# Multiple Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
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# Outline

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# Central Limit Theorem

## Definition

the central limit theorem says that
the probability distribution function of
the sum of large number of random variables
approaches a **Gauassian** distribution.
Although this theorem is known to be applicable to
some cases of statistically dependent random variables,
generally applicable to statistically independent random variables.

# Unequal and Equal Distributions

- **Unequal Distributions**

$$\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$$

$$\sigma^2_{Y_N} = \sigma^2_{X_1} + \sigma^2_{X_2} + \cdots + \sigma^2_{X_N}$$

- **Equal Distributions**

$$\begin{aligned} W_N &= (Y_N - \overline{Y}_N)/\sigma_{Y_N} \\ &= \sum_{i=1}^{N}(X_i - \overline{X}_i) \bigg/ \left[\sum_{i=1}^{N}\sigma^2_{X_i}\right]^{1/2} \\ &= \frac{1}{\sqrt{N}\sigma_X}\sum_{i=1}^{N}(X_i - \overline{X}_i) \end{aligned}$$

# Central Limit Theorem
## Uneqaul Distribution Case

### Definition

the sum $Y$ of $N$ independent random variables $X_1, X_2, ..., X_N$
which may have arbitrary probability densities
Let $Y = X_1 + X_2 + \cdots + X_N$, then

$$\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$$

$$\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$$

the probability distribution of $Y$ asymptotically approaches
to Gaussian distribution function as $N \to \infty$

# Sufficient Conditions

## Uneqaul Distribution Case

### Definition

$$\sigma_{X_i}^2 > B_1 > 0 \qquad i = 1, 2, ..., N$$

$$E[|X_i - \overline{X}_i|^3] < B_2 \qquad i = 1, 2, ..., N$$

whre $B_1$ and $B_2$ are positive numbers
these conditions guarantee that no one random variable in the sum dominates

# Distribution vs density functions
## Uneqaul Distribution Case

the central limit theorem guarantees

- only that the distribution of the sum of random variables become Gaussian
- the density of the sum of random variables is not always Gaussian

- the sum of continuous random variables :
  - under certain conditions on individual random variables the density of the sum is always Gaussian
- the sum of discrete random variables :
  - the density function may contain impulses and thus is not Gaussian.

# Discrete Random Variable Examples
## distribution may contain impulses

the sum $Y$ of $N$ independent discrete random variables

$Y = X_1 + X_2 + ... + X_N$

- discrete random variable
- density function may contain impulses
- therefore the density function is not Gaussian
- although the distribution approaches Gaussian

- when the possible discrete values of each random variable are $kb, k = 0, \pm 1, \pm 2, ...,$ where $b$ is a constant
  - the envelope of the impulses in the density of the sum will be Gaussian
  - with the mean $Y_N$ and variance $\sigma^2_{Y_N}$

# Central Limit Theorem (1)

## Eqaul Distribution Case

### Definition

the sum $Y$ of $N$ independent random variables $X_1, X_2, ..., X_N$
assume that $X_1, X_2, ..., X_N$ have the same distribution function.
Let $Y_N = X_1 + X_2 + \cdots + X_N$,
then $W_N = (Y_N - \overline{Y}_N)/\sigma_{Y_N}$ is
the zero-mean, unit-variance random variable

# Central Limit Theorem (2)

## Eqaul Distribution Case

### Definition

Let $Y_N = X_1 + X_2 + \cdots + X_N$ and $W_N = (Y_N - \overline{Y}_N)/\sigma_{Y_N}$

$$
\begin{aligned}
W_N &= (Y_N - \overline{Y}_N)/\sigma_{Y_N} \\
&= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Big/ \left[\sum_{i=1}^{N} \sigma_{X_i}^2\right]^{1/2} \\
&= \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}_i)
\end{aligned}
$$

where $\overline{X}_i = \overline{X}$ and $\sigma_{X_i}^2 = \sigma_X^2$

# Central Limit Theorem (3)

## Eqaul Distribution Case

$$
\begin{aligned}
W_N &= (Y_N - \overline{Y}_N) \Big/ \sigma_{Y_N} \\
&= \left( X_i - \sum_{i=1}^{N} \overline{X}_i \right) \Big/ \left[ \sum_{i=1}^{N} \sigma_{X_i}^2 \right]^{1/2} \\
&= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Big/ \left[ \sum_{i=1}^{N} \sigma_{X_i}^2 \right]^{1/2} \\
&= \sum_{i=1}^{N} (X_i - \overline{X}_i) \Big/ \left[ N\sigma_X^2 \right]^{1/2} \\
&= \frac{1}{\sqrt{N}\sigma_X} \sum_{i=1}^{N} (X_i - \overline{X}_i)
\end{aligned}
$$

where $\overline{X}_i = \overline{X}$ and $\sigma_{X_i}^2 = \sigma_{\overline{X}}$
$\overline{Y}_N = \overline{X}_1 + \overline{X}_2 + \cdots + \overline{X}_N$ and $\sigma_{Y_N}^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$

# Characteristic Function(1)

## Eqaul Distribution Case

the characteristic function of $W_N$
a zero mean, unit variance Gaussian random variable

$$\Phi_{W_N}(\omega) = exp(-\omega^2/2)$$

$W_N$is the density of the Gaussian random variable
Fourier transforms are unique

$$\Phi_{W_N}(\omega) = E[e^{j\omega W_N}]$$

# Characteristic Function(2)

## Eqaul Distribution Case

$$W_N = \frac{1}{\sqrt{N}\sigma_X}\sum_{i=1}^{N}(X_i - \overline{X}_i)$$

$$\Phi_{W_N}(\omega) = E[e^{j\omega W_N}] = E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}\sum_{i=1}^{N}(X_i - \overline{X})\right)\right]$$

$$= E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X})\right)\cdots exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_N - \overline{X})\right)\right]$$

$$= \left\{E\left[exp\left(\frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X})\right)\right]\right\}^N$$

$$E[X_1] = E[X_2] = \cdots = E[X_N] = \overline{X}$$

$$E[(X_1 - \overline{X})^2] = E[(X_1 - \overline{X})^2] = \cdots = E[(X_N - \overline{X})^2] = \sigma_X^2$$

# Characteristic Function (3)

## Eqaul Distribution Case

$$\Phi_{W_N}(\omega) = \left\{ E\left[ exp\left( \frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right] \right\}^N$$

$$\ln[\Phi_{W_N}(\omega)] = N \ln \left\{ E\left[ exp\left( \frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right] \right\}$$

$$E\left[ exp\left( \frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) \right) \right]$$

$$= E\left[ 1 + \frac{j\omega}{\sqrt{N}\sigma_X}(X_1 - \overline{X}) + \left( \frac{j\omega}{\sqrt{N}\sigma_X} \right)^2 (X_1 - \overline{X})^2 + \frac{R_N}{N} \right]$$

$$= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}$$

where $E[R_N]$ approaches zero as $N \to \infty$

# Characteristic Function (4)

## Eqaul Distribution Case

$$\ln[\Phi_{W_N}(\omega)] = N \ln\left[1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]$$

$$\ln[1-z] = -\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right], |z| < 1$$

$$\ln[\Phi_{W_N}(\omega)] = -\frac{\omega^2}{2} + E[R_N] - \frac{N}{2}\left[\frac{\omega^2}{2N} + \frac{E[R_N]}{N}\right]^2 + \cdots$$

$$\lim_{N\to\infty} \ln[\Phi_{W_N}(\omega)] = \ln\left[\lim_{N\to\infty} \Phi_{W_N}(\omega)\right] = -\frac{\omega^2}{2}$$

$$\lim_{N\to\infty} \Phi_{W_N}(\omega) = e^{-\frac{\omega^2}{2}}$$