Differentiation of Continuous Functions

Young W Lim

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Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

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Forward Difference Approximation (1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

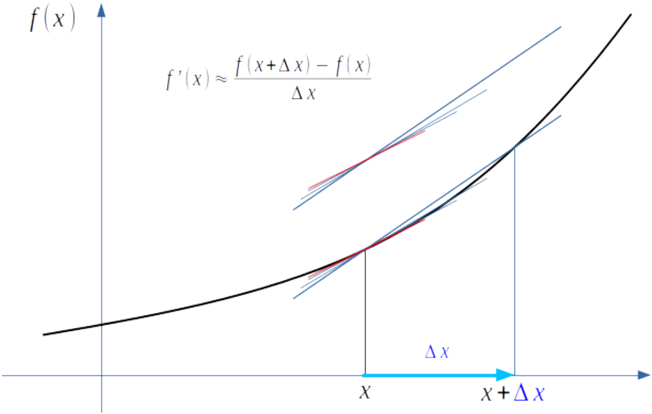

Figure: forward difference approximation

Forward Difference Approximation (3)

a forward difference approximation
as you are taking a point forward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx forwad as $x = x_{i+1}$.

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \\ &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \end{aligned}$$

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Backward Difference Approximation (1a)

forward difference approximation
for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation
for a finite $\Delta x < 0$, then $-\Delta x > 0$,

$$\begin{aligned} f'(x) &\approx \frac{f(x-\Delta x) - f(x)}{-\Delta x} \\ &= \frac{f(x) - f(x-\Delta x)}{\Delta x} \end{aligned}$$

Backward Difference Approximation (1b)

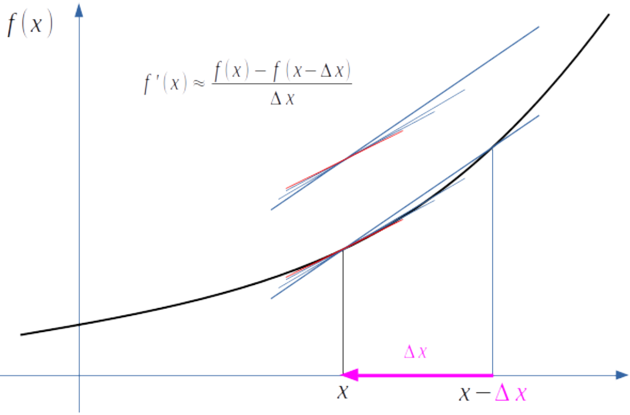

Figure: backward difference approximation (a)

Backward Difference Approximation (2a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$\begin{aligned} f'(x) &\approx \frac{f(x) - f(x-\Delta x)}{x - (x - \Delta x)} \\ &= \frac{f(x) - f(x-\Delta x)}{\Delta x} \end{aligned}$$

Backward Difference Approximation (2b)

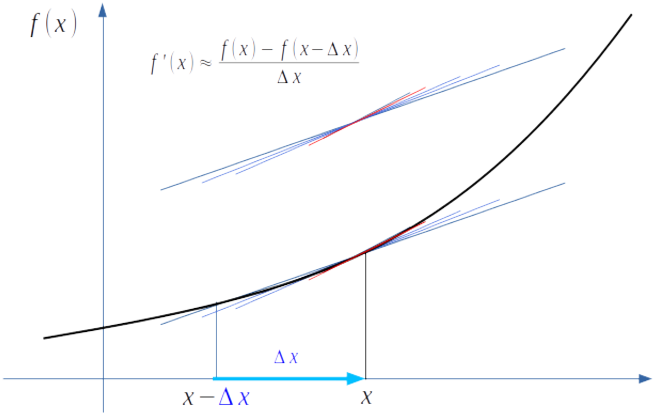

Figure: backward difference approximation (b)

Backward Difference Approximation (3)

a backward difference approximation
as you are taking a point backward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx backwad as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_i) - f(x_{i-1})}{\Delta x} \\ &= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \end{aligned}$$

Outline

Taylor Series (1)

the Taylor series of a function $f(x)$,
that is infinitely differentiable at a point a is the power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series (2)

If $f(x)$ is given by a convergent power series
in an open disk centred at a,
it is said to be *analytic* in this region.

Thus for x in this region,
f is given by a convergent power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Approximating first deivatives

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- for forward difference approximatin

  $$x_i = a, \quad x_{i+1} = x, \quad \Delta x = x_{i+1} - x_i$$

- for forward difference approximatin

  $$x_i = a, \quad x_{i-1} = x, \quad \Delta x = x_i - x_{i-1}$$

Deriving Forward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots,$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots = f'(x_i)(\Delta x)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) - \cdots = f'(x_i)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x) = f'(x_i)$$

Deriving Forward Difference Approximation (3)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

the $O(\Delta x)$ term shows that

the error in the approximation is of the order of Δx

both forward and backward divided difference approximation
of the first derivative are accurate on the order of $O(\Delta x)$

to get better approximations?
another method to approximate the first derivative is called
the Central divided difference approximation of the first derivative.

Deriving Backward Difference Approximation (1)

A Taylor expansion approximate $f(x)$, using $f(a), f'(a), f''(a), \cdots$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i)(\Delta x) = f(x_i) - f(x_{i-1}) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + \frac{f''(x_i)}{2!}(\Delta x) - \cdots$$

$=$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Deriving Central Divide Approximation (1)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$, and substitute $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$, and substitute $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Deriving Central Divide Approximation

$$\boxed{\begin{aligned} f(x_{i+1}) &= f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots \\ f(x_{i-1}) &= f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \cdots \end{aligned}}$$

subtracting eq(2) from eq(1)

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$

$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O\left((\Delta x)^2\right)$$

Outline

Central Divided Approximation

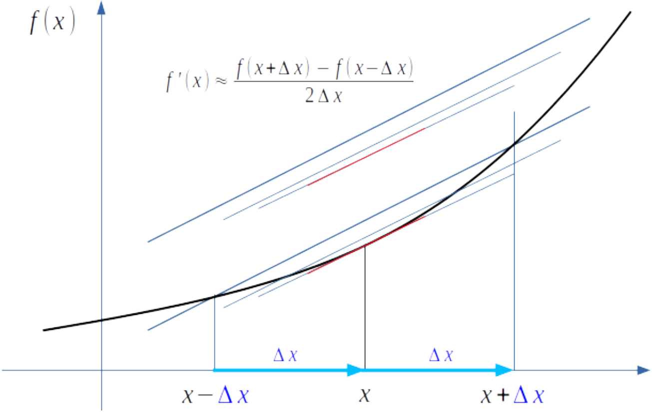

Figure: central difference approximation

Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line → the tangent line
- the slope of the tangent line

$$\begin{aligned} m_{tangent} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$