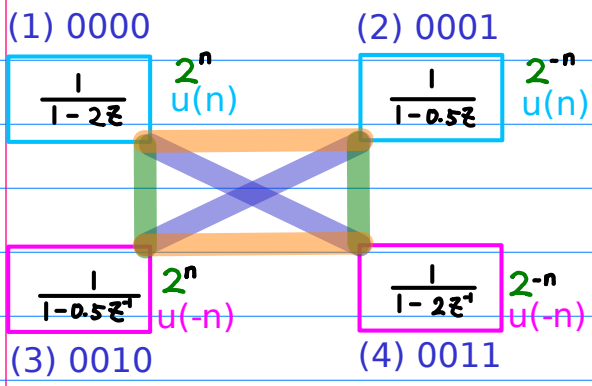
$$a^n R(n) \overset{(1)}{\Longrightarrow} a^{-n} R(n)$$
$$\overset{(2)}{\Longrightarrow} a^{-n}\overline{R(n)}$$

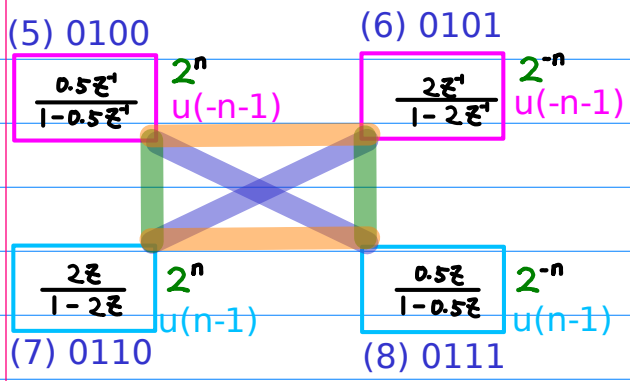
$$a^n R(n) \overset{(2)}{\Longrightarrow} a^n \overline{R(n)}$$
$$\overset{(1)}{\Longrightarrow} a^{-n}\overline{R(n)}$$



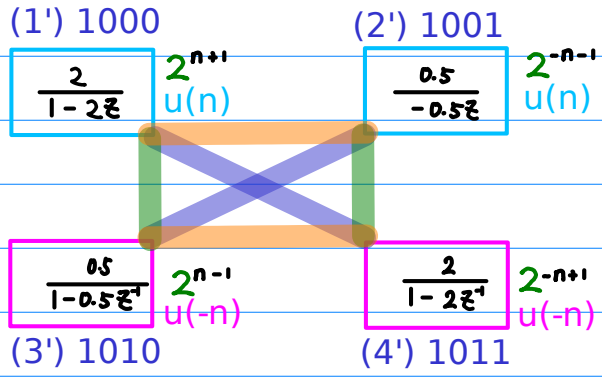



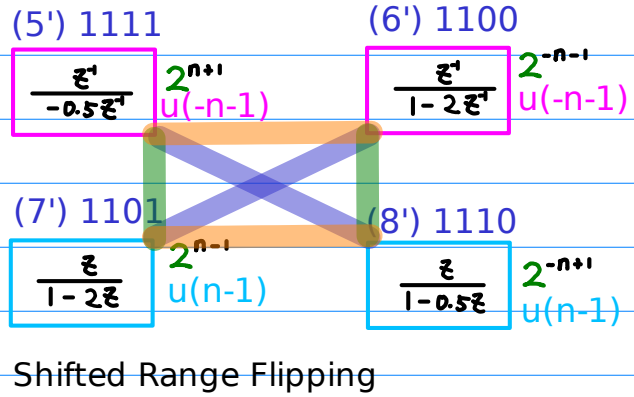
## A.I Flipping
### Base Inverting
### Range Flipping




## D.I Flipping2
### Base Inverting
### Shifted Range Flipping



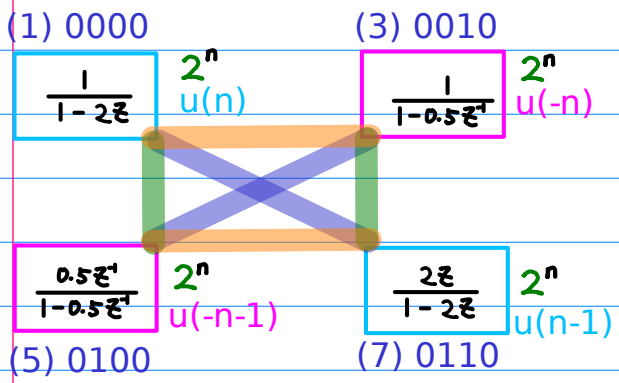



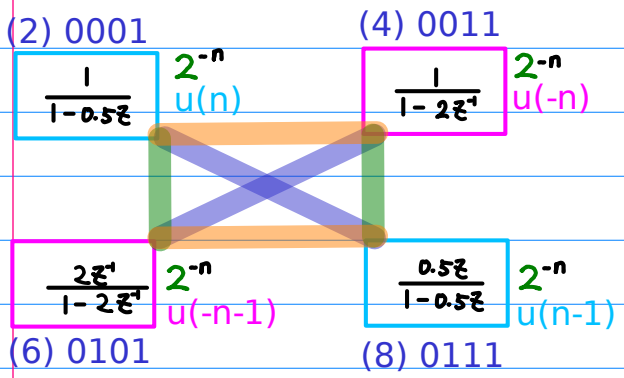

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n \quad \Longleftrightarrow \quad a^{-n}$$
$$R(n) \quad \Longleftrightarrow \quad R(-n)$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{-n} R(-n)$$

$$a^n \quad \Longleftrightarrow \quad a^{-n}$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{sh2(n)} R(-n)$$
$$a^n R(n) \quad \Longleftrightarrow \quad a^{-sh2(n)} R(-n)$$

## B.I Range Shifting
## Range Flipping
## Range Complementing

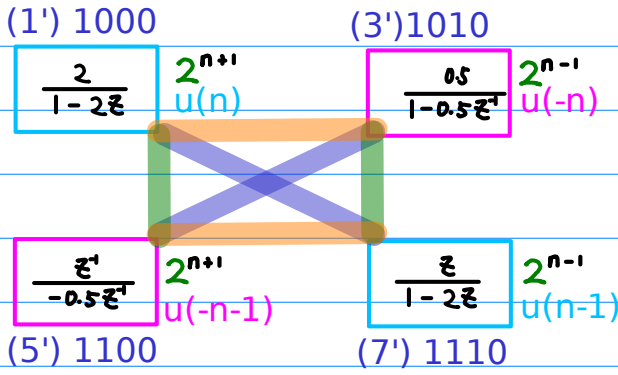


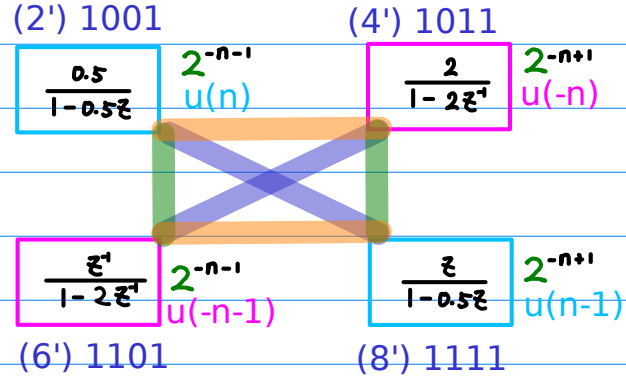




Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \Longleftrightarrow R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$R(n) \Longleftrightarrow \overline{R(-n)}$$

## E.I Shifting2
## Shifted Range Flipping
## Range Complementing




Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^n R(n) \Longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

## C.I Complementary Inverting
## Base Inverting
## Range Complementing

(1) 0000 $\frac{1}{1-2z}$ $2^{n}$ u(n)

(2) 0001 $\frac{1}{1-0.5z}$ $2^{-n}$ u(n)

(5) 0100 $\frac{0.5z^{-1}}{1-0.5z^{-1}}$ $2^{n}$ u(-n-1)

(6) 0101 $\frac{2z^{-1}}{1-2z^{-1}}$ $2^{-n}$ u(-n-1)

(3) 0010 $\frac{1}{1-0.5z^{-1}}$ $2^{n}$ u(-n)

(4) 0011 $\frac{1}{1-2z^{-1}}$ $2^{-n}$ u(-n)

(7) 0110 $\frac{2z}{1-2z}$ $2^{n}$ u(n-1)

(8) 0111 $\frac{0.5z}{1-0.5z}$ $2^{-n}$ u(n-1)

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

## F.I Complementary Inverting
## Base Inverting
## Range Complementing

(1') 1000 $\frac{2}{1-2z}$ $2^{n+1}$ u(n)

(2') 1001 $\frac{0.5}{1-0.5z}$ $2^{-n-1}$ u(n)

(5') 1100 $\frac{z^{-1}}{1-0.5z^{-1}}$ $2^{n+1}$ u(-n-1)

(6') 1101 $\frac{z^{-1}}{1-2z^{-1}}$ $2^{-n-1}$ u(-n-1)

(3') 1010 $\frac{0.5}{1-0.5z^{-1}}$ $2^{n-1}$ u(-n)

(4') 1011 $\frac{2}{1-2z^{-1}}$ $2^{-n+1}$ u(-n)

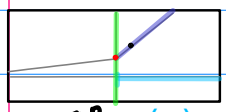
(7') 1110 $\frac{z}{1-2z}$ $2^{n-1}$ u(n-1)

(8') 1111 $\frac{z}{1-0.5z}$ $2^{-n+1}$ u(n-1)

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$
$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

## A.II Flipping
### Base Inverting
### Range Flipping

(1) 0000



$a^n$ u(n)

(4) 0011

$a^{-n}$ u(-n)

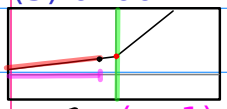
(3) 0010

$a^n$ u(-n)

(2) 0001

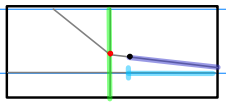
$a^{-n}$ u(n)

(5) 0100



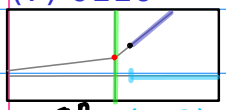
$a^n$ u(-n-1)

(8) 0111



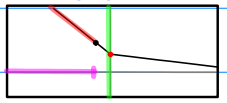
$a^{-n}$ u(n-1)

(7) 0110



$a^n$ u(n-1)

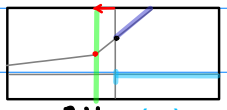
(6) 0101



$a^{-n}$ u(-n-1)

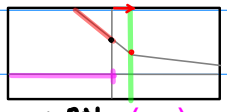
## D.II Flipping2
### Base Inverting
### Shifted Range Flipping

(1') 1000



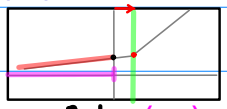
$a^{n+1}$ u(n)

(4') 1011



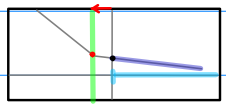
$a^{-n+1}$ u(-n)

(3') 1010



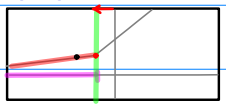
$a^{n-1}$ u(-n)

(2') 1001



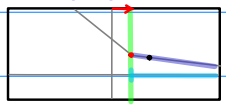
$a^{-n-1}$ u(n)

(5') 1100



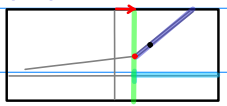
$a^{n+1}$ u(-n-1)

(8') 1111



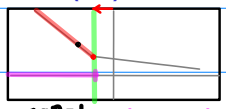
$a^{-n+1}$ u(n-1)

(7') 1110



$a^{n-1}$ u(n-1)

(6') 1101



$a^{-n-1}$ u(-n-1)

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

$$a^n R(n) \Longleftrightarrow a^{-n} R(-n)$$

$$a^n \Longleftrightarrow a^{-n}$$

$$R(n) \Longleftrightarrow R(-n)$$

$$a^n R(n) \Longleftrightarrow a^{-sh2(n)} R(-n)$$

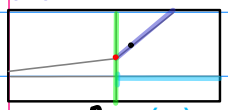
$$a^n \Longleftrightarrow a^{-n}$$

$$a^n R(n) \Longleftrightarrow a^{sh2(n)} R(-n)$$

## B.II Range Shifting
## Range Flipping
## Range Complementing

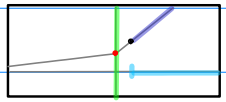
## E.II Shifting2
## Shifted Range Flipping
## Range Complementing

(1) 0000



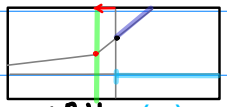
$a^n$ u(n)

(7) 0110



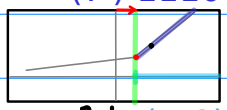
$a^n$ u(n-1)

(1') 1000



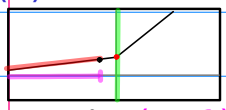
$a^{n+1}$ u(n)

(7') 1110



$a^{n-1}$ u(n-1)

(5) 0100

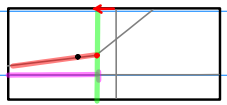


$a^n$ u(-n-1)

(3) 0010

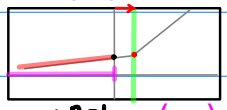
$a^n$ u(-n)

(5') 1100



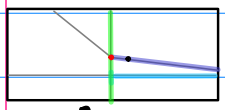
$a^{n+1}$ u(-n-1)

(3') 1010



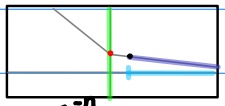
$a^{n-1}$ u(-n)

(2) 0001



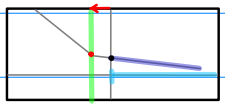
$a^{-n}$ u(n)

(8) 0111



$a^{-n}$ u(n-1)

(2') 1001

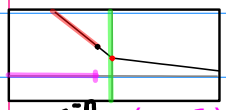


$a^{-n-1}$ u(n)

(8') 1111

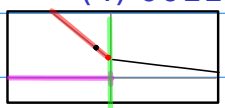
$a^{-n+1}$ u(n-1)

(6) 0101



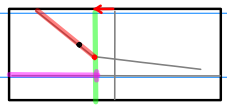
$a^{-n}$ u(-n-1)

(4) 0011



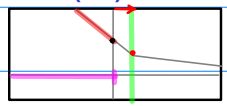
$a^{-n}$ u(-n)

(6') 1101



$a^{-n-1}$ u(-n-1)

(4') 1011



$a^{-n+1}$ u(-n)

Range Shifting
= Range Flipping
+ Range Complementing

$$R(n) \Longleftrightarrow \overline{R(-n)}$$

$$R(n) \Longleftrightarrow R(-n)$$

$$R(n) \Longleftrightarrow \overline{R(n)}$$

Shifted Range Flipping
= Exponent Shifting2
+ Range Flippping

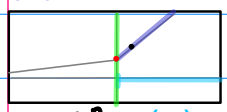
$$a^n R(n) \Longleftrightarrow a^{sh\,2(n)} \overline{R(-n)}$$

$$a^n R(n) \Longleftrightarrow a^{sh\,2(n)} R(-n)$$

$$R(n) \Longleftrightarrow \overline{R(n)}$$

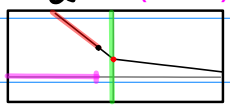
## C.II Complementary Inverting
## Base Inverting
## Range Complementing

(1) 0000



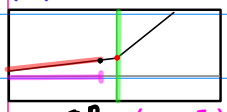
$a^{n}$ u(n)

$a^{-n}$ u(-n-1)



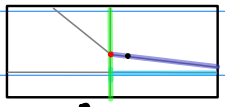
(6) 0101

(5) 0100



$a^{n}$ u(-n-1)

(2) 0001

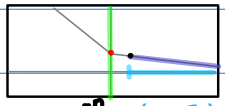


$a^{-n}$ u(n)

(3) 0010

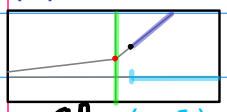
$a^{n}$ u(-n)

(8) 0111



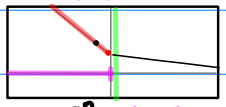
$a^{-n}$ u(n-1)

(7) 0110



$a^{n}$ u(n-1)

(4) 0011



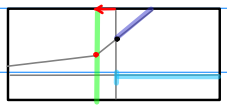
$a^{-n}$ u(-n)

$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$

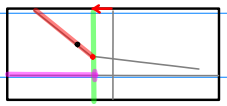
## F.II Complementary Inverting
## Base Inverting
## Range Complementing

(1') 1000



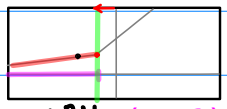
$a^{n+1}$ u(n)

$a^{-n-1}$ u(-n-1)



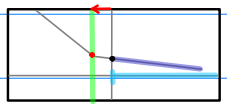
(6') 1101

(5') 1100



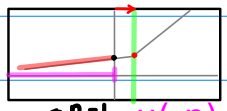
$a^{n+1}$ u(-n-1)

(2') 1001



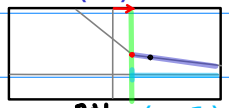
$a^{-n-1}$ u(n)

(3') 1010



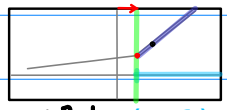
$a^{n-1}$ u(-n)

(8') 1111



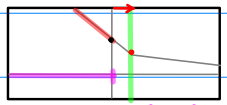
$a^{-n+1}$ u(n-1)

(7') 1110



$a^{n-1}$ u(n-1)

(4') 1011



$a^{-n+1}$ u(-n)

$$a^{n}R(n) \Longleftrightarrow a^{-n}\overline{R(n)}$$

$$a^{n} \Longleftrightarrow a^{-n}$$
$$R(n) \Longleftrightarrow \overline{R(n)}$$