Abstract Algebra Overview I (H.1)

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Algebraic Structure

In mathematics, and more specifically in abstract algebra, the term **algebraic structure** generally refers to a set (called **carrier set** or **underlying set**) with one or more finitary operations defined on it that satisfies a list of axioms.[1]

Examples of algebraic structures include groups, rings, fields, and lattices. More complex structures can be defined by introducing multiple operations, different underlying sets, or by altering the defining axioms. Examples of more complex algebraic structures include vector spaces, modules, and algebras.

groups
rings
fields
lattices

Vector space
modules
algebras

Set, Group, Ring, Field

A **set** is a collection of unique elements. The definition of a specific set determines which elements are members of the set. Elements not specifically defined as members of a set are not in the set.

A **group** is an algebraic system consisting of **a set**,
an identity element for each operation,
one operation and its inverse operation.

A **ring** is an algebraic system consisting of **a set**,
an identity element for each operation,
two operations and the inverse operation of the first operation.

A **field** is an algebraic system consisting of **a set**,
an identity element for each operation,
two operations and their respective inverse operations.

http://www.csee.umbc.edu/portal/help/theory/group_def.shtml

Group Definition

A group is a set, G, together with an operation $\bullet$ (called the *group law* of G) that combines any two elements a and b to form another element, denoted $a \bullet b$ or ab. To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy four requirements known as the *group axioms*.[5]

Closure

For all a, b in G, the result of the operation, $a \bullet b$, is also in G.[b]

Associativity

For all a, b and c in G, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

Identity element

There exists an element e in G, such that for every element a in G, the equation $e \bullet a = a \bullet e = a$ holds. Such an element is unique (see below), and thus one speaks of *the* identity element.

Inverse element

For each a in G, there exists an element b in G, commonly denoted a^{-1} (or $-a$, if the operation is denoted "+"), such that $a \bullet b = b \bullet a = e$, where e is the identity element.

> [T]he axioms for a group are short and natural... Yet somehow hidden behind these axioms is the monster simple group, a huge and extraordinary mathematical object, which appears to rely on numerous bizarre coincidences to exist. The axioms for groups give no obvious hint that anything like this exists.
>
> Richard Borcherds in *Mathematicians: An Outer View of the Inner World* [4]

Abelian group

From Wikipedia, the free encyclopedia

For the group described by the archaic use of the related term "Abe
group.

In abstract algebra, an **abelian group**, also called a **commutative group**, is a group in which the result of applying the group operation to two group elements does not depend on the order in which they are written (the axiom of commutativity). Abelian groups generalize the arithmetic of addition of integers. They are named after Niels Henrik Abel.[1]

A **group** is an algebraic system consisting of **a set**,
an identity element for each operation,
one operation and its inverse operation.

Abelian Group Definition

An abelian group is a set, A, together with an operation $\bullet$ that combines any two elements a and b to form another element denoted $a \bullet b$. The symbol $\bullet$ is a general placeholder for a concretely given operation. To qualify as an abelian group, the set and operation, $(A, \bullet)$, must satisfy five requirements known as the *abelian group axioms*:

Closure

For all a, b in A, the result of the operation $a \bullet b$ is also in A.

Associativity

For all a, b and c in A, the equation $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ holds.

Identity element

There exists an element e in A, such that for all elements a in A, the equation $e \bullet a = a \bullet e = a$ holds.

Inverse element

For each a in A, there exists an element b in A such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Commutativity

For all a, b in A, $a \bullet b = b \bullet a$.

More compactly, an abelian group is a commutative group. A group in which the group operation is not commutative is called a "non-abelian group" or "non-commutative group".

Ring Definition

A **ring** is a set R equipped with binary operations[1] $+$ and $\cdot$ satisfying the following three sets of axioms, called the **ring axioms**[2][3][4]

1. R is an abelian group under addition, meaning that
   - $(a + b) + c = a + (b + c)$ for all a, b, c in R ($+$ is associative).
   - $a + b = b + a$ for all a, b in R ($+$ is commutative).
   - There is an element 0 in R such that $a + 0 = a$ for all a in R (0 is the additive identity).
   - For each a in R there exists $-a$ in R such that $a + (-a) = 0$ ($-a$ is the additive inverse of a).
2. R is a monoid under multiplication, meaning that:
   - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in R ($\cdot$ is associative).
   - There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ for all a in R (1 is the multiplicative identity).[5]
3. Multiplication is distributive with respect to addition:
   - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all a, b, c in R (left distributivity).
   - $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in R (right distributivity).

algebra

In mathematics, an **algebra** is one of the fundamental algebraic structures used in abstract algebra. An **algebra over a field** is a vector space (a module over a field) equipped with a bilinear product. Thus, an algebra over a field is a set, together with operations of multiplication, addition, and scalar multiplication by elements of the underlying field, that satisfy the axioms implied by "vector space" and "bilinear".[1]

The multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras and nonassociative algebras. Given an integer *n*, the ring of real square matrices of order *n* is an example of an associative algebra over the field of real numbers under matrix addition and matrix multiplication. Three-dimensional Euclidean space with multiplication given by the vector cross product is an example of a nonassociative algebra over the field of real numbers.